Publ. Math. Debrecen
83/1-2 (2013), 97–121
DOI: 10.5486/PMD.2013.5521

On an S-unit variant of Diophantine m-tuples

By LÁSZLÓ SZALAY (Sopron) and VOLKER ZIEGLER (Graz)

Abstract. Let S be a fixed set of primes and let $a_1, \dots, a_m$ be positive distinct integers. We call the m-tuple $(a_1, \dots, a_m)$ S-Diophantine, if for all $i \neq j$ the integers $a_i a_j + 1 = s_{i,j}$ are S-integers. In this paper we show that if $|S| = 2$, then under some technical restrictions no S-Diophantine quadruple exists.

1. Introduction

An m-tuple $(a_1, \dots, a_m)$ of positive distinct integers is called Diophantine if

$$a_i a_j + 1 = \square \tag{1}$$

for $i \neq j$. Diophantine m-tuples have been studied since ancient times by several authors. Most notable is DUJELLA's result [8] that no Diophantine six-tuple exists and that there are only finitely many quintuples. It is widely believed that there exist no quintuples at all.

Not only Diophantine m-tuples have been considered, but also various variants. For instance, BUGEAUD and DUJELLA [3] examined m-tuples, where $\square$ in (1) is replaced by k-th power, DUJELLA and FUCHS [9] investigated a polynomial version, and FUCHS, LUCA and SZALAY [11] replaced $\square$ by terms of given binary recurrence sequences. For a complete overview we suggest DUJELLA's web page on Diophantine tuples [7].

In this paper we mean to consider an S-unit version of Diophantine m-tuples. Let S be a fixed set of primes. Then we call an m-tuple $(a_1, \dots, a_m)$, with positive

Mathematics Subject Classification: 11D61.
Key words and phrases: Diophantine tuples, S-Diophantine tuples, S-unit equations.
The second author was supported by the Austrian Science Found (FWF) under the project J2886-NT.

integers $0 < a_1 < \cdots < a_m$ an S-Diophantine m-tuple, if for all $1 \leq i < j \leq n$ we have $a_i a_j + 1 = s_{i,j}$ to be an S-unit. A closely related problem was studied by Győry, Sárközy and Tijdeman [12], who considered the largest prime factor of the products

$$\prod_{a \in A, b \in B} (ab + 1),$$

where A and B are fixed sets. This problem goes back to Erdős and Turán [10], who considered the number of prime factors in the product

$$\prod_{a \in A, b \in B} (a + b).$$

In particular, Győry, Sárközy and Tijdeman conjectured that for positive integers $a < b < c$ the greatest prime factor of

$$(ab + 1)(ac + 1)(bc + 1)$$

tends to infinity as $c \to \infty$. Effective but only partial results have been proved by Bugeaud [2] and Stewart and Tijdeman [20]. Finally this conjecture has been proved by Corvaja and Zannier [6] and independently by Hernández and Luca [13], which means in our context that there exist only finitely many S-Diophantine triples for a fixed set of primes S. Since both proofs depend on Schmidt's subspace theorem (see e.g. [19][Theorem 1E, p. 178]), this result is ineffective. A third proof that for a fixed set of primes S there are only finitely many S-Diophantine triples is given by Bugeaud and Luca [4] who proved that the greatest prime factor of

$$\prod_{\substack{a,b,c \in A \\ a<b<c}} (ab + 1)(ac + 1)(bc + 1),$$

where A is a finite set of positive integers, is larger than

$$\kappa \log |A| \log \log |A|,$$

with κ a constant effectively computable. Again this result does not yield an effective upper bound for c.

On the other hand Stewart and Tijdeman [20] proved an effective result, i.e. they showed that for a fixed set of primes there are only finitely many S-Diophantine quadruples which are effectively computable.

In this paper we consider the following problem. Fix the size of S, but not S itself. Does there exist an integer m such that no Diophantine m-tuple exists? In the case of $|S| = 2$ we conjecture that one can choose $m = 4$. Unfortunately, we were able to proof this conjecture only under some technical restrictions. Using the notation $\mathrm{ord}_p(q)$ for the multiplicative order of q modulo p, the main theorem in this paper is the following.

Theorem 1. *Let $S = \{p, q\}$ be a set of two primes with $p < q$ and assume that $p^2 \nmid q^{\mathrm{ord}_p(q)} - 1$, $q^2 \nmid p^{\mathrm{ord}_q(p)} - 1$, further that $q < p^\xi$ holds with some $\xi > 1$. Then there exists a constant $C = C(\xi)$ such that for all such $p, q > C$ no S-Diophantine quadruple exists. In particular we can choose*

$$C = C(\xi) = \Psi(9; 2.142 \cdot 10^{22}\xi^3),$$

where $\Psi(k; x)$ denotes the largest solution $y > 0$ to the equation $x = \frac{y}{(\log y)^k}$.

Remark 1. In case of $\xi = 2$ we obtain $C = C(2) = 1.023 \cdot 10^{41}$.

Let p be a large prime. Then there exists some $b \in \mathbb{Z}$, $1 < b < p$ such that $q = b + p$ is also prime. Put $g = \mathrm{ord}_p(q)$ and $g' = \mathrm{ord}_q(p)$. Then we have

$$q^g \equiv b^g + gpb^{g-1} \mod p^2 \quad \text{and} \quad p^{g'} \equiv \pm\big(b^{g'} - g'qb^{g'-1}\big) \mod q^2.$$

Let us assume that $q^g \equiv 1 \mod p^2$ or $p^{g'} \equiv 1 \mod q^2$, then we replace q by $q' = ap + b$ and obtain

$$q'^g \equiv b^g + gapb^{g-1} \mod p^2 \quad \text{and} \quad p^{g'} \equiv \pm(b^{g'} - g'aqb^{g'-1}) \mod q^2.$$

Since $b^g \equiv 1 + Ap \mod p^2$ for some A or $b^{g'} \equiv 1 + Bq \mod q^2$ and $p \nmid g$ with $q \nmid g'$ we deduce that if q' satisfies the assumptions of Theorem 1 then we have $a \not\equiv s_1 \mod p$ and $a \not\equiv s_2 \mod q$ for some s_1, s_2. Hence, $a \equiv r \mod pq$ for some $r \in (\mathbb{Z}_{pq})^*$. For technical reasons we also exclude the case $a \equiv 1 \mod q$ and we therefore assume that $(p-1)(q-2)$ possiblities for choosing a are left. I.e. a pair of primes (p, q') with

$$q' = b + ap = b + (r + kpq)p = b + rp + kp^2q$$

satisfies the assumptions of Theorem 1. Furthermore $b + pr$ and p^2q are coprime provided $r \not\equiv 1 \mod q$ and we may apply Dirichlet's prime number theorem. We have

$$\sharp\big\{q' \in \mathbb{P} : p^2 \nmid q^{\mathrm{ord}_p(q)} - 1,\ q^2 \nmid p^{\mathrm{ord}_q(p)} - 1,\ q' \le x\big\} \gg \frac{x}{\log x}\frac{(p-1)(q-2)}{\phi(p^2q)} \gg \frac{x}{p\log x}$$

primes $q' < x$ such that the pair (p, q') satisfies the assumptions of Theorem 1. Now, we choose $x = p^{1+\delta}$ for some $\delta > 0$ and we deduce that there exists a prime $q' < p^{1+\delta}$ such that the assumptions of Theorem 1 are fulfilled provided p is large. In particular, we obtain

Corollary 1. *There are infinitely many pairs p, q such that no non-trivial S-Diophantine quadruples exist.*

As mentioned above we conjecture that even more is true:

Conjecture 1. *There exist at most finitely many (respectively no) pairs of primes (p, q) such that $\{p, q\}$-Diophantine quadruples exist.*

2. Plan of the paper

In the next section we provide some useful lemmas that will be used frequently through the rest of the paper. These lemmas contain divisibility properties for the possible solutions in an explicit version of Stewart's and Tijdeman's result [20]. In our case we only have two primes to consider and we can therefore sharpen their result by using lower bounds for linear forms of logarithms in two variables due to Laurent, Mignotte and Nesterenko [15]. Moreover, we show that, assuming (a, b, c, d) is an S-Diophantine quadruple, yields three S-unit equations. In two subsequent sections we will consider two of these S-unit equations and will obtain restrictions for the exponents appearing in the S-units according to the assumptions of Theorem 1. These restrictions are in many cases contradictory and only finally 3 cases remain to handle. In Section 6 we consider the third S-unit equation and show that its possible solutions are not consistent with the restrictions found in the previous sections. In the last section we discuss open problems and questions. In particular, we discuss the case $|S| = 3$.

