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On p-adic T-numbers

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Abstract. Denote by w_n^* and w_n the exponents of Diophantine approximation defined in Mahler's and Koksma's classifications of transcendental numbers, respectively. We prove that there are p-adic T-numbers ξ such that $w_n(\xi) - w_n^*(\xi)$ is any number chosen in the segment $[0, (n-1)/n]$ for all positive integers n and for $w_n(\xi)$ large enough. Thus we improve SCHLICKEWEI's result [10] that p-adic T-numbers do exist.

1. Introduction

MAHLER [6] introduced in 1932 a classification of complex transcendental numbers according to how small the value of an integer polynomial at the given number can be with regards to the the height and degree of this polynomial. In 1939 KOKSMA [5] devised another classification which looks at how closely the complex transcendental number can be approximated by algebraic numbers of bounded height and degree. Koksma proved that the two classifications are identical and thus we have three classes consisting of S-numbers or S^*-numbers, T-numbers or T^*-numbers and U-numbers or U^*-numbers. Here the nonstared letters refer to Mahler's classification whereas the stared ones refer to Koksma's. See [3] for all references.

While almost all numbers in the sense of Lebesgue measure are S-numbers and U-numbers contain for example Liouville numbers, it was only in 1968 that SCHMIDT [11] proved the existence of T-numbers.

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Schlickewei [10] adapted this result to the p-adic setting. After this informal introduction, we give the necessary definitions in order to explain how the main result of this paper improves Schlickewei's result. We take inspiration from a paper by R. C. Baker [2] on complex T-numbers in order to establish similar results for p-adic T-numbers.

In the sequel, we use the following notation.

Let p be a prime number. We denote by $\mathbb{Q}_p$ the completion of the field of rational numbers $\mathbb{Q}$ with respect to the p-adic absolute value $|\cdot|_p$ which is normalised in such a way that $|p|_p = p^{-1}$. We also use $\mathbb{C}_p$ for the (metric) completion of an algebraic closure of $\mathbb{Q}_p$. The field $\mathbb{Q}_p$ of p-adic numbers is usually considered an analogue of the field $\mathbb{R}$ of real numbers, while the field $\mathbb{C}_p$ is analogous to the field $\mathbb{C}$ of complex numbers. Basic facts about p-adic theory will be tacitly used in this article, interested reader can consult e.g. [4].

The notation $\mathrm{H}(P)$ stands for the naive height of the polynomial P, i.e. the maximum of the absolute values of its coefficients and the height $\mathrm{H}(\alpha)$ of a p-adic algebraic number α is that of its minimal polynomial over $\mathbb{Z}$.

In analogy with his classification of complex numbers, Mahler proposed a classification of p-adic numbers. Let $\xi \in \mathbb{Q}_p$ and given $n \geq 1$, $H \geq 1$, define the quantity

$$w_n(\xi, H) := \min\{|P(\xi)|_p : P(X) \in \mathbb{Z}[X],\ \deg(P) \leq n,\ \mathrm{H}(P) \leq H,\ P(\xi) \neq 0\}.$$

We set

$$w_n(\xi) := \limsup_{H\to\infty} \frac{-\log(H w_n(\xi, H))}{\log H} \quad \text{and} \quad w(\xi) := \limsup_{n\to\infty} \frac{w_n(\xi)}{n},$$

and thus $w_n(\xi)$ is the upper limit of the real numbers w for which there exist infinitely many integer polynomials $P(X)$ of degree at most n satisfying

$$0 < |P(\xi)|_p \leq \mathrm{H}(\alpha)^{-w-1}.$$

In analogy with Koksma's classification of complex numbers, for $\xi \in \mathbb{Q}_p$ and given $n \geq 1$, $H \geq 1$, we define the quantity

$$w_n^*(\xi, H) := \min\{|\xi - \alpha|_p : \alpha \text{ algebraic in } \mathbb{Q}_p,\ \deg(\alpha) \leq n,\ \mathrm{H}(\alpha) \leq H,\ \alpha \neq \xi\}.$$

We set

$$w_n^*(\xi) := \limsup_{H\to\infty} \frac{-\log(H w_n^*(\xi, H))}{\log H} \quad \text{and} \quad w^*(\xi) := \limsup_{n\to\infty} \frac{w_n^*(\xi)}{n},$$

and thus $w_n^*(\xi)$ is the upper limit of the real numbers w for which there exist infinitely many algebraic numbers α in $\mathbb{Q}_p$ of degree at most n satisfying

$$0 < |\xi - \alpha|_p \le \mathrm{H}(\alpha)^{-w-1}.$$

We say that a transcendental number $\xi \in \mathbb{Q}_p$ is an

- *S-number* if $w(\xi) < \infty$;
- *T-number* if $w(\xi) = \infty$ and $w_n(\xi) < \infty$ for any integer $n \ge 1$;
- *U-number* if $w(\xi) = \infty$ and $w_n(\xi) = \infty$ for some integer $n \ge 1$.

S^*-, T^*- and U^*- numbers are defined as above, using w_n^* in place of w_n.

Actually, the definition of the quantity $w_n(\xi)$ given here differs from the one used by Schlickewei [10]. Indeed, for him, the numerator of the defining fraction is $-\log(w_n(\xi, H))$ instead of $-\log(Hw_n(\xi, H))$. This means that there is a shift by 1 in the value of the critical exponent, which however does not imply any change regarding the class of a given p-adic number. We have adopted the same choice as in [3] since then $w_n(\xi) = w_n^*(\xi) = n$ holds for almost all p-adic numbers ξ, with respect to the Haar measure on $\mathbb{Q}_p$. See [3] for details and further results on the exponents w_n and w_n^*.

The central result of Schlickewei's paper [10] is his Theorem 2:

Theorem S. *Let $(B_n)_{n\ge1}$ be a sequnce of real numbers such that*

$$B_1 > 9, \quad B_n > 3n^2 B_{n-1} \text{ for } n > 1.$$

There exist numbers $\xi \in \mathbb{Q}_p$ with

$$w_n^*(\xi) = B_n \text{ for any } n \ge 1.$$

While Schlickewei showed that p-adic T-numbers do exist, his proof only gave numbers ξ such that $w_n(\xi) = w_n^*(\xi)$ for all integers $n \ge 1$. Since for any p-adic transcendental number ξ we have

$$w_n^*(\xi) \le w_n(\xi) \le w_n^*(\xi) + n - 1$$

(see Theorem 9.3 in [3]), it is natural to ask whether there exist p-adic numbers ξ such that $w_n(\xi) \ne w_n^*(\xi)$ for some integer n and how large can $w_n(\xi) - w_n^*(\xi)$ really be. Although the second question is, as in the more extensively studied real case, far from being resolved, our main result (cf. [2] or [3, Theorem 7.1, p. 140]) gives a positive answer to the first question and it means a progress towards answering the second one.

