

The equation $SL(n, K) = C^4 \cup Z$ for $n = 2, 3, 4$; $K = \mathbb{R}, \mathbb{C}$

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Abstract. In the paper it has been proved that $SL(n, K) = C^4 \cup Z$ for $n = 2, 3, 4$; $K = \mathbb{R}, \mathbb{C}$; $PSL(2, \mathbb{R}) = C^4$ and $PSL(n, K) = C^8$ ($K = \mathbb{R}, \mathbb{C}$); $n = 3, 4$, where C denotes noncentral conjugacy class.

In the paper [3, IV] it has been proved that $G = PSL(2, \mathbb{R}) = C^s$ for some $s \geq 3$, for all conjugacy classes $C \neq \{1\}$ of G .

In this paper we shall prove that $SL(n, K) = C^4 \cup Z$ for $n = 2, 3, 4$; $K = \mathbb{R}, \mathbb{C}$.

By above results it follows that $PSL(2, \mathbb{R}) = C^4$ and $PSL(n, K) = C^8$ for $n = 3, 4$; $K = \mathbb{R}, \mathbb{C}$.

The following notations will be used.

C_V denotes the conjugacy class of the matrix V , \mathbb{R} — the field of real numbers, \mathbb{C} — the field of complex numbers and Z denotes the center of a group G . The remaining notations are standard.

The following lemmas will be used.

Lemma 1 (see [1]). *If $V = \text{diag}(v_1, \dots, v_n)$, $W = \text{diag}(w_1, \dots, w_n)$, $v_i \neq v_j$, $w_i \neq w_j$ for $i \neq j$ and $V, W \in SL(n, K)$, then $SL(n, K) = C_V C_W \cup Z$.*

From Lemma 1 it follows that in the special case $W = V$ and V, V^{-1} are similar, $PSL(n, K) = C_V^2$ (see [1]).

If all eigenvalues of V are not distinct, the equality $PSL(n, K) = C_V^2$ does not necessarily hold. This problem was investigated by a few authors, chiefly by J. L. BRENNER (see his papers cited in the book [4] and his reviews: Math. Rev. 1987h, 20001 and Zbl. 561, 20004, 1985).

Lemma 2. *If $A \in GL(2, K)$ ($K = \mathbb{R}, \mathbb{C}$), $A \notin Z$, then there exists $S, T \in SL(2, K)$ such that the eigenvalues λ_1, λ_2 of $A^S A^T$ are distinct and $\lambda_i \neq k$, where k is an arbitrary number $\neq 0$.*

PROOF. Let $N = A^P = P^{-1}AP$ denotes the rational canonical form of A in the group $GL(n, K)$.

Note that if all eigenvalues of NN^X are distinct, where $X \in SL(n, K)$, then all eigenvalues of $A^{PU}A^{PXU}$ also are distinct and we can choose the matrix U such that $\det(PU) = \det(PXU) = 1$.

Therefore it suffices to prove that there exists $X \in SL(n, K)$ such that all eigenvalues of NN^X are distinct.

Any noncentral matrix $A \in GL(2, K)$ is similar to the matrix

$$N = \begin{bmatrix} 0 & 1 \\ a & b \end{bmatrix}.$$

For N and $X = \begin{bmatrix} 0 & 1 \\ -1 & x \end{bmatrix}$,

$$(1) \quad |NN^X - \lambda E| = \lambda^2 - s\lambda + t,$$

where

$$s = ax^2 + b(1-a)x - a^2 - 1, \quad t = a^2.$$

It is clear that there are infinitely many $x \in K$ such that $\Delta = s^2 - 4t > 0$ or $\Delta \neq 0$, respectively to the case $K = \mathbb{R}$ or $K = \mathbb{C}$. Hence there are infinitely many matrices X such that all eigenvalues of NN^X are distinct.

The second assertion follows from the fact that the equation (1) = 0 in unknown x has at most two solution for given λ .