3. Preliminaries

At the beginning of this section we introduce and fix the following notations and assumptions for the rest of the paper. Let $(a, b, c, d) \in \mathbb{Z}^4$ be an S-Diophantine quadruple with $S = \{p, q\}$ and $p < q$. We assume $0 < a < b < c < d$ and write

$$ab + 1 = s_1, \qquad ac + 1 = s_2,$$
$$ad + 1 = s_3, \qquad bc + 1 = s_4,$$
$$bd + 1 = s_5, \qquad cd + 1 = s_6,$$

where $s_i = p^{\alpha_i} q^{\beta_i}$ are S-units for $i = 1, \ldots, 6$. Moreover, we note that

$$\begin{aligned} abcd &= s_2 s_5 - ac - bd - 1 = s_2 s_5 - s_2 - s_5 + 1 \\ &= s_3 s_4 - ad - bc - 1 = s_3 s_4 - s_3 - s_4 + 1 \end{aligned}$$

and therefore we obtain the unit equation

$$s_2 s_5 - s_3 s_4 = s_2 + s_5 - s_3 - s_4. \tag{2}$$

Similarly we also get the unit equations

$$s_1 s_6 - s_3 s_4 = s_1 + s_6 - s_3 - s_4 \tag{3}$$

and

$$s_2 s_5 - s_1 s_6 = s_2 + s_5 - s_1 - s_6. \tag{4}$$

The solution of these unit equations, under some conditions, plays a crucial role in the proof. Since our proof heavily depends on computing p-adic and q-adic valuations, therefore the following lemma provides a useful tool.

Lemma 1. *Let p and q be odd primes and assume that $q^c \| p^{\mathrm{ord}_q(p)} - 1$ and $q^z | p^x - 1$. Then $x \geq \mathrm{ord}_q(p) q^{z-c}$. Moreover, if $q^c \| p^{\mathrm{ord}_q(p)} - 1$ and $q^z | p^x + 1$ then $x \geq \frac{\mathrm{ord}_q(p)}{2} q^{z-c}$.*

Proof. The lemma is elementary and some related versions can be found in [5, Section 2.1.4]. For completeness we give a sketch of the proof.

First, note that by the assumption above we have

$$p^{\mathrm{ord}_q(p)} \equiv 1 + aq^c \mod q^{c+1}$$

holds for some a relatively prime to q. Now let us assume $p^x \equiv 1 + aq^m \mod q^{m+2}$ with $q \nmid a$ and $m \geq c \geq 1$. Taking the q-th power we obtain

$$p^{xq} \equiv 1 + aq^{m+1} + q^{2m+1} B \equiv 1 + aq^{m+1} \mod q^{m+2},$$

since $m \geq 1$. Clearly, B denotes some appropriate integer. Similarly, we see that $q^{m+1} \nmid p^{xk} - 1$ follows if $q \nmid k$. Now, by induction, the first statement of the lemma is obvious.

Note that the smallest positive solution to $p^z \equiv -1 \mod p^c$ is at least $\frac{\mathrm{ord}_q(p)}{2}$. Therefore $p^{\mathrm{ord}_q(p)/2} \equiv -1 + aq^c \mod q^{c+1}$ holds for some a. Indeed, squaring both sides, it shows that $q^c \| p^{\mathrm{ord}_q(p)} - 1$. Now the proof runs along similar lines as in the case above. □

Next we consider the case when the S-units on the right side fulfill some divisibility properties.

Lemma 2. *Assume that $\{a,b,c\}$ is an S-Diophantine triple with $a<b<c$. If $ac+1=s$ and $bc+1=t$ then $s\nmid t$.*

Proof. Let us assume $s|t$. Then

$$\mathbb{Z}\ni m=\frac{bc+1}{ac+1}=\frac{b}{a}+\frac{a-b}{a^2c+a}=\frac{b}{a}+\frac{\theta}{a^2}$$

with $|\theta|<1$. Therefore m is an integer if and only if $\theta=0$. Thus $a=b$ leads to a contradiction. □

Corollary 2. *If $|S|=1$, then there does not exist an S-Diophantine triple.*

Proof. Apply Lemma 2 and note that in case of $|S|=1$ we immediately have $s|t$ using the notation of the lemma. □

We can immediately see that $s_2\nmid s_4$, $s_3\nmid s_5$, $s_5\nmid s_6$ and $s_3\nmid s_6$, in particular none of the equations $\alpha_2=\alpha_4$, $\alpha_3=\alpha_5$, $\alpha_5=\alpha_6$, $\alpha_3=\alpha_6$, $\beta_2=\beta_4$, $\beta_3=\beta_5$, $\beta_5=\beta_6$ and $\beta_3=\beta_6$ hold.

Lemma 3. *We have*

$$\begin{aligned}
a|\gcd&\left(\frac{s_2-s_1}{\gcd(s_2,s_1)},\frac{s_3-s_1}{\gcd(s_3,s_1)},\frac{s_3-s_2}{\gcd(s_3,s_2)}\right),\\
b|\gcd&\left(\frac{s_4-s_1}{\gcd(s_4,s_1)},\frac{s_5-s_1}{\gcd(s_5,s_1)},\frac{s_5-s_4}{\gcd(s_5,s_4)}\right),\\
c|\gcd&\left(\frac{s_4-s_2}{\gcd(s_4,s_2)},\frac{s_6-s_2}{\gcd(s_6,s_2)},\frac{s_6-s_4}{\gcd(s_6,s_4)}\right),\\
d|\gcd&\left(\frac{s_5-s_3}{\gcd(s_5,s_3)},\frac{s_6-s_3}{\gcd(s_6,s_3)},\frac{s_6-s_5}{\gcd(s_6,s_5)}\right).
\end{aligned}$$

Proof. We prove only the divisibility property for a since the other cases run completely analogously. First note that $a|a(c-b)=s_2-s_1$. Since $\gcd(a,s_1)=1$ and $\gcd(a,s_2)=1$ we deduce $a|\frac{s_2-s_1}{\gcd(s_2,s_1)}$. Similarly we get the other relations $a|\frac{s_3-s_1}{\gcd(s_3,s_1)}$ and $a|\frac{s_3-s_2}{\gcd(s_3,s_2)}$, hence the proof of the lemma is complete. □

The next lemma is a useful consequence of Lemma 3.

Lemma 4. *Let $(a,b,c,d)\in\mathbb{Z}^4$ be an S-Diophantine quadruple. Then $\gcd(s_4,s_2)\gcd(s_4,s_1)<s_4$.*

PROOF. By the lemma above we have $b \leq \frac{s_4}{\gcd(s_4,s_1)} - 1$ and $c \leq \frac{s_4}{\gcd(s_4,s_2)} - 1$. It yields

$$s_4 = bc + 1 < \frac{s_4^2}{\gcd(s_4, s_1)\gcd(s_4, s_2)}. \qquad \square$$

Now we prove a lemma which is very helpful in the last two sections of the paper, after collecting enough information on the exponents α_i and β_i, $i = 1, 2 \ldots, 6$.