Theorem 1. *Let $(w_n)_{n\geq 1}$ and $(w_n^*)_{n\geq 1}$ be two non-decreasing sequences in $[1,+\infty]$ such that*

$$w_n^* \leq w_n \leq w_n^* + (n-1)/n, \quad w_n > n^3 + 2n^2 + 5n + 2, \quad \text{for any } n \geq 1. \tag{1}$$

Then there exists a p-adic transcendental number ξ such that

$$w_n^*(\xi) = w_n^* \quad \text{and} \quad w_n(\xi) = w_n, \quad \text{for any } n \geq 1.$$

It is also important to notice that we impose much milder growth requirements on the sequence $(w_n)_{n\geq 1}$ than in Theorem S. Thus our Theorem 1 considerably improves the range of attainable values for w_n^* and w_n.

The next section brings together all the necessary auxiliary results which are for the most part analogues of the results in the real and complex case. In Section 3 we give the main proposition together with its proof and in the last section we use this proposition to prove our Theorem 1.

2. Auxiliary results

We will be using the following lemmas which are included here for the convenience of reader.

Lemma 1 (Gelfond's Lemma)**.** *Let $P_1(x_1,\dots,x_k),\dots,P_r(x_1,\dots,x_k)$ be non-zero polynomials of total degree $n_1,\dots,n_r$, respectively, and set $n = n_1+\cdots+n_r$. We then have*

$$2^{-n}\,\mathrm{H}(P_1)\cdots\mathrm{H}(P_r) \leq \mathrm{H}(P_1\cdots P_r) \leq 2^n\,\mathrm{H}(P_1)\cdots\mathrm{H}(P_r).$$

Proof. See [3, Lemma A.3, p. 221]. □

Lemma 2. *Let $\alpha \in \mathbb{Q}_p$ be a non-zero algebraic number of degree n. Let a, b and c be integers with $c \neq 0$. We then have*

$$\mathrm{H}\left(\frac{a\alpha+b}{c}\right) \leq 2^{n+1}\,\mathrm{H}(\alpha)\max\{|a|,|b|,|c|\}^n,$$

where $\mathrm{H}(\alpha)$ is the height of the minimal polynomial of α over $\mathbb{Z}$.

Proof. (cf. [3, Lemma A.4, p. 222]) Let $P(X)$ and $Q(X)$ be minimal polynomials over $\mathbb{Z}$ of α and $\frac{a\alpha+b}{c}$, respectively. Since $Q(\frac{aX+b}{c})$ is a polynomial in $\mathbb{Q}[X]$

vanishing at α, we must have $\deg Q \geq \deg P$. Likewise, since $a^n P(\frac{cX-b}{a})$ is a polynomial in $\mathbb{Z}[X]$ vanishing at $\frac{a\alpha+b}{c}$, we must have $\deg Q \leq \deg P$. From the minimality of P and Q using Gauss's lemma, we conclude that $a^n P(\frac{cX-b}{a}) = d \cdot Q(X)$, where d is an integer. Therefore,

$$\begin{aligned}\mathrm{H}\left(\frac{a\alpha+b}{c}\right) &= \mathrm{H}(Q(X)) \leq \mathrm{H}\left(a^n P\Big(\frac{cX-b}{a}\Big)\right)\\ &\leq \max_i \sum_{k=i}^{n} \binom{k}{i} \cdot \max\{|a|,|b|,|c|\}^n \cdot \mathrm{H}(P) < 2^{n+1} \max\{|a|,|b|,|c|\}^n \cdot \mathrm{H}(P),\end{aligned}$$

because $\sum_{k=i}^{n} \binom{k}{i} = \binom{n+1}{i+1} < 2^{n+1}$. □

Lemma 3. *Let $P(X) = a_n X^n + \cdots + a_1 X + a_0 = a_n(X-\alpha_1)\cdots(X-\alpha_n) \in \mathbb{C}_p[X]$. Then for any set $I \subseteq \{1,\ldots,n\}$, it holds*

$$\prod_{i\in I} |\alpha_i|_p \leq \frac{\max_{j\in\{0,1,\ldots,n\}} |a_j|_p}{|a_n|_p}.$$

Proof. This is shown in [7, p. 341]. □

Lemma 4. *Let $P(X)$ be a non-constant, separable, integer polynomial of degree n. Let $\xi \in \mathbb{C}_p$ and α be a root of $P(X)$ in $\mathbb{C}_p$ such that $|\xi-\alpha|_p$ is minimal. Then*

$$n^{-3n/2}\,\mathrm{H}(P)^{-n+1}|\xi-\alpha|_p \leq |P(\xi)|_p.$$

Proof. (cf. [3, Lemma A.8, p. 231]) Let $P(X) = a_n X^n + \cdots + a_1 X + a_0 = a_n(X-\alpha_1)\cdots(X-\alpha_n) \in \mathbb{Z}[X]$, where we order roots $\alpha_1,\ldots,\alpha_n \in \mathbb{C}_p$ in such a way that

$$\alpha = \alpha_1 \quad \text{and} \quad |\xi-\alpha_1|_p \leq \cdots \leq |\xi-\alpha_n|_p.$$

First we bound the discriminant of $P(X)$.

$$\begin{aligned}|\operatorname{Disc}(P)| &= \left|\frac{1}{a_n}(-1)^{\frac{n(n-1)}{2}} \operatorname{Res}(P,P')\right|\\ &= \frac{1}{|a_n|}\left|\det\begin{bmatrix} a_n & \cdots & a_0 & & \\ & \ddots & & \ddots & \\ & & a_n & \cdots & a_0 \\ na_n & \cdots & a_1 & & \\ & \ddots & & \ddots & \\ & & na_n & \cdots & a_1 \end{bmatrix}\right| = \left|\det\begin{bmatrix} 1 & \cdots & a_0 & & \\ & \ddots & & \ddots & \\ & & a_n & \cdots & a_0 \\ n & \cdots & a_1 & & \\ & \ddots & & \ddots & \\ & & na_n & \cdots & a_1 \end{bmatrix}\right|\\ &\leq (n+1)^{n-1} n^n \cdot n^n\,\mathrm{H}(P)^{2n-2} \leq n^{3n}\,\mathrm{H}(P)^{2n-2}.\end{aligned}$$

Now we have,

$$n^{-3n/2}\,\mathrm{H}(P)^{-n+1} \le \sqrt{|\operatorname{Disc}(P)|_p} = |a_n|_p|\alpha_1-\alpha_2|_p\cdots|\alpha_1-\alpha_n|_p\sqrt{|\operatorname{Disc}(Q)|_p},$$

where $Q(X) = a_n(X-\alpha_2)\cdots(X-\alpha_n)$. Applying Lemma 3 to the integer polynomial $P(X)$, we get the last bound below

$$\sqrt{|\operatorname{Disc}(Q)|_p} = |a_n|_p^{n-2}\cdot\left|\begin{vmatrix}\alpha_2^0 & \cdots & \alpha_2^{n-2}\\ \vdots & \ddots & \vdots\\ \alpha_n^0 & \cdots & \alpha_n^{n-2}\end{vmatrix}\right|_p$$

