

Weakly-symmetry of the Sasakian lifts on tangent bundles

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*This paper is dedicated to Professor Lajos Tamássy on the occasion
of 90th birthday*

Abstract. The weakly symmetry of the Sasakian lift G of a Riemannian metric g is characterized in terms of flatness for g and G . The cases of recurrent or pseudo-symmetric G studied by Binh and Tamássy are obtained in particular.

1. Introduction

The notion of *weakly symmetric Riemannian manifold* was introduced by LAJOS TAMÁSSY and TRAN QUOC BINH in [9]. Since then, this type of Riemannian geometry was the subject of several papers: [3]–[8], [12]. For example, the authors of this concept study the case of Einstein and Sasaki manifold in [10], respectively the situation of Kähler manifolds in [11]; the case of decomposable (i.e. product) space appears in [1].

Two weaker variants of weakly symmetries, namely recurrence and pseudo-symmetry, are considered, again by TAMÁSSY and BINH, in [2] having as prescribed metric the Sasakian lift G to TM of a Riemannian metric g on the base manifold M . Their result is as follows: *If (TM, G) is recurrent or pseudo-symmetric then (M, g) must be flat and thus (TM, G) must be flat too. The converse is trivially true.* The aim of this short note is to extend this reduction result to the general case of weakly symmetry for G .

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2. The Sasaki lift of a Riemannian metric

Fix a pair (M, g) with M_n a smooth $n(\geq 3)$ -dimensional manifold and g a Riemannian metric on M . Let $\pi : TM \rightarrow M$ its tangent bundle. Let $q = (q^i) = (q^1, \dots, q^n)$ be the coordinates on the base manifold M and the corresponding bundle coordinates $(q, v) = (q^i, v^i) = (q^1, \dots, q^n, v^1, \dots, v^n)$ on TM ; then the metric g has the local coefficients $g_{ij} = g(\frac{\partial}{\partial q^i}, \frac{\partial}{\partial q^j})$. For $t \in TM$ let $V_t TM = \ker \pi_{*,t}$ be the vertical subspace of $T_t TM$. The basis of the vertical distribution $V(TM)$ is given by $\{\frac{\partial}{\partial v^i}; 1 \leq i \leq n\}$. The Levi-Civita connection ∇ of g has the Christoffel symbols (Γ_{ij}^k) and yields the decomposition:

$$T_t TM = V_t TM \oplus H_t TM \quad (2.1)$$

with $H_t TM$ the horizontal subspace spanned by $\{\frac{\delta}{\delta q^i}; 1 \leq i \leq n\}$ where:

$$\frac{\delta}{\delta q^i} = \frac{\partial}{\partial q^i} - \Gamma_{ij}^k v^j \frac{\partial}{\partial v^k}. \quad (2.2)$$

Then every vector field $\tilde{X}$ on TM has the decomposition: $\tilde{X} = X^v + X^h$ with respect to (2.1). Also, a vector field $X = X^i(q) \frac{\partial}{\partial q^i}$ on M has the following lifts: a vertical one, $X^V = X^i \frac{\partial}{\partial v^i}$, respectively a horizontal one, $X^H = X^i \frac{\delta}{\delta q^i}$. Let us denote by R the tensor field of curvature of g .

The Sasaki lift of g to TM is the Riemannian metric G of diagonal form:

$$G = \begin{pmatrix} g_{ij} & 0 \\ 0 & g_{ij} \end{pmatrix} \quad (2.3)$$

with respect to the decomposition (2.1). Let $\tilde{\nabla}$ and $\tilde{R}$ be the Levi-Civita connection and respectively the curvature of G ; we have in the point $t \in TM$:

$$\begin{cases} (\tilde{\nabla}_{X^H} Y^H)|_t = (\nabla_X Y)^H|_t - \frac{1}{2}(R(X, Y)t)^V|_t, \\ (\tilde{\nabla}_{X^H} Y^V)|_t = (\nabla_X Y)^V|_t + \frac{1}{2}(R(t, Y)X)^H|_t, \\ (\tilde{\nabla}_{X^V} Y^H)|_t = \frac{1}{2}(R(t, X)Y)^H|_t, (\tilde{\nabla}_{X^V} Y^V)|_t = 0. \end{cases} \quad (2.4)$$