Lemma 5. *Let the notations be as above and assume that $q > p \geq 5$. Put $\delta = \max\{0, \alpha_4 - \alpha_1 - \alpha_2\}$ and $\epsilon = \max\{0, \beta_4 - \beta_1 - \beta_2\}$. Then we have*

$$p^\delta q^\epsilon a^2 = p^{\alpha_1+\alpha_2+\delta-\alpha_4} q^{\beta_1+\beta_2+\epsilon-\beta_4} - r,$$

with $0 < r < 2p^\delta q^\epsilon$ and $r \in \mathbb{Z}$. If we additionally assume that

$$p^{\alpha_4-\alpha_2} q^{\beta_4-\beta_2} > p^\delta q^\epsilon \quad \text{or} \quad \delta = \epsilon = 0$$

then

$$p^{\alpha_4} q^{\beta_4} - 2p^{\alpha_1+\alpha_2+2\delta-\alpha_4} q^{\beta_1+\beta_2+2\epsilon-\beta_4} < p^{\alpha_2+\delta} q^{\beta_2+\epsilon} < p^{\alpha_4} q^{\beta_4}.$$

The essential part in the proof of the Lemma is the computation of a good approximation of the quantity a^2. To quantify our approximations we will use the so called L-notation (cf. [14]). This allows us to keep track of how large the constants of the usual O-terms get. The L-notation is defined as follows. For two functions $g(t)$ and $h(|t|)$ we write $g(t) = L(h(|t|))$ if $|g(t)| \leq h(|t|)$. In view of applications the estimate

$$\frac{1}{x-1} = \frac{1}{x} + L\left(\frac{1.25}{x^2}\right) = \frac{1}{x} + \frac{1}{x^2} + L\left(\frac{1.25}{x^3}\right)$$

for $|x| \geq 5$ becomes useful. We obtain it by a formal Laurent expansion of $\frac{1}{x-1}$ at infinity.

PROOF OF LEMMA 5. We compute

$$\begin{aligned} a^2 &= \frac{(s_1-1)(s_2-1)}{s_4-1} \\ &= \frac{s_1 s_2}{s_4} - \frac{s_1+s_2}{s_4} + \frac{1}{s_4} + \frac{s_1 s_2}{s_4^2} + L\left(1.25\frac{s_1+s_2+1+s_1s_2/s_4}{s_4^2}\right) \end{aligned}$$

and therefore we obtain

$$\begin{aligned} p^\delta q^\epsilon a^2 &= p^{\alpha_1+\alpha_2+\delta-\alpha_4} q^{\beta_1+\beta_2+\epsilon-\beta_4} - p^{\alpha_1+\delta-\alpha_4} q^{\beta_1+\epsilon-\beta_4} - p^{\alpha_2+\delta-\alpha_4} q^{\beta_2+\epsilon-\beta_4} \\ &+ p^{\delta-\alpha_4} q^{\epsilon-\beta_4} + p^{\alpha_1+\alpha_2+\delta-2\alpha_4} q^{\beta_1+\beta_2+\epsilon-2\beta_4} + L\left(\frac{3.93}{p^{2\alpha_4-\alpha_2-\delta} q^{2\beta_4-\beta_2-\epsilon}}\right). \end{aligned} \tag{5}$$

It implies

$$p^\delta q^\epsilon a^2 = p^{\alpha_1+\alpha_2+\delta-\alpha_4} q^{\beta_1+\beta_2+\epsilon-\beta_4} - r$$

with $0 < r < 2p^\delta q^\epsilon$ and $r \in \mathbb{Z}$. Note that the Diophantine problems

$$\frac{s_1+s_2}{s_4} + \frac{3.93 s_2}{s_4^2} - \frac{1}{s_4} - \frac{s_1 s_2}{s_4^2} > 2, \qquad s_1 \geq 5, s_4 \geq 35$$

and

$$\frac{s_1+s_2}{s_4} - \frac{3.93 s_2}{s_4^2} - \frac{1}{s_4} - \frac{s_1 s_2}{s_4^2} < 0, \qquad s_1 \geq 5, s_4 \geq 35$$

have no integer solutions. On the other hand, if $r \geq 1$ we deduce that

$$1 < p^{\alpha_1+\delta-\alpha_4} q^{\beta_1+\epsilon-\beta_4} + p^{\alpha_2+\delta-\alpha_4} q^{\beta_2+\epsilon-\beta_4}$$

since $1/s_4 + s_1 s_2/s_4^2 > 3.93 s_2/s_4^2$. In the case of $\delta = \epsilon = 0$ we obtain

$$1 - p^{\alpha_1-\alpha_4} q^{\beta_1-\beta_4} < p^{\alpha_2-\alpha_4} q^{\beta_2-\beta_4} < 1$$

and

$$1 - p^{\alpha_1+\delta-\alpha_4} q^{\beta_1+\epsilon-\beta_4} < p^{\alpha_2+\delta-\alpha_4} q^{\beta_2+\epsilon-\beta_4} < p^{\alpha_2+\delta-\delta-\alpha_2} q^{\beta_2+\epsilon-\epsilon-\beta_2} = 1$$

otherwise. Some simple computations yield now the second part of the lemma. □

Next, we mean to find appropriate lower bounds for b and c. When $ac+1$ and $bc+1$ are perfect powers of p we may apply Lemma 2. Therefore q divides either $ac+1$ or $bc+1$, and we have $(c-1)c+1 \geq bc+1 \geq q$. Hence $c > \sqrt{q}$. Knowing that $p \leq ab+1 < b^2$ we derive $b > \sqrt{p}$ and therefore we established

Lemma 6. *We have $b > \sqrt{p}$ and $c > \sqrt{q}$.*

The rest of this section is devoted to bring the result due to Stewart and Tijdeman [20] in a more accurate form according to our intentions. In particular, we need suitable upper bounds for d.

Lemma 7. *Let $S = \{p, q\}$, and suppose that (a, b, c, d) is an S-Diophantine quadruple with $a < b < c < d$. Assuming that $10^{10} < p < q$ we have*

$$\frac{\log d}{(\log\log d)^4} < 7.969 \cdot 10^{21} (\log p \log q)^3.$$

Proof. In order to keep the constants as small as possible we use the theorems on linear forms of logarithms due to Matveev [16] and Laurent, Mignotte and Nesterenko [15]. First recall Matveev's result.

Theorem 2 (Matveev 2000)**.** *Denote by $\gamma_1, \ldots, \gamma_n$ algebraic numbers, nor 0 neither 1, by $\log\gamma_1, \ldots, \log\gamma_n$ determinations of their logarithms, by D the degree over $\mathbb{Q}$ of the number field $K = \mathbb{Q}(\gamma_1, \ldots, \gamma_n)$, and by $b_1, \ldots, b_n$ rational integers. Furthermore let $\kappa = 1$ if K is real and $\kappa = 2$ otherwise. Choose*

$$A_i \geq \max\{Dh(\gamma_i), |\log\gamma_i|\} \quad (1 \leq i \leq n),$$

where $h(\gamma)$ denotes the absolute logarithmic Weil height of γ and

$$B = \max\{1, \max\{|b_j|A_j/A_n : 1 \leq j \leq n\}\}.$$

Assume that $b_n \neq 0$ and $\log\gamma_1, \ldots, \log\gamma_n$ are linearly independent over $\mathbb{Z}$. Then

$$\log|b_1\log\gamma_1 + \cdots + b_n\log\gamma_n| \geq -C(n)C_0W_0D^2\Omega,$$

with

$$\Omega = A_1 \cdots A_n,$$

$$C(n) = C(n,\kappa) = \frac{16}{n!\kappa}e^n(2n+1+2\kappa)(n+2)(4(n+1))^{n+1}\left(\frac{1}{2}en\right)^{\kappa},$$

$$C_0 = \log\left(e^{4.4n+7}n^{5.5}D^2\log(eD)\right), \quad W_0 = \log(1.5eBD\log(eD)).$$

In the case of linear forms in two logarithms we can use a sharper bound due to Laurent et al. [15]:

Theorem 3 (Laurent, Mignotte, Nesternko 1995)**.** *Let γ_1 and γ_2 be two positive, real, multiplicatively independent elements in a number field of degree D over $\mathbb{Q}$. For $i = 1, 2$, let $\log\gamma_i$ be any determination of the logarithm of α_i, and let $A_i > 1$ be a real number satisfying*

$$\log A_i \geq \max\{h(\gamma_i), |\log\gamma_i|/D, 1/D\}.$$

Further, let b_1 and b_2 be two positive integers. Define

$$b' = \frac{b_1}{D\log A_2} + \frac{b_2}{D\log A_1} \quad \text{and} \quad \log b = \max\left\{\log b' + 0.14, 21/D, \frac{1}{2}\right\}.$$

Then

$$|b_2\log\gamma_2 - b_1\log\gamma_1| \geq \exp\left(-24.34D^4(\log b)^2\log A_1\log A_2\right).$$