Using $|\xi-\alpha_1|_p \le \cdots \le |\xi-\alpha_n|_p$, we get for $1 \le i < j \le n$

$$|\alpha_i-\alpha_j|_p \le \max\{|\xi-\alpha_i|_p, |\xi-\alpha_j|_p\} = |\xi-\alpha_j|_p$$

which combined with the previous bound gives

$$\begin{aligned} n^{-3n/2}\,\mathrm{H}(P)^{-n+1} &\le |a_n|_p|\xi-\alpha_2|_p\cdots|\xi-\alpha_n|_p,\\ n^{-3n/2}\,\mathrm{H}(P)^{-n+1}|\xi-\alpha|_p &\le |P(\xi)|_p. \end{aligned}$$

□

Lemma 5. *Let ξ be an algebraic number in $\mathbb{Q}_p$ and n be a positive integer. Then, for any positive real number ε, there exists a positive (ineffective) constant $\kappa(\xi,n,\varepsilon)$ such that*

$$|\xi-\alpha|_p > \kappa(\xi,n,\varepsilon)\,\mathrm{H}(\alpha)^{-n-1-\varepsilon}$$

for any algebraic number α of degree at most n.

Proof. See [10, Theorem 3, p. 183]. □

Lemma 6. *Let n be a positive integer.*

(a) *Let p be an odd prime and d be the smallest prime in the arithmetic progression $1, p+1, 2p+1, \ldots$.*
   (i) *If $p \nmid n$, the polynomial $X^n - d$ is irreducible over $\mathbb{Q}$ and has a root in $\mathbb{Q}_p$.*
   (ii) *If $p|n$, the polynomial $X^n - dX^{n-1} + dX - d$ is irreducible over $\mathbb{Q}$ and has a root in $\mathbb{Q}_p$.*

(b) *If $p = 2$, then*
   (iii) *If n is odd, the polynomial $X^n - 3$ is irreducible over $\mathbb{Q}$ and has a root in $\mathbb{Q}_2$.*

(iv) *If n is even, the polynomial $X^n + X + 2$ is irreducible over $\mathbb{Q}$ and has a root in $\mathbb{Q}_2$.*

Moreover, in each of the four cases we can take the root to be in $1 + p\mathbb{Z}_p$.

PROOF. The statement of this Lemma is given in [10, Lemma 1, p. 184]). Proof of irreducibility uses Eisenstein's criterion (see e.g. [9, Theorem 2.1.3, p. 50]) in cases (i), (ii) and (iii). For the irreducibility in case (iv) we use another result by OSADA [8], see [9, Theorem 2.2.7, p. 58].

Hensel's lemma shows that each of the specified polynomials has a root in $1 + p\mathbb{Z}_p$. □

Lemma 7. *If $\alpha \in \mathbb{C}_p$ is a root of the polynomial $P(X) = a_n X^n + \cdots + a_1 X + a_0 \in \mathbb{Z}_p[X]$, then $|a_0|_p \le |\alpha|_p \le 1/|a_n|_p$.*

PROOF. If $\alpha = 0$, then both inequalities obviously hold. Therefore, we assume $\alpha \neq 0$. From $P(\alpha) = 0$ we get

$$a_0 \alpha^{-1} = -(a_n \alpha^{n-1} + \cdots + a_1).$$

If $|\alpha|_p \le 1$ this implies $|a_0 \alpha^{-1}|_p \le 1$. If $|\alpha|_p > 1$, $|a_0 \alpha^{-1}|_p \le 1$ obviously holds. Either way, we get $|a_0|_p \le |\alpha|_p$ and the other inequality follows from this one by noting that $1/\alpha$ is a root of the polynomial $X^n P(1/X) = a_0 X^n + a_1 X^{n-1} + \cdots + a_n$. □

For the prime p and any positive integer n we denote by $\eta_n \in 1 + p\mathbb{Z}_p$ the root defined in the appropriate case of Lemma 6.

Lemma 8. *If η_n' is a conjugate of η_n over $\mathbb{Q}$ different from η_n itself, then $|\eta_n' - \eta_n|_p = 1$.*

PROOF. Obviously, η_n and η_n' are both roots of a polynomial $P(X)$ mentioned in Lemma 6. We denote by $\delta = \eta_n' - \eta_n$ and then easily establish that it satisfies

$$0 = \frac{P(\eta_n') - P(\eta_n)}{\delta} = \sum_{k=1}^{n} \frac{P^{(k)}(\eta_n)}{k!} \delta^{k-1}$$

where $P^{(k)}(\eta_n)/k! \in \mathbb{Z}_p$ since $P(X) \in \mathbb{Z}[X]$ and $\eta_n \in \mathbb{Z}_p$. It follows from Lemma 7 that

$$|P'(\eta_n)|_p \le |\delta|_p \le \frac{1}{|P^{(n)}(\eta_n)/n!|_p}.$$

But with reference to Lemma 6,

$$P'(\eta_n) \equiv n \cdot 1 \not\equiv 0 \pmod{p} \qquad \text{(case (i))},$$

$$P'(\eta_n) \equiv n \cdot 1 - 1 \cdot (n-1) \cdot 1 + 1 \equiv 2 \not\equiv 0 \pmod{p} \qquad \text{(case (ii))},$$
$$P'(\eta_n) \equiv n \cdot 1 \equiv 1 \not\equiv 0 \pmod{p} \qquad \text{(case (iii))},$$
$$P'(\eta_n) \equiv n \cdot 1 + 1 \equiv 1 \not\equiv 0 \pmod{p} \qquad \text{(case (iv))},$$

while $P^{(n)}(\eta_n)/n! = 1$ in all four cases. This shows that $|\delta|_p = 1$ which is what we wanted to prove. □

Remark 1. In order to minimize cumbersome repetition, we will be assuming that p is an odd prime which does not divide the degree n of the algebraic number η_n we defined earlier, in other words, the situation from case (i) of Lemma 6. Modifications which are needed to deal with the other three cases from this Lemma will be briefly mentioned at the appropriate places.

Later on, we will define $\xi_j = \frac{c_j + v_j \eta_{m_j}}{g_j}$, where c_j, v_j, g_j will be integers such that $P_j(X) = (g_j X - c_j)^{m_j} - d v_j^{m_j}$ really is the minimal polynomial of ξ_j over $\mathbb{Z}$. If $\xi_j = \theta_{j,1}, \theta_{j,2}, \ldots, \theta_{j,m_j} \in \mathbb{C}_p$ are roots of $P_j(X)$ then we obviously have

$$\xi_j - \theta_{j,k} = \frac{c_j + v_j \eta_{m_j}}{g_j} - \frac{c_j + v_j \eta'_{m_j}}{g_j} = \frac{v_j}{g_j}(\eta_{m_j} - \eta'_{m_j}),$$

where we denoted by η'_{m_j} a conjugate of η_{m_j}. But Lemma 8 now implies $|\xi_j - \theta_{j,k}|_p = |\frac{v_j}{g_j}|_p$ for all $k = 2, \ldots, m_j$.