Lemma 3. *If $A \in GL(3, K)$ ($K = \mathbb{R}, \mathbb{C}$), $A \notin Z$, then there exists $X \in SL(3, K)$ such that the eigenvalue $\lambda_1, \lambda_2, \lambda_3$ of AA^X are mutually distinct and $\lambda_i \neq k$, where k is an arbitrary number $\neq 0$.*

PROOF. Any noncentral matrix $A \in GL(3, K)$ is similar by a matrix of determinant equal to an arbitrary number $\neq 0$ to the one of the following matrices

$$(2) \quad A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & b & c \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 & 0 \\ a & b & 0 \\ 0 & 0 & c \end{bmatrix}.$$

For A_1 and $X = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & x & x\alpha \end{bmatrix}$ (α -parameter),

$$(3) \quad |A_1 A_1^X - \lambda E| = -\lambda^3 + s\lambda^2 - t\lambda + r,$$

where

$$\begin{aligned} s &= -a\alpha^2x^2 + x(c\alpha + a\alpha + 1 - a) + b + ab, \\ t &= -ax^2 + x(b + ab - a^2\alpha + a\alpha) - c - ac, \quad r = a^2. \end{aligned}$$

It is clear that there are infinitely many $x \in K$ such that for $|x|$ large enough,

$$D = 27q^2 + 4p^3 = -s^2t^2 + 4t^3 + 4s^3r - 18str + 27r^2 < 0$$

or $D \neq 0$, respectively to the case $K = \mathbb{R}$ or $K = \mathbb{C}$. Hence there are infinitely many matrices X such that all eigenvalues of $A_1A_1^X$ are different.

For A_2 and

$$X = \begin{bmatrix} 0 & 1 & 0 \\ 0 & \beta x & 1 \\ 1 & x & 0 \end{bmatrix} \quad (\beta - \text{parameter}),$$

$$(4) \quad |A_2A_2^X - \lambda E| = -\lambda^3 + s\lambda^2 - t\lambda + r,$$

where

$$\begin{aligned} s &= -a\beta x^2 + (b\beta + ac - \beta c)x + cb, \\ t &= -ac^2\beta x^2 + ac(-\beta + bc + a)x - abc, \quad r^2 = a^2c^2. \end{aligned}$$

It is clear that there are infinitely many $x \in K$ such that $D < 0$ or $D \neq 0$ for $|x|$ large enough, respectively to the case $K = \mathbb{R}$ or $K = \mathbb{C}$. Thus there exist infinitely many matrices X such that all eigenvalues of $A_2A_2^X$ are distinct.

A simple calculation shows that we can choose the parameters α, β such that each equation (3) = 0, (4) = 0 has at most two solutions for given λ . This proves the second assertion of Lemma 3.

Lemma 4. *If $A \in SL(4, K)$ ($K = \mathbb{R}, \mathbb{C}$) $A \notin Z$, then there exists $S, T \in SL(4, K)$ such that all eigenvalues of $A^S A^T$ are distinct.*

PROOF. As in the proof of Lemma 2 it suffices to prove that there exists $X \in SL(4, K)$ such that all eigenvalues of NN^X are distinct, where N denotes the rational canonical form of A in the group $GL(4, K)$.

Any noncentral matrix $A \in SL(4, K)$ is similar to the one of following matrices:

$$N_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & a & b & c \end{bmatrix}; \quad N_2 = \begin{bmatrix} A_i & 0 \\ 0 & d \end{bmatrix}, \quad A_i \in (2), \quad i = 1, 2;$$

$$N_3 = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix}, \quad C_i = \begin{bmatrix} 0 & 1 \\ a_i & b_i \end{bmatrix}.$$

For N_1 and $X = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & y & 0 \end{bmatrix}$,

$$(6) \quad |N_1 N_1^X - \lambda E| = (\lambda^2 - y\lambda + 1)[\lambda^2 - (2b - y)\lambda + 1].$$

It is clear that there exists $y \in K$ such that the polynomial (6) has four distinct roots.

From Lemma 3 it follows that for A_i there exists $Y \in SL(3, K)$ such all eigenvalues of $A_i A_i^Y$ are distinct which can be chosen different from d^2 . Hence there exists the matrix

$$X = \begin{bmatrix} Y & 0 \\ 0 & 1 \end{bmatrix}$$

such that all eigenvalues of $N_2 N_2^X$ are distinct.

By Lemma 2 it follows that for C_i ($i = 1, 2$) there exist $Y, U \in SL(2, K)$ such that the eigenvalues v_1, v_2 and w_1, w_2 of $C_1 C_1^Y$ and $C_2 C_2^U$, respectively are different and that Y and U can be chosen such that $v_i \neq w_j$. Hence there exists the matrix

$$X = \begin{bmatrix} Y & 0 \\ 0 & U \end{bmatrix},$$

such that all eigenvalues of $N_3 N_3^X$ are distinct.

Theorem 1. *If C is any noncentral conjugacy class of $SL(2, \mathbb{R})$, then $SL(2, \mathbb{R}) = C^4 \cup Z$.*

PROOF. In the set $A^S A^T$, where $A, S, T \in SL(2, \mathbb{R})$, there exists the matrix B with two distinct eigenvalues v, v^{-1} , by Lemma 2.