We use the same linear forms as in [20] and consider

$$T_1 = \frac{c}{b} \cdot \frac{bd+1}{cd+1} = \frac{c}{b} p^{\alpha_5 - \alpha_6} q^{\beta_5 - \beta_6}.$$

Similarly we find (see also Stewart and Tijdeman [20])

$$\log(T_1) = \log\left(1 + \frac{c-b}{dcb+b}\right) \leq \log\left(1 + \frac{1}{2d}\right) < \frac{1}{d}.$$

On the other hand, Matveev's result (Theorem 2) yields a lower bound. We bring up this lower bound now. First, choose $A_1 = \log p, A_2 = \log q$ and $A_3 = \log c > \frac{\log q}{2}$. Obviously we have $0 \leq \alpha_5, \alpha_6 \leq \frac{\log(d^2-d+1)}{\log p} < \frac{2\log d}{\log p}$ and $0 \leq \beta_5, \beta_6 \leq \frac{2\log d}{\log q}$. Therefore we obtain $B < \frac{2\log d}{\log c}$, hence we have

$$1.690182 \cdot 10^{10} \log c \log p \log q \left(2.1 + \log\left(\frac{\log d}{\log c}\right)\right) > \log d. \tag{6}$$

In the case of

$$T_2 = \frac{(bd+1)(ac+1)}{(cd+1)ab}$$

we compute

$$\log(T_2) = \log\left(1 + \frac{db+ac-ab+1}{abcd+ab}\right) < \log\left(1 + \frac{2}{ac}\right) < \frac{4}{c},$$

and therefore by Theorem 2

$$1.690182 \cdot 10^{10} \log(ab) \log p \log q \left(2.8 + \log\left(\frac{\log d}{\log(ab)}\right)\right) > \log c - \log 4 \tag{7}$$

follows.

In case of

$$T_3 = \frac{(ab+1)(cd+1)}{(ac+1)(bd+1)}$$

we find

$$\log(T_3) = \log\left(1 + \frac{(d-a)(c-b)}{abcd+db+ac+1}\right) < \log\left(1 + \frac{1}{ab}\right) < \frac{2}{ab}.$$

Assume for a moment that $b' + 0.14 \geq 21$. Thus we may apply Theorem 3. First,

$$b' \leq \frac{8\log d}{\log p \log q},$$

therefore we have

$$24.34 \log p \log q \left(2.08 + \log\left(\frac{\log d}{\log p \log q}\right)\right)^2 > \log(ab) - \log 2. \tag{8}$$

If we even suppose that p, q are large, say $10^{10} < p < q$, by combining the inequalities (6), (7) and (8), and using the lower bounds for b and c derived in Lemma 6, we obtain

$$7.969 \cdot 10^{21} (\log p \log q)^3 (\log \log d)^4 > \log d. \tag{9}$$

Since the bound $\frac{21}{8} \log p \log q > \log d$ is much sharper than (9), we proved the lemma completely. □

The previous result gives us upper bounds for d. On the other hand, we will find by Lemma 1 lower bounds for d. In particular, the following lemma provides bounds for p under some restrictions.

Lemma 8. *Assume* $\max_{i=1,\dots,6}\{\alpha_i + \beta_i\} > p$. *Then we deduce* $p < C(\xi)$ *with*

$$C(\xi) = \Psi(9; 2.142 \cdot 10^{22} \xi^3),$$

where $\Psi(k; x)$ *denotes the largest solution* $y > 0$ *to the equation* $x = \frac{y}{(\log y)^k}$.

Proof. Note that $C(\xi)$ is increasing with $\xi \geq 1$ and note that $C(1) = 1.02 \cdot 10^{40}$. Therefore we may assume $p, q > 10^{40}$. By

$$d^2 > cd + 1 > p^{\max_{i=1,\dots,6}\{\alpha_i + \beta_i\}} > p^p,$$

Lemma 7 and the conditions of the lemma we get

$$c\xi^3 (\log p)^6 (\log \log d)^4 > \log d > \frac{1}{2} p \log p,$$

where $c = 8.478 \cdot 10^{21}$. Therefore

$$c\xi^3 (\log p)^6 > \frac{\log d}{(\log \log d)^4} > \frac{p \log p}{2(\log \log p + \log p)^4} > \frac{p}{2.687842 (\log p)^3},$$

since $\frac{\log x}{(\log \log x)^4}$ is increasing if $x > 5.15 \cdot 10^{23}$. Solving the last inequality for p, it gives the required result. □

The following proposition will be frequently used.

Proposition 1. *Assume that one of the equations* (2), (3) *and* (4) *is written in the form*

$$p^{e_1}q^{f_1} - p^{e_2}q^{f_2} = p^{e_3}q^{f_3} + p^{e_4}q^{f_4} - p^{e_5}q^{f_5} - p^{e_6}q^{f_6},$$

further let e be the difference of the third to least exponent and the least exponent of the e_i, with $i = 1, \dots, 6$, and let f be defined in the obvious similar way. Then we deduce $e, f \leq 1$, provided that $p > C(\xi)$. Moreover, the two least exponents are equal.

Proof. Let us consider, say, unit equation (2). We obtain

$$p^{\alpha_2+\alpha_5}q^{\beta_2+\beta_5} - p^{\alpha_3+\alpha_4}q^{\beta_3+\beta_4} = p^{\alpha_5}q^{\beta_5} + p^{\alpha_2}q^{\beta_2} - p^{\alpha_3}q^{\beta_3} - p^{\alpha_4}q^{\beta_4}.$$

Suppose that all exponents α_i with $i = 2, 3, 4, 5$ are distinct. Computing the p-adic valuations on the left and right hand sides we see that

$$v_p\left(p^{\alpha_5}q^{\beta_5} + p^{\alpha_2}q^{\beta_2} - p^{\alpha_3}q^{\beta_3} - p^{\alpha_4}q^{\beta_4}\right) = \min\{\alpha_i\}.$$

Say, the minimum is α_2. But, in this case we have $\alpha_2 < \alpha_2+\alpha_5$ and $\alpha_2 < \alpha_3+\alpha_4$, i.e. the p-adic valuation on the left side does not fit to the p-adic valuation on the right. Therefore in any case the two least exponents are equal. Observe, that all other cases can be deduced by the same method.

Now divide the equation by the least occurring powers of p and q, respectively. Consider (2) and assume $\alpha_2 = \alpha_5$ and $\beta_4 = \beta_3$ are the smallest exponents. Then

$$p^{\alpha_2}q^{\beta_2+\beta_5-\beta_3} - p^{\alpha_3+\alpha_4-\alpha_2}q^{\beta_3} - q^{\beta_2-\beta_3}(q^{\beta_5-\beta_2}+1) = -p^{\min\{\alpha_3,\alpha_4\}-\alpha_2}(p^{|\alpha_3-\alpha_4|}+1)$$

holds. Clearly, in all other cases we obtain similar equations. In particular, in any case we obtain that for some x the quantity $1 \pm p^x$ is divided by q^f. Since x is at most $\max\{\alpha_i + \beta_i\}$, due to Lemma 8 we obtain that $x < p$ or $p < C(\xi)$. Hence Lemma 1 yields $f \leq 1$ for large p. By similar arguments we also deduce $e \leq 1$. □

4. Unit equation (2)

In this section we deal with equation (2), and our main result is to deduce some relations for the exponents appearing in (2). In particular, this section is devoted to the proof of the following proposition.

Table 1. List of the possible solutions to equation (2)

Case	α	β
1	$\alpha_2 = \alpha_5 \leq 1$	$\beta_3 = \beta_4 \leq 1$
2	$\alpha_2 = \alpha_5 \leq 1$	$\beta_3 = \beta_4 = \beta_2 - 1$
3	$\alpha_3 = \alpha_4 = \alpha_2 - 1$	$\beta_2 = \beta_5 = \beta_3 - 1$
4	$\alpha_3 = \alpha_4 = \alpha_2 - 1$	$\beta_2 = \beta_5 \leq 1$
5	$\alpha_3 = \alpha_4 \leq 1$	$\beta_2 = \beta_5 = \beta_3 - 1$
6	$\alpha_3 = \alpha_4 \leq 1$	$\beta_2 = \beta_5 = \beta_4 - 1 = 0$
7	$\alpha_3 = \alpha_4 \leq 1$	$\beta_2 = \beta_5 \leq 1$

Proposition 2. *Let $C(\xi)$ be defined as in Lemma 8. If $p > C(\xi)$ then one of the seven cases in Table 1 holds.*

By Proposition 1 we may assume that $\alpha_i = \alpha_j$ is minimal for some distinct $i, j \in \{2, 3, 4, 5\}$, i.e. we have to consider six cases. If $\alpha_i = \alpha_j$ and $\beta_i = \beta_j$ hold we deduce that either $s_i | s_j$ or $s_j | s_i$. Therefore we can exclude, by Lemma 2 the cases $\alpha_2 = \alpha_4$ and $\alpha_3 = \alpha_5$ and also when $\beta_2 = \beta_4$ and $\beta_3 = \beta_5$. So four subcases remain to consider.