In our construction we will have $\xi = \lim_{j\to\infty} \xi_j$ and $|\xi_j - \theta_{j,k}|_p > |\xi_j - \xi|_p$, so $|\xi - \theta_{j,k}|_p = |\xi_j - \theta_{j,k}|_p = |\frac{v_j}{g_j}|_p$ which gives

$$|P_j(\xi)|_p = |g_j^{m_j}|_p \prod_{k=1}^{m_j} |\xi - \theta_{j,k}|_p = |v_j|_p^{m_j - 1} |g_j|_p |\xi - \xi_j|_p. \tag{2}$$

(It is easily seen that the same equality holds in other three cases from Lemma 6 as well.)

Remark 2. Let us consider what happens if we take $\xi_j = \frac{a_j}{b_j} \eta_{m_j}$ where $a_j, b_j \in \mathbb{Z}$ and $\gcd(a_j, b_j) = 1$. Schlickewei even has $|a_j|_p = |b_j|_p = 1$ in [10], but we do not take these additional assumptions. Now, because $\eta_{m_j}^{m_j} - d = 0$, where $d \equiv 1 \pmod{p}$, we have $|\eta_{m_j}|_p = 1$ and thus $|\xi_j|_p = |a_j|_p |b_j|_p^{-1}$. If $|a_j|_p$ is not bounded by a positive number from below then $\gcd(a_j, b_j) = 1$ implies $|\xi_{j_k}|_p \to 0$ $(k \to \infty)$ for some subsequence (ξ_{j_k}) which gives $\xi = 0$ and this is not possible. If $|b_j|_p$ is not bounded by a positive number from below then $\gcd(a_j, b_j) = 1$ implies $|\xi_{j_k}|_p \to \infty$ $(k \to \infty)$ for some subsequence (ξ_{j_k}) which gives $\xi = \infty$ and this is not possible either.

Therefore, $(|a_j|_p)_{j\geq 1}$ and $(|b_j|_p)_{j\geq 1}$ are both bounded from below by a positive number and since they are trivially bounded from above by 1, we conclude that for all positive integers n and for all j such that $m_j = n$, $|a_j|_p^{m_j-1}|b_j|_p$ is bounded so that (2) implies $|P_j(\xi)|_p \asymp |\xi - \xi_j|_p$. This gives (after an analysis we later give for the general case we study) $w_n(\xi) = w_n^*(\xi)$. That is why we have to construct ξ_j in a more complicated manner analogous to the real case.

3. Main proposition

We now follow the exposition of R. C. Baker's theorem as given in [3, §7.2, p. 141]. Some lines where the proof is identical to the real case will be briefly mentioned, while places where a modification is necessary will be more thoroughly explained.

For the real number r, we denote by $\lfloor r \rfloor$ the largest integer not greater than r.

Proposition 1 (cf. [3, Proposition 7.1, p. 142]). *Let $\nu_1, \nu_2, \ldots$ be real numbers > 1 and $\mu_1, \mu_2, \ldots$ be real numbers in $[0,1]$. Let $m_1, m_2, \ldots$ be positive integers and $\chi_1, \chi_2, \ldots$ be real numbers satisfying $\chi_n > n^3 + 2n^2 + 4n + 3$ for any $n \geq 1$. Then, there exist positive real numbers $\lambda_1, \lambda_2, \ldots$, prime numbers $g_1, g_2, \ldots$, and integers $c_1, c_2, \ldots$ such that the following conditions are satisfied.*

(I_j) $g_j \nmid c_j^{m_j} + (-1)^{m_j+1} d v_j^{m_j}$, *where* $v_j = p^{\lfloor \mu_j \log_p g_j \rfloor}$ $(j \geq 1)$.
$(g_j \nmid c_j^{m_j} + d v_j c_j^{m_j-1} + (-1)^{m_j+1} d v_j^{m_j-1} c_j + (-1)^{m_j+1} d v_j^{m_j}$,
$g_j \nmid c_j^{m_j} + (-1)^{m_j+1} 3 v_j^{m_j}$,
$g_j \nmid c_j^{m_j} + (-1)^{m_j+1} v_j^{m_j-1} c + (-1)^{m_j} 2 v_j^{m_j}$
in cases (ii), (iii) and (iv) of Lemma 6, respectively)

(II_1) $\xi_1 = \dfrac{c_1 + v_1 \eta_{m_1}}{g_1}$.

(II_j) $\xi_j = \dfrac{c_j + v_j \eta_{m_j}}{g_j}$ *belongs to the annulus $I_{j-1} \subseteq \mathbb{Q}_p$ defined by*

$$\frac{1}{2p} g_{j-1}^{-\nu_{j-1}} \leq |x - \xi_{j-1}|_p < g_{j-1}^{-\nu_{j-1}}.$$

(III_j) $|\xi_j - \alpha_n|_p \geq \lambda_n H(\alpha_n)^{-\chi_n}$ *for any algebraic number α_n of degree $n \leq j$ which is distinct from $\xi_1, \ldots, \xi_j$ $(j \geq 1)$.*

Proof. In what follows, we denote by α_n a p-adic algebraic number of degree exactly n. We fix a sequence $(\varepsilon_n)_{n\geq 1}$ in $]0,1[$ such that, for any $n \geq 1$, we have

$$\chi_n > n^3 + 2n^2 + 4n + 3 + 20n^2\varepsilon_n. \tag{3}$$

We add four extra conditions $(IV_j),\dots,(VII_j)$ to be satisfied by the numbers ξ_j.

Set $J_0=I_0$ and

$$J_j:=\{x\in I_j:|x-\alpha_n|_p\ge 2\lambda_n\,\mathrm{H}(\alpha_n)^{-\chi_n}\text{ for any algebraic }\alpha_n \\ \text{of degree } n\le j,\ \alpha_n\ne\xi_1,\dots,\xi_j,x,\ \mathrm{H}(\alpha_n)\ge(\lambda_n g_j^{\nu_j})^{1/\chi_n}\}.$$

The extra conditions are:

(IV_j) $\xi_j\in J_{j-1}$ $(j\ge 2)$.

(V_j) $|\xi_j-\alpha_j|_p\ge 2\lambda_j\,\mathrm{H}(\alpha_j)^{-\chi_j}$ for any $\alpha_j\ne\xi_j$ $(j\ge 1)$.

(VI_j) $n\le j,\ \mathrm{H}(\alpha_n)\le g_j^{1/(n+1+\varepsilon_n)}\ \Rightarrow\ |\xi_j-\alpha_n|_p\ge 1/g_j$ $(j\ge 1)$.

(VII_j) $\mu(J_j)\ge\mu(I_j)/2$ $(j\ge 1)$.

Here, μ denotes the Haar measure ($\mu(\{x\in\mathbb{Q}_p:|x-a|_p\le p^{-\lambda}\})=p^{-\lambda}$).