The matrix B is similar in the group $SL(2, \mathbb{R})$ to the matrix $V = \text{diag}(v, v^{-1})$. Hence

$$C_V = C_B \subseteq C_A^2 \quad \text{and} \quad C_V^2 \subseteq C_A^4.$$

By Lemma 1 it follows that $SL(2, \mathbb{R}) - Z \subseteq C_V C_V \subseteq C_A^4$ for any $A \notin Z$. Hence $SL(2, \mathbb{R}) \subseteq C_A^4 \cup Z$. The inverse inclusion is obvious.

Corollary 1.1. *If C is any nonidentity conjugacy class of $PSL(2, \mathbb{R})$, then $PSL(2, \mathbb{R}) = C^4$.*

PROOF. The matrices $V = \text{diag}(v, v^{-1})$ and V^{-1} are similar in the group $PSL(2, \mathbb{R})$, so $Z \subseteq C_V C_V \subseteq C^4$.

Note that $PSL(2, \mathbb{C}) = C^2$. It follows by ([2] or Lemma 1) and the fact that the equation $A = V^X V^Y$ in unknowns $X, Y \in SL(2, \mathbb{C})$, where $V = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $A = \text{diag}(a, a^{-1})$ ($a \neq 1$) or $A = V$, has a solution. The proof given in [4] is wrong (see p. 242, row 2).

Theorem 2. *If $K = \mathbb{R}, \mathbb{C}$ and C - noncentral conjugacy class of $SL(3, K)$, then $SL(3, K) = C^4 \cup Z$.*

PROOF. In the set AA^X , where $A, X \in SL(3, K)$, $A \notin Z$, there exists the matrix B with all distinct eigenvalues v_1, v_2, v_3 , by Lemma 3. The matrix B is similar to the diagonal matrix $V = \text{diag}(v_1, v_2, v_3)$ in $SL(3, K)$ i. e. $V^T = B$. Hence

$$C_V = C_B \subseteq C_A^2 \quad \text{and} \quad C_V^2 \subseteq C_A^4.$$

From Lemma 1 it follows that

$$SL(3, K) - Z \subseteq C_V^2 \subseteq C_A^4, \quad \text{so} \quad SL(3, K) \subseteq C^4 \cup Z.$$

Thus $SL(3, K) = C_A^4 \cup Z$ for any $A \notin Z$.

Theorem 3. *If C is any noncentral conjugacy class of $SL(4, K)$ and $K = \mathbb{R}, \mathbb{C}$, then $SL(4, K) = C^4 \cup Z$.*

The proof is the same as in Theorem 2 but instead of Lemma 3, Lemma 4 is used.

Corollary 2.1. *If C is any noncentral conjugacy class of $PSL(3, K)$, then $PSL(3, K) = C^8$.*

PROOF. From our proof of Lemma 3 it follows that the matrix E does not necessarily belongs to the set C^4 . But it is clear that the diagonal matrices $V = \text{diag}(1, v, v^{-1})$ ($v \neq \pm 1$) and V^{-1} belong to the set C^4 , so $Z \subseteq C_V C_V \subseteq C^8$. Hence $PSL(3, K) = C^8$.

In the same way we can prove the following

Corollary 3.1. *If C is any nonidentity conjugacy class of $PSL(4, K)$, then $PSL(4, K) = C^8$.*

Note that our method do not permit to state whether the exponent of C is minimal.

The method of proofs of Lemmas 2–4 suggests the conjecture: if $K = \mathbb{R}, \mathbb{C}$, then $SL(n, K) = C^4 \cup Z$ for any $n \geq 2$.

Remark. One of reviewers of my paper informed the editor that recently he received a paper (which probably did not appear yet) of Mr. Arieh Lev: “The covering number of the group $PSL_n(F)$ ” in which it has been proved the following two theorems:

Theorem I. Let G be the group $PSL_n(F)$, where $n \geq 3$, F is a field and $|F| \geq 4$. Denote by $\text{cn}(G)$ the minimal value of k such that $C^k = G$ for every nontrivial conjugacy class C of G . Then $\text{cn}(G) = n$.

Theorem II. Let C be a nonscalar SL_n — conjugacy class of $GL_n(F)$, where $n \geq 3$, F is a field and $|F| \geq 4$. Let $\mathbf{M} = \{M \in GL_n(F) - Z(GL_n(F)) \mid \det M = (\det C)^n\}$. Then $\mathbf{M} \subseteq C^n$. In particular, if $\det C = 1$, then $C^n \supseteq SL_n(F) - Z(SL_n(F))$.

References

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