Before we discuss them we write down again equation (2) explicitly:

$$p^{\alpha_2+\alpha_5} q^{\beta_2+\beta_5} - p^{\alpha_3+\alpha_4} q^{\beta_3+\beta_4} = p^{\alpha_2} q^{\beta_2} + p^{\alpha_5} q^{\beta_5} - p^{\alpha_3} q^{\beta_3} - p^{\alpha_4} q^{\beta_4}. \tag{10}$$

4.1. The case when $\alpha_2 = \alpha_5$ is minimal. First, observe that $\beta_2 < \beta_5$ and we also note that $\beta_4 < \beta_2$ otherwise $s_2 | s_4$ would contradict Lemma 2. Since a sole minimum cannot exist we deduce that $\beta_3 = \beta_4$. The third smallest exponent of q in equation (10) is either $2\beta_3$ or β_2. Hence, by Proposition 1 we have $\beta_3 = \beta_4 \leq 1$ or $\beta_3 = \beta_4 = \beta_2 - 1$. Note that $\beta_4 = \beta_2$ would yield a contradiction by $s_2 | s_4$.

The third smallest exponent of p in equation (10) is either $2\alpha_2$, α_3 or α_4. Therefore we have either $\alpha_2 = \alpha_5 \leq 1$, $\alpha_2 = \alpha_5 = \alpha_3 - 1$ or $\alpha_2 = \alpha_5 = \alpha_4 - 1$. Note that only the first case may hold since by assumption $\beta_2 > \beta_3 = \beta_4$, consequently $s_2 > s_3$ or $s_2 > s_4$ fulfills because of $p < q$. Therefore we deduce that one of the first two cases in Table 1 holds.

4.2. The case when $\alpha_2 = \alpha_3$ is minimal. Again $\beta_4 < \beta_2$ since $s_2 \nmid s_4$. Thus we have $\beta_4 = \beta_5 < \beta_2 < \beta_3$. Therefore the third smallest exponent of q in equation (10) is β_2, subsequently $\beta_4 = \beta_5 = \beta_2 - 1$.

Similarly, by considering the exponents of p in equation (10), we obtain that $\alpha_2 = \alpha_3 = \alpha_4 - 1$ because $\alpha_4 < \alpha_5$. But together with the relations of the β's we arrived at the contradiction $s_2 > s_4$.

4.3. The case when $\alpha_4 = \alpha_5$ is minimal. We immediately see that $\beta_2 < \beta_4$ and $\beta_4 < \beta_5$, since otherwise $s_2 | s_4$ and $s_4 > s_5$, respectively. Therefore $\beta_2 = \beta_3$ is minimal. Consider the exponents of q in equation (10) to obtain $\beta := \beta_2 = \beta_3 = \beta_4 - 1$. Since we have $\beta_2 = \beta_3$ we deduce $\alpha_2 < \alpha_3$ and therefore Proposition 1 in view of p-exponents yields $\alpha := \alpha_5 = \alpha_4 = \alpha_2 - 1$.

In the virtue of $c | s_4 - s_2$ Lemma 3 yields $c < q$. On the other hand, we have $s_4 = p^\alpha q^{\beta+1} = bc + 1 < c^2 < q^2$, and therefore $\beta = 0$ and $p^\alpha < q$. Consider now s_1. We have

$$qp > p^{\alpha+1} = s_2 = ac + 1 > ab + 1 = p^{\alpha_1} q^{\beta_1}.$$

Therefore we have either $\beta_1 = 0$ and $b < p^\alpha$ or $ab + 1 = q$.

First suppose $\beta_1 = 0$. Then we have

$$\mathbb{Z} \ni \frac{ps_4}{s_2} = \frac{pb}{a} - \frac{1}{a} \cdot \frac{p(b-a)}{ac+1} = \frac{pb}{a} - \frac{1}{a} \cdot \frac{\overbrace{b-a}^{<1}}{p^\alpha}.$$

Since the left hand side is an integer we deduce that the "braced" quantity is zero, hence $b = a$, which is a contradiction.

In the case of $ab + 1 = q$, by assumption $c < q$ and $ab + 1 = q$ we get

$$\mathbb{Z} \ni \frac{s_4}{s_1} = \frac{c}{a} - \frac{1}{a} \cdot \frac{\overbrace{c-a}^{<1}}{ab+1}.$$

But $c = a$ is again a contradiction.

4.4. The case when $\alpha_3 = \alpha_4$ is minimal. We have $\beta_2 < \beta_3, \beta_4$ since otherwise we would have $s_2 \geq s_3, s_4$. Because no sole minimum exists we deduce $\beta_2 = \beta_5$. Applying Proposition 1 we obtain either $\beta_2 = \beta_5 \leq 1$ or $\beta_2 = \beta_5 = \beta_3 - 1$ or $\beta_2 = \beta_5 = \beta_4 - 1$. Now we may assume $\alpha_2 < \alpha_5$ and again applying Proposition 1, it provides either $\alpha_3 = \alpha_4 = \alpha_2 - 1$ or $\alpha_3 = \alpha_4 \leq 1$. The combination of the relations of the α's and β's yields either cases listed in Table 1 or the case $\alpha := \alpha_3 = \alpha_4 = \alpha_2 - 1$ and $\beta := \beta_2 = \beta_5 = \beta_4 - 1$ or the case $\alpha_3 = \alpha_4 \leq 1$ and $\beta := \beta_2 = \beta_5 = \beta_4 - 1$.

When $\alpha := \alpha_3 = \alpha_4 = \alpha_2 - 1$ and $\beta := \beta_2 = \beta_5 = \beta_4 - 1$, similarly to the subsection above, it leads to a contradiction. Note that only the relations between s_2 and s_4 have been used there.

Therefore it remains to prove $\beta = \beta_2 = 0$ in the last case. By $c | s_4 - s_2$ and Lemma 3 we have $c < q$ and therefore $q^2 > bc + 1 = s_4$. Hence $\beta_4 \leq 1$. But $\beta_4 = 0$ would lead to a negative β_2, hence $\beta_4 = \beta_2 + 1 = 1$.

5. Unit equation (4)

In this section we consider the unit equation (4) more closely, in particular we prove the following proposition.

Proposition 3. *Let $C(\xi)$ be defined as in Lemma 8. If $p > C(\xi)$ then one of the three cases in Table 2 holds.*

Table 2. List of the possible solutions to the system of equations (2) and (4)

Case	α	β
I	$\alpha_3 = \alpha_4 \leq 1$; $\alpha_1 = \alpha_6 \leq 1$	$\beta_2 = \beta_5 \leq 1$
II	$\alpha_2 = \alpha_5 \leq 1$	$\beta_3 = \beta_4 \leq 1$; $\beta_1 = \beta_6 \leq 1$
III	$\alpha_2 = \alpha_5 \leq 1$	$\beta_3 = \beta_4 = \beta_2 - 1$; $\beta_1 = \beta_6 \leq 1$

Since none of the α's take a sole minimum in Proposition 1, and $\alpha_5 = \alpha_6$ induces $s_5|s_6$ (a contradiction to Lemma 2) we are left to five subcases. Note that equation (4) takes the form

$$p^{\alpha_2+\alpha_5}q^{\beta_2+\beta_5} - p^{\alpha_1+\alpha_6}q^{\beta_1+\beta_6} = p^{\alpha_2}q^{\beta_2} + p^{\alpha_5}q^{\beta_5} - p^{\alpha_1}q^{\beta_1} - p^{\alpha_6}q^{\beta_6}. \tag{11}$$

5.1. The case when $\alpha_1 = \alpha_2$ is minimal. Since $\beta_5 = \beta_6$ implies $s_5|s_6$ and $\beta_1 < \beta_2$ we are left to the two possibilities $\beta_1 = \beta_5$ and $\beta_1 = \beta_6$.