We construct the numbers $\xi_1,\lambda_1,\xi_2,\lambda_2,\dots$ by induction with description of steps the same as in [3, p. 144]. At the j-th stage, there are two steps. Step (A_j) consist in building an algebraic number

$$\xi_j=\frac{c_j+v_j\eta_{m_j}}{g_j}$$

satisfying conditions (I_j) to (VI_j). In step (B_j), we show that the number ξ_j constructed in (A_j) satisfies (VII_j) as well, provided that g_j is chosen large enough in terms of

$$\nu_1,\dots,\nu_j,\mu_1,\dots,\mu_j,m_1,\dots,m_j,\chi_1,\dots,\chi_j, \\ \varepsilon_1,\dots,\varepsilon_j,\xi_1,\dots,\xi_{j-1},\lambda_1,\dots,\lambda_{j-1}. \quad (4)$$

The symbols o, $\gg$ and $\ll$ used throughout steps (A_j) and (B_j) mean that the numerical implicit constants depend (at most) on quantities (4). Furthermore, the symbol o implies 'as g_j tends to infinity'.

Note that we will have $v_j,c_j\in\{1,\dots,g_j-1\}$ and therefore $g_j=\max\{g_j,v_j,c_j\}$.

Step (A_1) is easy. Let $P(X):=X^{m_1}-dv_1^{m_1}$ denote the minimal polynomial of $\eta_{m_1}v_1$ over $\mathbb{Z}$ and observe that (I_1) is satisfied if, and only if, g_1 does not divide $P(-c_1)$. But if a prime number g divides $P(k),P(k+1),\dots,P(k+m_1)$ for some integer k then using the divided differences or the argument from [3] we get that g divides $m_1!$. Hence, if we take the prime number g_1 larger than m_1 (and p), there are $\gg g_1$ possible numbers $\xi_1=(c_1+v_1\eta_{m_1})/g_1$ and since $c_1\le g_1$, the distance between such numbers is

$$\left|\frac{c_1'+v_1\eta_{m_1}}{g_1}-\frac{c_1''+v_1\eta_{m_1}}{g_1}\right|_p=\left|\frac{c_1'-c_1''}{g_1}\right|_p\ge\frac{1}{g_1}. \quad (5)$$

There are only $o(g_1)$ rational numbers α_1 satisfying $\mathrm{H}(\alpha_1) \le g_1^{1/(2+\varepsilon_1)}$, so we are able to choose ξ_1 such that (VI_1) is verified. Moreover, by Lemma 5 with $n=1$, there exist λ_1 in $]0,1[$ such that both (III_1) and (V_1) hold.

We continue exactly as in [3] making only the necessary and obvious changes. Let $j \ge 2$ be an integer and assume that $\xi_1, \dots, \xi_{j-1}$ have been constructed. Step (A_j) is much harder to verify, since we have no control on the set J_{j-1}. Thus, it is difficult to check that the condition (IV_j) holds, so we introduce a new set J'_{j-1} which contains J_{j-1}.

Set $\xi_j = (c_j + v_j\eta_{m_j})/g_j$ for some positive integers c_j and g_j with

$$g_j^{\nu_j} > 8 g_{j-1}^{\nu_{j-1}}, \tag{6}$$

and denote by J'_{j-1} the set of p-adic numbers x in I_{j-1} satisfying $|x-\alpha_n|_p \ge 2\lambda_n \mathrm{H}(\alpha_n)^{-\chi_n}$ for any algebraic number α_n of degree $n \le j-1$, distinct from $\xi_1, \dots, \xi_{j-1}, x$ and whose height $\mathrm{H}(\alpha_n)$ satisfies the inequalities

$$(\lambda_n g_{j-1}^{\nu_{j-1}})^{1/\chi_n} \le \mathrm{H}(\alpha_n) \le (2\lambda_n g_j^{n^2+n+1+2n\varepsilon_n})^{1/(\chi_n - n - 1 - \varepsilon_n)}. \tag{7}$$

Since

$$\chi_n - n - 1 - \varepsilon_n > n^3 + 2n^2 + 2n + 1 + 5n^2\varepsilon_n > (n+1)(n^2+n+1+2n\varepsilon_n), \tag{8}$$

the exponent of g_j in the right of (7) is strictly less than $1/(n+1)$. Thus, there are $o(g_j)$ algebraic numbers α_n satisfying (7). We will prove that for g_j large enough we have $\gg g_j$ suitable choices for c_j such that the conditions (I_j) to (V_j) are fulfilled.

Denote by $B(c,r)$ the ball $\{x \in \mathbb{Q}_p : |x-c|_p < r\}$. By introducing

$$\hat{B}_{j-1} = B(\xi_{j-1}, g_{j-1}^{-\nu_{j-1}}) \quad \text{and} \quad \check{B}_{j-1} = B\big(\xi_{j-1}, g_{j-1}^{-\nu_{j-1}}/(2p)\big),$$

we can write $I_{j-1} = \hat{B}_{j-1} \setminus \check{B}_{j-1}$.

Because in ultrametric space every two balls are either disjoint or one is a subset of the other, we can take a subfamily $\mathcal{F}$ of the balls defined by (7) and the text that immediately precedes it, i.e. a subfamily of

$$\{B(\alpha_n, 2\lambda_n \mathrm{H}(\alpha_n)^{-\chi_n}) : \alpha_n \text{ algebraic of degree } n \le j-1, \\ \alpha_n \ne \xi_1, \dots, \xi_{j-1}, x, \text{ and } \mathrm{H}(\alpha_n) \text{ satisfies (7)}\}$$

such that every two balls in $\mathcal{F}$ are disjoint, each of them is contained in I_{j-1} and $J'_{j-1} = I_{j-1} \setminus \bigcup_{B\in\mathcal{F}} B$. If $\check{B}_{j-1}$ is not already a subset of some ball in $\mathcal{F}$, then we add $\check{B}_{j-1}$ to the family $\mathcal{F}$ so that

$$J'_{j-1} = I_{j-1} \setminus \bigcup_{B\in\mathcal{F}} B = \hat{B}_j \setminus \bigcup_{B\in\mathcal{F}} B.$$

We look at the numbers from the set

$$S_j := \Big\{\xi_j = \frac{c_j + v_j\eta_{m_j}}{g_j} : c_j = 1, \dots, g_j - 1\Big\}.$$

For any ball $B = B(s,r) \in \mathcal{F}$, we have $r = p^{-k}$ for some $k \in \mathbb{Z}_{\geq 0}$ (depending on B) and we can take s to be the smallest nonnegative integer in B. Consider when $\xi_j \in B$:

$$|(g_j s - v_j\eta_{m_j}) - c_j|_p = \left|\frac{c_j + v_j\eta_{m_j}}{g_j} - s\right|_p = |\xi_j - s|_p < r = p^{-k}.$$