The expression of $\tilde{R}$ is, [2, p. 556–557]:

$$\left\{ \begin{array}{l} \tilde{R}(X^V, Y^V)Z^V|_t = 0, \\ \tilde{R}(X^V, Y^V)Z_t^H = \left[R(X, Y)Z + \frac{1}{4}R(t, X)R(t, Y)Z \right. \\ \quad \left. - R(t, Y)R(t, X)Z \right]_t^H, \\ \tilde{R}(X^H, Y^V)Z^V|_t = - \left[\frac{1}{2}R(Y, Z)X + \frac{1}{4}R(t, Y)R(t, Z)X \right]^H|_t, \\ 2\tilde{R}(X^H, Y^V)Z_t^H = [(\nabla_X R)(t, Y, Z)]_t^H \\ \quad + \left[R(X, Z)Y + \frac{1}{2}R(R(t, Y)Z, X)t \right]_t^V, \\ \tilde{R}(X^H, Y^H)Z^V|_t = \frac{1}{2}[(\nabla_X R)(t, Z, Y) - (\nabla_Y R)(t, Z, X)]^H|_t \\ \quad + \left[R(X, Y)Z + \frac{1}{4}R(R(t, Z)Y, X)t - \frac{1}{4}R(R(t, Z)X, Y)t \right]^V|_t, \\ \tilde{R}(X^H, Y^H)Z_t^H|_t = \frac{1}{2}[(\nabla_Z R)(X, Y, t)]^V|_t \\ \quad + \left[R(X, Y)Z + \frac{1}{4}R(t, R(Z, Y)t)X + \frac{1}{4}R(t, R(X, Z)t)Y \right. \\ \quad \left. + \frac{1}{2}R(t, R(X, Y)t)Z \right]_t^H \end{array} \right. \quad (2.5)$$

where t from the above expressions $R(t, \cdot) \cdot$, $R(\cdot, \cdot)t$ is thought as a vector field on M , namely $t = v^i \frac{\partial}{\partial q^i}$. Then $t^V = v^i \frac{\partial}{\partial v^i}$ is the Liouville vector field while $t^H = v^i \frac{\delta}{\delta q^i}$ is exactly the geodesic spray of the metric g .

3. Weakly symmetric Sasakian lifts

Definition ([9]). The Riemannian manifold (M, g) is called *weakly symmetric* if there exist four 1-forms $\alpha_1, \dots, \alpha_4$ and a vector field A , all on M , such that:

$$\begin{aligned} (\nabla_W R)(X, Y, Z) &= \alpha_1(W)R(X, Y)Z + \alpha_2(X)R(W, Y)Z + \alpha_3(Y)R(X, W)Z \\ &\quad + \alpha_4(Z)R(X, Y)W + g(R(X, Y)Z, W)A. \end{aligned} \quad (3.1)$$

De and Bandyopadhyay proved in [3] that the following relations are necessary:

$$\left\{ \begin{array}{l} \alpha_2 = \alpha_3 = \alpha_4 \\ A = (\alpha_2)^\sharp \end{array} \right. \quad (3.2)$$

i.e. A is the g -dual vector field of the 1-form α_2 . Therefore a weakly symmetric Riemannian manifold is characterized by:

$$\begin{aligned} (\nabla_W R)(X, Y, Z) &= \alpha_1(W)R(X, Y)Z + \alpha_2(X)R(W, Y)Z + \alpha_2(Y)R(X, W)Z \\ &\quad + \alpha_2(Z)R(X, Y)W + g(R(X, Y)Z, W)(\alpha_2)^\sharp. \end{aligned} \quad (3.3)$$

The aim of this note is to study the weakly symmetry of the Sasakian lift (2.3). More precisely, we have:

Theorem. *The Riemannian manifold (TM, G) is weakly symmetric if and only if the base manifold (M, g) is flat. Hence, (TM, G) is flat.*

PROOF. If $R = 0$ it results that $\tilde{R} = 0$ and we have (3.3) as null equality. For a proof of the first part we use several times the techniques of [2] using the formulae (2.5). Firstly, we consider the condition (3.3) for W^H, X^H, Y^V and Z^V and we get:

$$\begin{aligned} &\alpha_1(W^H)\tilde{R}(X^H, Y^V)Z^V + \alpha_2(X^H)\tilde{R}(W^H, Y^V)Z^V + \alpha_2(Y^V)\tilde{R}(X^H, W^H)Z^V \\ &\quad + \alpha_2(Z^V)\tilde{R}(X^H, Y^V)W^H + G(\tilde{R}(X^H, Y^V)Z^V, W^H)(\alpha_2)^\sharp \\ &= -\tilde{\nabla}_{W^H} \left[\frac{1}{2}R(Y, Z)X + \frac{1}{4}R(t, Y)R(t, Z)X \right]_t^H \\ &\quad - \tilde{R} \left[(\nabla_W X)_t^H - \frac{1}{2}(R(W, X)t)_t^V Y^V \right] Z^V \\ &\quad - \tilde{R}(X^H, \frac{1}{2}(R(t, Y)W)_t^H + (\nabla_W Y)_t^V) Z^V \\ &\quad - \tilde{R}(X^H, Y^V) \left[\frac{1}{2}(R(t, Z)W)_t^H + (\nabla_W Z)_t^V \right]. \end{aligned} \quad (3.4)$$