5.1.1. *The subcase when $\beta_1 = \beta_5$ is minimal.* Note that $\alpha_1 = \alpha_2 = \alpha_5$ cannot hold since otherwise $s_1 = s_5$ is a contradiction. Therefore we deduce $\alpha_2 < \alpha_5$, but this yields by Proposition 2 $\beta_2 = \beta_5 = \beta_1$, again a contradiction.

5.1.2. *The subcase when $\beta_1 = \beta_6$ is minimal.* By the assumption $\beta_1 = \beta_6 < \beta_5$ we deduce $\alpha_5 < \alpha_6$. Hence Proposition 1 yields $\alpha_1 = \alpha_2 = \alpha_5$ or $\alpha_1 = \alpha_2 = \alpha_5 - 1$ for the exponents of p. Since $\alpha_5 \leq \alpha_2 + 1$ we deduce $\beta_2 \leq \beta_5$ and Proposition 1 yields in view of exponents of q that either $\beta_1 = \beta_6 \leq 1$ or $\beta_1 = \beta_6 = \beta_2 - 1$.

Let us assume $\alpha_1 = \alpha_2 = \alpha_5$ and $\beta_1 = \beta_6 \leq 1$. Then only the first two cases of Table 1 hold, i.e. these are cases II and III of Table 2.

Now let us assume $\alpha_1 = \alpha_2 = \alpha_5$ and $1 < \beta_1 = \beta_6 = \beta_2 - 1$. Again only the first two cases of Table 1 hold. In the first case we have $\alpha_3 > \alpha_6$ since $s_3 \nmid s_6$ and obviously $\beta_3 < \beta_6$ and we also have $\alpha_6 > \alpha_5 = \alpha_2$ since otherwise $s_5|s_6$. Therefore Lemma 3 in view of the pairs (s_6, s_3) and (s_6, s_2) yields $d|p^{\beta_6-\beta_3} - p^{\alpha_3-\alpha_6}$ thus $d < q^{\beta_6}$, and $c|p^{\alpha_6-\alpha_2} - q$ thus $c < p^{\alpha_6}$. Therefore $p^{\alpha_6}q^{\beta_6} = cd + 1 < p^{\alpha_6}q^{\beta_6}$

shows a contradiction. In the second case we obtain $\beta_1 = \beta_6 = \beta_2 - 1 = \beta_3 = \beta_4$, hence $s_3 | s_6$ again is a contradiction.

Assume now that $\alpha_1 = \alpha_2 = \alpha_5 - 1$. Since $\alpha_2 \neq \alpha_5$, we may exclude the first two cases of Table 1.

Next we consider the cases 3 and 4 in Table 1 and we may assume $\alpha := \alpha_3 = \alpha_4 = \alpha_1 - 1 = \alpha_2 - 1 = \alpha_5 - 2$. Since $\beta_2 = \beta_5$ we have

$$s_2 = p^{\alpha+1}q^{\beta_2} < p^{\alpha}q^{\beta_3}, p^{\alpha}q^{\beta_4} < p^{\alpha+2}q^{\beta_2} = s_5,$$

and therefore we may suppose $\beta := \beta_2 = \beta_5 = \beta_3 - 1 = \beta_4 - 1$ and $\beta_1 < \beta$. Now Lemma 3 yields in view of the pair (s_3, s_5) that $d | p^2 - q$ and therefore $p^{\alpha+2}q^{\beta} = bd + 1 < p^4$ which is impossible unless $\alpha = 0$, $\beta = 1$ and $\beta_1 = 0$. But the later assumption leads to $ab + 1 = p$, hence $b < p$ and $p^2 q = bd + 1 < p^3$ mean again a contradiction.

Now let us assume that either case 5 or case 6 of Table 1 holds. Write $\alpha := \alpha_1 = \alpha_2 = \alpha_5 - 1$. Since $\alpha_3 = \alpha_4 \leq 1$ and $s_2 < s_3, s_4 < s_5 = ps_2$ we deduce $\beta_3 = \beta_4$. Therefore we have $\beta_1 < \beta_2 = \beta_5 = \beta_3 - 1 = \beta_4 - 1 =: \beta$ and Lemma 3 in view of the pairs (s_4, s_2) and (s_5, s_3) yields $b < c < q$ and $d < p^{\alpha+1-\alpha_4}$. Hence $bd + 1 < qp^{\alpha+1}$ which yields a contradiction unless $\beta = 0$. But $\beta = 0$ yields $\beta_1 < 0$.

We turn now to the case $\alpha := \alpha_1 = \alpha_2 = \alpha_5 - 1$, $\alpha' := \alpha_3 = \alpha_4 \leq 1$, $\beta_1 = \beta_6 = 0$ and $\beta_2 = \beta_5 = 1$ which corresponds to case 7 of Table 1. Since $p^{\alpha}q = s_2 < s_3, s_4 < p^{\alpha+1}q$ and $\alpha_3 = \alpha_4$ we deduce that $\beta_3 = \beta_4 =: \beta$. Next, in view of the pairs $(s_2, s_1), (s_5, s_1), (s_4, s_2)$ and (s_6, s_5) and Lemma 3 we obtain

$$a < q, \quad b < pq, \quad c < q^{\beta-1}, \quad d \leq p^{\alpha_6-\alpha-1} - q.$$

Therefore $pq^{\beta} > bc + 1 = p^{\alpha'}q^{\beta}$, which can only hold if $\alpha' = 0$. We reconsider now the unit equation (11) and solve it for p^{α_6}. We get

$$p^{\alpha_6} = \left(1 - \frac{1}{p^{\alpha}}\right)^{-1} \left(p^{\alpha+1}q^2 - q(p+1) + 1\right) = p^{\alpha+1}q^2 + L(2pq^2). \tag{12}$$

Together with the estimations above, (12) implies

$$d \leq q^2 + \frac{2q^2}{p^{\alpha}} - q.$$

Furthermore, we have

$$q^{\beta} = bc + 1 < d^2 \leq q^4 \left(1 + \frac{4}{p^{\alpha}} + \frac{4}{p^{2\alpha}}\right) - 2q^3 \left(1 + \frac{2}{p^{\alpha}}\right) + q^2 + 1 < q^5, \tag{13}$$

i.e. $\beta \leq 4$. Since $s_2 > s_1$ and $s_2 \nmid s_4$ we deduce $\beta \geq 2$. In case of $\beta = 2$ we have $c < q$, i.e. $q^2 = bc + 1 < q^2$ is a contradiction. Therefore we consider the case $\beta = 4$ next. Note that we have $\frac{1}{p^\alpha} < \frac{p}{q^3}$ since $s_3 < s_5$. Using this estimate in (13), it yields

$$q^4 = bc + 1 < d^2 < q^4 + 4pq + \frac{4p^2}{q^2} - 2q^3 + q^2 + 1 < q^4.$$

Therefore we can restrict ourselves to the case $\beta = 3$. Since $s_3 < s_5$ we deduce $\frac{1}{p^\alpha} < \frac{p}{q^2}$ and by the estimations for d we obtain

$$d \leq q^2 + 2p - q \leq q^2,$$

provided $q \geq 2p$. Recall that $a < q$, hence $q^3 = ad + 1 < q^3$ leads to a contradiction. Consequently, we may assume $q < 2p$. In this case we have

$$q^3 = bc + 1 > ac + 1 > \frac{q^{\alpha+1}}{2^\alpha}$$

which is again a contradiction unless $\alpha \leq 2$. Obviously, $\alpha = 0$ is impossible. Thus we consider the case $\alpha = 1$, which provides a contradiction by $q^3 = bc + 1 < bd + 1 = p^2 q$. So only $\alpha = 2$ remains to investigate. Recall (12) to obtain

$$p^{\alpha_6} = p^3 q^2 + L(2pq).$$

It gives $\alpha_6 = 5$. Note that we assume that $p < q < 2p$ and p is large. Hence by the estimate $d < p^{\alpha_6 - \alpha - 1} = p^2$ we have $p^5 = cd + 1 < p^4$. This is a contradiction.