Thus we see that all the associated c_j for such ξ_j are of the form $c_j = \tilde{s} + p^{k+1}l$, where $l = 0, 1, 2, \dots$, $\tilde{s}$ is an integer, $0 \leq \tilde{s} < p^{k+1}$, and $c_j < g_j$. The measure of B is obviously $\mu(B) = p^{-k-1}$ and if we define

$$N_j(B) := \#\{\xi_j : \xi_j \in S_j \cap B\},$$

it follows that

$$g_j\mu(B) - 1 = \frac{g_j}{p^{k+1}} - 1 \leq N_j(B) \leq \frac{g_j}{p^{k+1}} + 1 = g_j\mu(B) + 1$$

$$g_j\mu\Big(\bigcup_{B\in\mathcal{F}} B\Big) - \#\mathcal{F} \leq \sum_{B\in\mathcal{F}} N_j(B) \leq g_j\mu\Big(\bigcup_{B\in\mathcal{F}} B\Big) + \#\mathcal{F}.$$

Analogously

$$g_j\mu(\hat{B}_{j-1}) - 1 \leq N_j(\hat{B}_{j-1}) \leq g_j\mu(\hat{B}_{j-1}) + 1.$$

Using

$$N_j(J'_{j-1}) = N_j(\hat{B}_{j-1}) - \sum_{B\in\mathcal{F}} N_j(B) \quad \text{and} \quad \mu(J'_{j-1}) = \mu(\hat{B}_{j-1}) - \mu\Big(\bigcup_{B\in\mathcal{F}} B\Big),$$

we get

$$\mu(J'_{j-1}) - \frac{\#\mathcal{F}+1}{g_j} \leq \frac{N_j(J'_{j-1})}{g_j} \leq \mu(J'_{j-1}) + \frac{\#\mathcal{F}+1}{g_j}.$$

As was explained right after the equation (8), $\#\mathcal{F} = o(g_j)$. Since (VII_{j-1}) with $J'_{j-1} \supset J_{j-1}$ implies $\mu(J'_{j-1}) \gg 1$, we conclude

$$N_j(J'_{j-1}) \gg g_j.$$

We now have $\gg g_j$ possible numbers $\xi_j = (c_j + v_j\eta_{m_j})/g_j \in J'_{j-1}$ which means that they trivially satisfy (II_j). We will prove that they also satisfy (IV_j) for g_j large enough.

Let α_n be an algebraic number of degree n. By Lemma 5, there exists a positive constant $\kappa(m_j, n, \varepsilon_n)$ such that

$$\begin{aligned}|\xi_j - \alpha_n|_p &= \left|\frac{c_j + v_j\eta_{m_j}}{g_j} - \alpha_n\right|_p = \left|\frac{v_j}{g_j}\right|_p \left|\eta_{m_j} - \left(\frac{g_j\alpha_n - c_j}{v_j}\right)\right|_p \\ &\geq |v_j|_p \kappa(m_j, n, \varepsilon_n)\,\mathrm{H}\left(\frac{g_j\alpha_n - c_j}{v_j}\right)^{-n-1-\varepsilon_n} \\ &\geq g_j^{-(n^2+n+1+2n\varepsilon_n)}\,\mathrm{H}(\alpha_n)^{-n-1-\varepsilon_n}, \end{aligned} \tag{9}$$

if g_j satisfies

$$g_j \geq \kappa(m_j, n, \varepsilon_n)^{-1/(n\varepsilon_n)} 2^{(n+1)(n+1+\varepsilon_n)/(n\varepsilon_n)}.$$

Here, we have used Lemma 2:

$$\mathrm{H}\left(\frac{g_j\alpha_n - c_j}{v_j}\right) \leq 2^{n+1}\,\mathrm{H}(\alpha_n)\max\{|c_j|, |g_j|, |v_j|\}^n = 2^{n+1}\,\mathrm{H}(\alpha_n) g_j^n.$$

In particular, if g_j is large enough, we have

$$|\xi_j - \alpha_n|_p \geq 2\lambda_n\,\mathrm{H}(\alpha_n)^{-\chi_n} \tag{10}$$

as soon as

$$\mathrm{H}(\alpha_n)^{\chi_n - n - 1 - \varepsilon_n} \geq 2\lambda_n g_j^{n^2+n+1+2n\varepsilon_n}. \tag{11}$$

This together with the definition of J'_{j-1} shows that all our $\xi_j \in J'_{j-1}$ also belong to J_{j-1}. Therefore, the condition (IV_j) is verified and we proceed with the condition (I_j).

Let us suppose that g_j divides all $P(x_j)$, where x_j $(j = 0, \ldots, m_j)$ takes $m_j + 1$ different values from $\{-g_j + 1, \ldots, -1\}$ and $P(X)$ is a polynomial of degree m_j, here $P(X) = X^{m_j} - dv_j^{m_j}$. Using either the formula

$$\begin{aligned}P(X) = P[x_0] + P[x_0, x_1](x - x_0) + \cdots \\ + P[x_0, \ldots, x_{m_j}](x - x_0)(x - x_1)\cdots(x - x_{m_j-1}),\end{aligned}$$

where

$$P[x_0] = P(x_0), \quad P[x_0, x_1] = \frac{P[x_1] - P[x_0]}{x_1 - x_0}, \quad \ldots$$

are the divided differences, or using the Lagrange interpolation formula

$$P(x) = \sum_{i=0}^{m_j} P(x_i)\frac{(x - x_0)\cdots(x - x_{i-1})(x - x_{i+1})\cdots(x - x_{m_j})}{(x_i - x_0)\cdots(x_i - x_{i-1})(x_i - x_{i+1})\cdots(x_i - x_{m_j})},$$

we see that the leading coefficient of $P(X)$ can be written as a fraction whose numerator is divisible by g_j and the denominator divides $((g_j-1)!)^l$ for some l large enough. But for a prime g_j that would imply that the leading coefficient, which is obviously 1, is divisible by g_j. This contradiction implies that there are $\gg g_j$ possible numbers $\xi_j = (c_j + v_j\eta_{m_j})/g_j$ which satisfy (I_j) as well. (Note the important thing here is that the polynomial $P(X)$ is monic. This is also true for other polynomials from Lemma 6 while proofs of all the other conditions from the proposition are obviously indifferent to the choice of case from this Lemma.)

Conditions (VI_j), (V_j), (III_j) and the step (B_j) are done mutatis mutandis just like in [3]. We are left with $\gg g_j$ suitable algebraic numbers ξ_j, mutually distant by at least g_j^{-1} (compare (5)). Only $o(g_j)$ algebraic numbers α_n satisfy

$$\mathrm{H}(\alpha_n) \le g_j^{1/(n+1+\varepsilon)}, \tag{12}$$

thus there are $\gg g_j$ algebraic numbers ξ_j such that $|\xi_j - \alpha_n|_p \ge 1/g_j$ for the numbers α_n verifying (12). Further, Lemma 5 ensures that there exists λ_j in $]0,1[$ such that (V_j) is satisfied. Consequently, there are $\gg g_j$ algebraic numbers ξ_j satisfying $(I_j), (II_j), (IV_j), (V_j)$ and (VI_j).