Secondly, in the above equation we consider only the four times the vertical part of both sides and then:

$$\begin{aligned} &\alpha_2(Y^V) [4R(X, W)Z + R(R(t, Z)W, X)t - R(R(t, Z)X, W)t] \\ &\quad + \alpha_2(Z^V)(2R(X, W)Y + R(R(t, Y)W, X)t) - g(2R(Y, Z)X \\ &\quad + R(t, Y)R(t, Z)X, W)\alpha_2^\sharp = \frac{1}{2}R[W, 2R(Y, Z)X + R(t, Y)R(t, Z)X]t \\ &\quad - 2R(X, R(t, Y)W)Z + \frac{1}{2}R[R(t, Z)R(t, Y)W, X]t - \frac{1}{2}R[R(t, Z)X, R(t, Y)W] \\ &\quad - \frac{1}{2}R[R(t, Y)R(t, Z)W, X]t + R(X, R(t, Z)W)Y \end{aligned} \quad (3.5)$$

which is exactly four times the first relation on the page 559 of [2]. Thus in the following, the arguments are as in [2, p. 559-560]. We choose in (3.5) consequently:

I) $Y = t$ and then:

$$\begin{aligned} \alpha_2(t^V) [4R(X, W)Z + R(R(t, Z)W, X)t - R(R(t, Z)X, W)t] + 2\alpha_2(Z^V)R(X, W)t \\ - 2g(R(t, Z)X, W)(\alpha_2)^\sharp = R(W, R(t, Z)X)t + R(X, R(t, Z)W)t \end{aligned} \quad (3.6)$$

II) $Z = t$ and then:

$$\begin{aligned} 4\alpha_2(Y^V)R(X, W)t + \alpha_2(t^V)[2R(X, W)Y + R(R(t, Y)W, X)t] - 2g(R(Y, t)X, W)\alpha_2^\sharp \\ = R(W, R(Y, t)X)t - 2R(X, R(t, Y)W)t. \end{aligned} \quad (3.7)$$

In the last relation we replace Y with Z :

$$\begin{aligned} 4\alpha_2(Z^V)R(X, W)t + \alpha_2(t^V)[2R(X, W)Z + R(R(t, Z)W, X)t] - 2g(R(Z, t)X, W)\alpha_2^\sharp \\ = R(W, R(Z, t)X)t - 2R(X, R(t, Z)W)t \end{aligned} \quad (3.8)$$

and by adding (3.6) and (3.8) we derive:

$$\begin{aligned} 6\alpha_2(Z^V)R(X, W)t + \alpha_2(t^V)[6R(X, W)Z + 2R(R(t, Z)W, X)t - R(R(t, Z)X, W)t] \\ = -R(X, R(t, Z)W)t. \end{aligned} \quad (3.9)$$

With $Z = t$ we get:

$$\alpha_2(t^V)R(X, W)t = 0 \quad (3.10)$$

and if $\alpha_2(t^V) \neq 0$ we have the conclusion. Suppose now that $\alpha_2(t^V) = 0$ then $(\alpha_2)^\sharp = 0$; returning to (3.6) it results:

$$4\alpha_2(Z^V)R(X, W)t = 2R(X, R(t, Z)W)t + 2R(W, R(t, Z)X)t. \quad (3.11)$$

With $W = X$ we obtain:

$$R(X, R(t, Z)X)t = 0 \quad (3.12)$$

and we take the g -product with Z : $g(R(t, Z)X, R(t, Z)X) = 0$ which means:

$$R(t, Z)X = 0. \quad (3.13)$$

Again the g -product with an arbitrary Y gives:

$$g(R(X, Y)t, Z) = 0. \quad (3.14)$$

The vector field Z being arbitrary we get: $R(X, Y)t = 0$, for every X, Y and t . Thus, we have the Conclusion. $\square$

For $\alpha_2 = 0$ respectively $\alpha_1 = 2\alpha_2$ in (3.3) we get the Tamássy–Binh result of Introduction:

Corollary. *The Riemannian manifold (TM, G) is recurrent or pseudo-symmetric or locally symmetric ($\tilde{\nabla}\tilde{R} = 0$) if and only if the base manifold (M, g) is flat. Hence, (TM, G) is flat.*

The following *open problem* is natural: to extend the present Theorem to other classes of metrics on tangent bundles. The possible (first) candidates are from the natural metrics of [4] or [5].

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