5.2. The case when $\alpha_1 = \alpha_5$ is minimal. Since the case $\alpha_1 = \alpha_2$ has already treated, we may suppose $\alpha_1 = \alpha_5 < \alpha_2$. But by Proposition 2 we obtain $\beta_2 = \beta_5$, hence $s_2 > s_5$ which is an obvious contradiction.

5.3. The case when $\alpha_1 = \alpha_6$ is minimal. Note that $\beta_1 < \beta_6$, therefore we distinguish three subcases: $\beta_2 = \beta_5$, $\beta_1 = \beta_5$ and $\beta_1 = \beta_2$.

5.3.1. *The subcase when $\beta_2 = \beta_5$ is minimal.* Here $\beta_1 < \beta_6$ and $\alpha_2 < \alpha_5$. Applying Proposition 1, we obtain either $\beta_2 = \beta_5 \leq 1$ or $\beta_2 = \beta_5 = \beta_1$ or $\beta_2 = \beta_5 = \beta_1 - 1$. Meanwhile, for the $\alpha's$ we have either $\alpha_1 = \alpha_6 \leq 1$ or $\alpha_1 = \alpha_6 = \alpha_2 - 1$. Note that the case $\alpha_1 = \alpha_2$ has already been treated above.

Let us consider the case $\beta' := \beta_2 = \beta_5 \leq 1$ and $\alpha' := \alpha_1 = \alpha_6 \leq 1$ first. By Proposition 2, we deduce that either case I holds or we have $\alpha := \alpha_3 = \alpha_4 = \alpha_2 - 1$.

First, let us assume that $\beta_4 \leq \beta_1 + \beta'$. Applying Lemma 4 we see immediately that no solution exists in this case.

Therefore we may suppose $\beta_4 \geq \beta_1 + \beta' + 1$. Now Lemma 5 yields

$$a^2 = p^{1+\alpha'} q^{\beta_1+\beta'-\beta_4} - r$$

with $0 < r < 2$, where r is not necessarily an integer. By $a \geq 1$ we deduce $\beta_4 = \beta_1 + \beta' + 1$, i.e. $a^2 = \frac{p^{1+\alpha'}}{q} - r$, hence $\alpha' = 1$. In order to apply the inequality stated in Lemma 5, we have to show that

$$p^{\alpha_2+\delta} q^{\beta_2+\epsilon} < p^{\alpha_4} q^{\beta_4},$$

which is in our case equivalent to

$$p^{\alpha+1} q^{1+\beta'} < p^{\alpha} q^{\beta_1+\beta'+1}.$$

This is true unless $\beta_1 = 0$. Now Lemma 5 gives

$$p^{\alpha} q^{\beta_1+\beta'+1} - 2p^2 q < p^{\alpha+1} q^{1+\beta'} < p^{\alpha} q^{\beta_1+\beta'+1}$$

or

$$q^{\beta_1} - 2\frac{1}{p^{\alpha-2} q^{\beta'}} < p < q^{\beta_1}. \tag{14}$$

Unless $\beta' = 0$ and $\alpha \leq 1$ or $\beta' = 1$ and $\alpha = 0$ we have $q^{\beta_1} - 2 < p < q^{\beta_1}$ which is a contradiction to p is an odd prime. But $\alpha = 1$ leads to $\alpha_3 = \alpha_6$ and $\alpha = 0$ leads to $s_1 > s_2$, since we assume $\beta_1 > 0$.

If $\beta_1 = 0$ then, by the assumption $\beta_1 \geq \beta_2 = \beta'$ we deduce $\beta' = 0$ and therefore $\beta_4 = 1$. Since $c < q$ (apply Lemma 3 to the pair (s_2, s_4)) and $b < s_1 = p$ (note that $\alpha_1 = \alpha_6 \leq 1$) we have $bc + 1 < pq$, i.e. $\alpha = 0$. But $\alpha = 0$ entails $s_2 = s_1 = p$, and this is a contradiction.

Now, let us consider the case $\beta_2 = \beta_5 \leq 1$ and $\alpha_1 = \alpha_6 = \alpha_2 - 1$. We note that the cases 3 and 4 in Proposition 2 cannot hold since we would obtain $\alpha_1 = \alpha_6 = \alpha_2 - 1 = \alpha_3 = \alpha_4$ and then $s_3 | s_6$ is a contradiction. Therefore we may assume $\alpha_3 = \alpha_4 \leq 1$. Since $s_2 > s_1$ we deduce that $\beta_1 \leq \beta_2$ and therefore also $\beta_1 < \beta_3, \beta_4$. Considering the unit equation (3), we obtain $\beta_1 = \beta_6$ since a sole minimum cannot exist. So $s_1 = s_6$ is a contradiction.

Now we treat the case $\beta_2 = \beta_5 = \beta_1$. Proposition 2 shows us that $\beta_2 = \beta_5 < \beta_4$ and in view of our actual case $\beta_1 < \beta_4$ holds. Hence, by (3) we deduce that either $\beta_1 = \beta_6$ or $\beta_1 = \beta_3$, which yields either $s_5 | s_6$ or $s_3 | s_5$.

The next case is $\beta_2 = \beta_5 = \beta_1 - 1$. First note that $\alpha_1 = \alpha_6 = \alpha_2 - 1$ cannot hold since $s_1 > s_2$ would mean a contradiction. Therefore we may assume that

$\alpha_1 = \alpha_6 \leq 1$. Since the case $\beta_2 = \beta_5 \leq 1$ has already been treated, we deduce from Proposition 2 that $\beta_2 = \beta_5 = \beta_1 - 1 = \beta_3 - 1$ and either $\alpha_3 = \alpha_4 \leq 1$ or $\alpha_3 = \alpha_4 = \alpha_2 - 1$.

When $\beta = \beta_2 = \beta_5 = \beta_1 - 1 = \beta_3 - 1$, $\alpha_1 = \alpha_6 = 0$ and $\alpha_3 = \alpha_4 = 1$, by $a|s_3 - s_1$ and Lemma 3 we have $a < p$ and since $ab + 1 = q^{\beta+1}$ we deduce on the one hand $b < q^{\beta+1}$ and on the other hand $b > \frac{q^{\beta+1}}{p} > q^{\beta}$. Moreover, we have $s_2 < s_3$ and so $p^{\alpha_2 - 1} < q$ and $ac + 1 < pq^{\beta+1}$, i.e. $c < pq^{\beta+1}$. The bounds for b and c yield $pq^{\beta_4} = bc + 1 < pq^{2\beta+2}$, i.e. $\beta_4 \leq 2\beta + 1$. Now we consider the pairs (s_4, s_1) and (s_4, s_2) in view of Lemma 3. From the first pair we obtain $b|pq^{\beta_4 - \beta - 1} - 1$, hence $\beta_4 = 2\beta + 1$ because $b > q^{\beta}$. Then the second pair yields $c|q^{\beta+1} - p^{\alpha_2 - 1}$, i.e. $c \leq q^{\beta+1}$. Moreover since $s_4 = ad + 1 = pq^{\beta+1}$ and $d < pq^{\beta+1}$ we get $q^{\beta_6} = cd + 1 < pq^{2\beta+2}$ which results in $\beta_6 = 2\beta + 2$. Now the pair (s_6, s_4) yields a new bound for c, namely $c < q$ and together with $a < p$ we have $q^{\beta+1} = ab + 1 < ac + 1 < pq$ and therefore $\beta = 0$. Now we consider the pair (s_3, s_6) and obtain $d|q - p$. Thus $q^2 = cd + 1 < q^2$ is a contradiction finally.