It remains for us to show that such a ξ_j also satisfies (III_j). To this end, because of (IV_j) and (V_j), it suffices to prove that

$$|\xi_j - \alpha_n|_p \ge \lambda_n \,\mathrm{H}(\alpha_n)^{-\chi_n}$$

holds for any algebraic number α_n of degree $n < j$, which is different from $\xi_1, \dots, \xi_j$ and whose height $\mathrm{H}(\alpha_n)$ satisfies

$$\mathrm{H}(\alpha_n) < (\lambda_n g_{j-1}^{\nu_{j-1}})^{1/\chi_n}.$$

Since by (6) the sequence $(g_t^{\nu_t})_{t\ge1}$ is increasing, we either have

$$g_n^{-\nu_n} < \lambda_n \,\mathrm{H}(\alpha_n)^{-\chi_n}, \tag{13}$$

or there exists an integer t with $n < t < j$ such that

$$g_t^{-\nu_t} < \lambda_n \,\mathrm{H}(\alpha_n)^{-\chi_n} \le g_{t-1}^{-\nu_{t-1}}. \tag{14}$$

In the former case, we infer from (V_n), (6) and (13) that

$$|\xi_j - \alpha_n|_p \ge |\xi_n - \alpha_n|_p - |\xi_j - \xi_n|_p \ge 2\lambda_n \,\mathrm{H}(\alpha_n)^{-\chi_n} - g_n^{-\nu_n} > \lambda_n \,\mathrm{H}(\alpha_n)^{-\chi_n}.$$

In the latter case, (IV_t), (6) and (14) yield that

$$|\xi_j - \alpha_n|_p \geq |\xi_t - \alpha_n|_p - |\xi_j - \xi_t|_p \geq 2\lambda_n \,\mathrm{H}(\alpha_n)^{-\chi_n} - g_t^{-\nu_t} > \lambda_n \,\mathrm{H}(\alpha_n)^{-\chi_n}.$$

Thus condition (III_j) holds and the proof of step (A_j) is completed.

Before going on with the step (B_j), let us mention that the integer c_j is far from being uniquely determined. Indeed, at any step j we have $\gg g_j$ suitable choices for ξ_j which shows that the construction actually gives an uncountable set of T-numbers.

Let $j \geq 1$ be an integer. For the proof of step (B_j), we first establish that if g_j is large enough and if x lies in I_j, then we have

$$|x - \alpha_n|_p \geq 2\lambda_n \,\mathrm{H}(\alpha_n)^{-\chi_n} \tag{15}$$

for any algebraic number $\alpha_n \neq \xi_j$ of degree $n \leq j$ such that

$$(\lambda_n g_j^{\nu_j})^{1/\chi_n} \leq \mathrm{H}(\alpha_n) \leq g_j^{\nu_j/(\chi_n - n - 1 - \varepsilon_n)}. \tag{16}$$

Let, then, $\alpha_n \neq \xi_j$ be an algebraic number satisfying (16) and let x be in I_j, that is, such that

$$\frac{1}{2p} g_j^{-\nu_j} \leq |x - \xi_j|_p < g_j^{-\nu_j}. \tag{17}$$

If $\nu_j(n + 1 + \varepsilon_n) \leq \chi_n - n - 1 - \varepsilon_n$, then $\mathrm{H}(\alpha_n) \leq g_j^{1/(n+1+\varepsilon_n)}$ and it follows from (VI_j), (16), (17), and the assumption $\nu_j > 1$ that

$$|x - \alpha_n|_p \geq |\xi_j - \alpha_n|_p - |\xi_j - x|_p \geq g_j^{-1} - g_j^{-\nu_j} \geq 2g_j^{-\nu_j} \geq 2\lambda_n \,\mathrm{H}(\alpha_n)^{-\chi_n},$$

provided that g_j is large enough.

Otherwise, we have

$$\nu_j(n + 1 + \varepsilon_n) > \chi_n - n - 1 - \varepsilon_n, \tag{18}$$

and, by (9), we get

$$\begin{aligned} |x - \alpha_n|_p &\geq |\xi_j - \alpha_n|_p - |\xi_j - x|_p \geq g_j^{-(n^2+n+1+2n\varepsilon_n)} \,\mathrm{H}(\alpha_n)^{-n-1-\varepsilon_n} - g_j^{-\nu_j} \\ &\geq g_j^{-(n^2+n+1+2n\varepsilon_n)} \,\mathrm{H}(\alpha_n)^{-n-1-\varepsilon_n}/2. \end{aligned} \tag{19}$$

To check the last inequality, we have to verify that

$$2g_j^{-\nu_j} \leq g_j^{-(n^2+n+1+2n\varepsilon_n)} \,\mathrm{H}(\alpha_n)^{-n-1-\varepsilon_n}. \tag{20}$$

In view of (16), inequality (20) is true as soon as

$$2g_j^{\nu_j(n+1+\varepsilon_n)/(\chi_n-n-1-\varepsilon_n)} \le g_j^{\nu_j} g_j^{-(n^2+n+1+2n\varepsilon_n)},$$

which, by (18), holds for g_j large enough when

$$\frac{n+1+\varepsilon_n}{\chi_n-n-1-\varepsilon_n} < 1-(n^2+n+1+2n\varepsilon_n)\frac{n+1+\varepsilon_n}{\chi_n-n-1-\varepsilon_n}, \tag{21}$$

and in particular when χ_n satisfies (3). Furthermore, we have

$$g_j^{-(n^2+n+1+2n\varepsilon_n)}\,\mathrm{H}(\alpha_n)^{-n-1-\varepsilon_n} \ge 4\lambda_n\,\mathrm{H}(\alpha_n)^{-\chi_n}. \tag{22}$$

Indeed, by (16), $\lambda_n < 1$, and (18), we get

$$\begin{aligned}\mathrm{H}(\alpha_n)^{-\chi_n-n-1-\varepsilon_n} &\ge (\lambda_n g_j^{\nu_j})^{(\chi_n-n-1-\varepsilon_n)/\chi_n} \ge \lambda_n g_j^{\nu_j(\chi_n-n-1-\varepsilon_n)/\chi_n}\\ &> \lambda_n g_j^{(\chi_n-n-1-\varepsilon_n)^2/(\chi_n(n+1+\varepsilon_n))} \ge 4\lambda_n g_j^{n^2+n+1+2n\varepsilon_n},\end{aligned}$$

since we infer from (3) that

$$(\chi_n-n-1-\varepsilon_n)^2 > \chi_n(n+1+\varepsilon_n)(n^2+n+1+2n\varepsilon_n). \tag{23}$$

Combining (19) and (22), we have checked that

$$|x-\alpha_n|_p \ge 2\lambda_n\,\mathrm{H}(\alpha_n)^{-\chi_n}$$

holds under assumption (18). By (18), this implies that (15) is true if α_n satisfies (16) and is not equal to ξ_j. Consequently, for g_j large enough, the complement J_j^c of J_j in I_j is contained in the union of the balls

$$B(\alpha_n, 2\lambda_n\,\mathrm{H}(\alpha_n)^{-\chi_n}),$$

where $\alpha_n \in \mathbb{Q}_p$ runs over the algebraic numbers of degree $n \le j$ and height greater than $g_j^{\nu_j/(\chi_n-n-1-\varepsilon_n)}$. The Haar measure of J_j^c is then

$$\ll \sum_{n=1}^{j} \sum_{H > g_j^{\nu_j/(\chi_n-n-1-\varepsilon_n)}} H^{n-\chi_n} = o(g_j^{-\nu_j}) = o(\mu(I_j)),$$

since for any positive integers H and n there are at most

$$(2H+1)^{n+1} - (2(H-1)+1)^{n+1} < (8H)^n$$

algebraic numbers of height H and degree n. Thus, we conclude that we can find g_j large enough such that $\mu(J_j) \ge \mu(I_j)/2$. This completes step (B_j) as well as proof of Proposition 1. □

At this point, we summarize where the condition $\chi_n > n^3+2n^2+4n+3$ appears. There are three steps where it is needed, namely (8), (21) and (23). Asymptotically, these three inequalities reduce, respectively, to $\chi_n > (n+1)(n^2+n+2)$, $\chi_n > (n+1)(n^2+n+3)$, and $(\chi_n-n-1)^2 > \chi_n(n+1)(n^2+n+1)$. The most restricting condition is given by (21), hence, our assumption on χ_n.

4. Proof of main result

Let $(w_n)_{n\geq 1}$ and $(w_n^*)_{n\geq 1}$ be two sequences fulfilling the conditions of Theorem 1. We will define numbers which are needed to apply Proposition 1.

Let $(m_j)_{j\geq 1}$ be a sequence of positive integers taking infinitely many times each value $1, 2, \ldots$ For $j \geq 1$, we set $\nu_j = m_j(w^*_{m_j} + 1)$ and define μ_j in $[0, 1]$ by

$$w^*_{m_j} + \frac{m_j - 1}{m_j}\mu_j = w_{m_j}.$$

Moreover, for any integer $n \geq 1$, we set $\chi_n = w_n - n + 1$ so that $\chi_n > n^3 + 2n^2 + 4n + 3$. Let $\lambda_1, \lambda_2, \ldots, \xi_1, \xi_2, \ldots$ be as in Proposition 1 and denote by ξ the limit of sequence $(\xi_j)_{j\geq 1}$. This sequence obviously converges since it is a Cauchy sequence and $\mathbb{Q}_p$ is complete.

We fix an integer $n \geq 1$. Observe that the minimal polynomial of ξ_j over $\mathbb{Z}$ is the polynomial

$$P_j(X) = (g_j X - c_j)^{m_j} - dv_j^{m_j},$$

which is primitive since, by condition (I_j), its leading coefficient is coprime with its constant term. Thus, recalling that $c_j, v_j < g_j$, we get that $g_j^{m_j} \leq \mathrm{H}(\xi_j) \leq (2g_j)^{m_j}$. (We have completely analogous statements in the other three cases of Lemma 6.) Furthermore, for any $j \geq 1$ we have

$$|\xi - \xi_j|_p \in \left[\frac{1}{2p} g_j^{-\nu_j}, g_j^{-\nu_j}\right[$$

and the definition of ν_j implies that

$$|\xi - \xi_j|_p \in \left[\frac{1}{2p}\,\mathrm{H}(\xi_j)^{-w^*_{m_j}-1}, 2^{\nu_j}\,\mathrm{H}(\xi_j)^{-w^*_{m_j}-1}\right]. \tag{24}$$

Moreover, if α_m is an algebraic number of degree $m \leq n$ which is not equal to one of the ξ_j, then, by (III_j) we have

$$|\xi_j - \alpha_m|_p \geq \lambda_m\,\mathrm{H}(\alpha_m)^{-\chi_m},$$

hence, as j tends to infinity,

$$|\xi - \alpha_m|_p \geq \lambda_m\,\mathrm{H}(\alpha_m)^{-\chi_m} \geq \lambda_m\,\mathrm{H}(\alpha_m)^{-w^*_m-1}, \tag{25}$$

since $\chi_m = w_m - m + 1 \leq w^*_m + 1 - m + 1 \leq w^*_m + 1$. As $m_j = n$ for infinitely many integers j, it follows from (24), (25) and from the fact that the sequence $(w^*_m)_{m\geq 1}$ is increasing that

$$w^*_n(\xi) = w^*_n.$$

It remains for us to prove that $w_n(\xi) = w_n$. This is clear for $n = 1$, thus we assume $n \geq 2$. Until the end of this proof, we write $A \ll B$ when there is a positive constant $c(m_j)$, depending only on m_j, such that $|A| \leq c(m_j)|B|$, and we write $A \asymp B$ if both $A \ll B$ and $B \ll A$ hold. Since $\mathrm{H}(P_j) \asymp g_j^{m_j}$, we get from (2)

$$\begin{aligned} |P_j(\xi)|_p &= |v_j|_p^{m_j-1}|\xi - \xi_j|_p \\ \asymp g_j^{-\mu_j(m_j-1)} g_j^{-\nu_j} &\asymp g_j^{-m_j(w_{m_j}-w^*_{m_j})} g_j^{-m_j(w^*_{m_j}+1)} \asymp \mathrm{H}(P_j)^{-w_{m_j}-1}. \end{aligned} \tag{26}$$

Since $m_j = n$ for infinitely many j, we infer from (26) that $w_n(\xi) \geq w_n$. In order to show that we have equality, let $P(X)$ be an integer polynomial of degree at most n, which we write under the form

$$P(X) = aR_1(X)\cdots R_s(X)\cdot Q_1(X)\cdots Q_t(X),$$

where a is an integer and the polynomials $R_i(X)$ and $Q_j(X)$ are primitive and irreducible. We moreover assume that each $R_i(X)$ does not have a root equal to one of ξ_ℓs, but that each $Q_j(X)$ has a root equal to some ξ_ℓ. If k denotes the degree of the polynomial $R_i(X)$, then, by Lemma 4, it has a root θ satisfying

$$\begin{aligned} |R_i(\xi)|_p \gg \mathrm{H}(R_i)^{1-k}|\xi-\theta|_p &\gg \lambda_n \,\mathrm{H}(R_i)^{-\chi_k-k+1} \\ &= \lambda_n \,\mathrm{H}(R_i)^{-w_k} \gg \lambda_n \,\mathrm{H}(R_i)^{-w_n}. \end{aligned} \tag{27}$$

If ℓ denotes the degree of $Q_j(X)$, then (26) shows that

$$|Q_j(\xi)|_p \asymp \mathrm{H}(Q_j)^{-w_\ell-1} \geq \mathrm{H}(Q_j)^{-w_n-1}.$$

Together with (27) and Lemma 1, this gives

$$|P(\xi)|_p \gg \big(\mathrm{H}(R_1)\cdots\mathrm{H}(R_s)\,\mathrm{H}(Q_1)\cdots\mathrm{H}(Q_t)\big)^{-w_n-1} \gg \mathrm{H}(P)^{-w_n-1},$$

and we get $w_n(\xi) = w_n$, as claimed.

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