Only the case $\beta = \beta_2 = \beta_5 = \beta_1 - 1 = \beta_3 - 1$, $\alpha' = \alpha_1 = \alpha_6 \leq 1$ and $\alpha = \alpha_3 = \alpha_4 = \alpha_2 - 1$ is still open. Note that $\alpha > \alpha'$. We know that

$$\mathbb{Z} \ni \frac{p(bc+1)}{ac+1} = \frac{pb}{a} - \frac{1}{a} \cdot \overbrace{\frac{p(b-a)}{ac+1}}^{\theta}.$$

If $|\theta| < 1$ we obtain a similar contradictory argument as in Lemma 2. Therefore $c > b > p^{\alpha} q^{\beta}$ follows. From the inequlity $p^{\alpha} q^{\beta} < b < s_1 < s_2$ we get $p^{\alpha - \alpha'} < q < p^{\alpha + 1 - \alpha'}$. Using this inequality in $c < ac + 1 = p^{\alpha+1} q^{\beta}$ we get $c < q^{\beta+1} p^{\alpha'+1}$ and $d < q^{\beta+2} p^{\alpha'}$. Thus

$$p^{\alpha'} q^{\beta_6} = cd + 1 < p^{1+2\alpha'} q^{2\beta+3}$$

and $\beta_6 \leq 2\beta + 3 + e$. Using the upper bound $b < ab + 1 = p^{\alpha'} q^{\beta+1}$ we similarly obtain

$$p^{\alpha} q^{\beta_4} = bc + 1 < p^{1+2\alpha'} q^{2\beta+2}$$

hence $\beta_4 \leq 2\beta + 2 + e$. We apply Lemma 3 to the pair (s_4, s_1) and obtain

$$p^{\alpha} q^{\beta} < b < p^{\alpha - \alpha'} q^{\beta_4 - \beta - 1} < q^{\beta_4 - \beta}$$

which yields $p^{\alpha'} < q^{\beta_4 - 2\beta - 1}$. Thus $\beta_4 = 2\beta + 2$ if $\alpha' = 0$ and $\beta_4 = 2\beta + 2$ or $\beta_4 = 2\beta + 3$ if $\alpha' = 1$. We consider the pair (s_6, s_4) and obtain an upper bound $c < q$ if $\alpha' = 0$ and $c < q^2$ if $\alpha' = 1$. But

$$p^{1 - \alpha'} q^{\beta+1} < p^{\alpha} q^{\beta} < b < c < q^{1+\alpha'}$$

is a contradiction unless $\beta = 0$, $\alpha' = 1$, $\beta_6 = 4$ and $\beta_4 = 2$. Since in any other case we would obtain the sharper bound $c < q$. We remind that $d < q^{\beta+2}p^{\alpha'} = pq^2$, thus $pq^4 = cd + 1 < pq^4$ is a contradiction.

5.3.2. *The subcase when $\beta_1 = \beta_5$ is minimal.* Since the case above we have $\beta_2 > \beta_5$ and from Proposition 2 we deduce $\alpha_2 = \alpha_5$. Then $s_2 > s_5$, which is impossible.

5.3.3. *The subcase when $\beta_1 = \beta_2$ is minimal.* Now $\alpha_1 = \alpha_6 \leq \alpha_5$ implies $\beta_1 = \beta_2 \leq \beta_5 < \beta_6$, and Proposition 1 yields $\beta := \beta_2 = \beta_1 = \beta_5 - 1$. Note that the case $\beta_2 = \beta_5$ was treated above. Therefore we have $\alpha_2 = \alpha_5 = 1$, $\alpha_1 = \alpha_6 = 0$ and $\beta_3 = \beta_4 < \beta_2 = \beta_1$ by Proposition 2 and our assumptions. Considering $b|s_5 - s_1$, we obtain $b|qp-1$. Similarly, by $a|s_2 - s_1$ we gain $a|p-1$. Thus $ab+1 = q^\beta < p^2q$, hence $\beta \leq 2$. If $\beta = 2$ then we have $b|qp - 1$ and $b|q^2 - 1 = s_1 - 1$, and we obtain $b|q - p$, i.e. $q^2 > b^2 > ab + 1 = q^2$, a contradiction. Therefore we have $\beta = 1$ leading to $q^{\beta_6} = cd + 1 < (ac + 1)(bd + 1) = p^2q^3 < q^5$, i.e $\beta_6 = 3, 4$. Note that $\beta_6 \leq 2$ would yield $s_5 > s_6$. If we suppose $\beta_6 = 3$ we obtain, by $d|s_6 - s_5$ that $d|q-p$ and hence $q^3 = cd+1 < q^2$ is a contradiction. Similarly, we obtain $d|q^2-p$ in the case $\beta_6 = 4$, hence $q^4 = cd + 1 < q^4$ is also impossible. Note that $\beta = 0$ yields $\beta_3 < 0$, which is again a contradiction.

5.4. The case when $\alpha_2 = \alpha_5$ is minimal. By Proposition 2 we have $\alpha_2 = \alpha_5 \leq 1$. Obviously, the relations $\beta_1 < \beta_2 < \beta_5$ hold since otherwise it would lead to $s_1 < s_2 < s_5$. Therefore we conclude $\beta_1 = \beta_6$, and by Proposition 1 $\beta_1 = \beta_6 = \beta_2 - 1$ or $\beta_1 = \beta_6 \leq 1$ follows. The case $\beta_1 = \beta_6 \leq 1$, together with Proposition 2 yields the cases II and III. On the other hand, $\beta_1 = \beta_6 = \beta_2 - 1$, together with the second case of Proposition 2 immediately yields a contradiction. The remaining case $\alpha_2 = \alpha_5 \leq 1$, $\beta_1 = \beta_6 = \beta_2 - 1$ and $\beta_3 = \beta_4 \leq 1$ provides $\beta_3 = \beta_4 < \beta_1 = \beta_6$. But this implies $\alpha_1 < \alpha_6 < \alpha_3$. Therefore we obtain, in view of equation (3) and Proposition 1 that $\alpha_1 = \alpha_4$. Consequently, $\beta_1 < \beta_4$, which contradicts $\beta_3 < \beta_1$.

5.5. The case when $\alpha_2 = \alpha_6$ is minimal. Because of $s_1 < s_2, s_6$ and $\alpha_1 \geq \alpha_2, \alpha_6$ we gain $\beta_1 < \beta_2, \beta_6$. Therefore we have $\beta_1 = \beta_5 \leq \beta_2 < \beta_6$ and $\alpha_2 = \alpha_6 < \alpha_1 < \alpha_5$. Now, by Proposition 1, $\alpha_2 = \alpha_6 = \alpha_1 - 1$ and $\beta_1 = \beta_5 = \beta_2 - 1$ follow. Note that $\beta_1 = \beta_5 = \beta_2$ would imply the contradiction $s_2 < s_1$. Since $\beta_2 \neq \beta_5$ we deduce $\alpha_2 = \alpha_5$ and therefore in the actual case $\alpha_5 = \alpha_6$ holds. But $s_5|s_6$ is a contradiction again.

6. The unit equation (3)

In this section we concentrate on the equation

$$p^{\alpha_1+\alpha_6}q^{\beta_1+\beta_6} - p^{\alpha_3+\alpha_4}q^{\beta_3+\beta_4} = p^{\alpha_1}q^{\beta_1} + p^{\alpha_6}q^{\beta_6} - p^{\alpha_3}q^{\beta_3} - p^{\alpha_4}q^{\beta_4}. \tag{15}$$

As earlier, we have to distinguish several cases.

6.1. The case when $\alpha_1 = \alpha_3$ is minimal. Obviously, we have $\beta_1 < \beta_3$, therefore either $\beta_1 = \beta_4$ or $\beta_1 = \beta_6$ or $\beta_4 = \beta_6$ holds. But, both the cases $\beta_1 = \beta_4$ and $\beta_4 = \beta_6$ give case I in Proposition 3 since otherwise $s_3|s_6$. But case I contradicts our assumption $\alpha_1 = \alpha_3$, since othewise $s_3|s_6$ again.

Therefore we may assume $\beta_1 = \beta_6$ and either case II or III holds. Since by assumption $\beta_4 \geq \beta_6$ we deduce that $\alpha_1 = \alpha_3 \leq \alpha_4 \leq \alpha_6$. Therefore Proposition 1 results in $\alpha_1 = \alpha_3 = \alpha_4 - 1$ or $\alpha_1 = \alpha_3 = \alpha_4$.

First suppose that case II holds. Then we have $\beta_1 = \beta_6 = 0$ and $\beta_3 = \beta_4 = 1$. Put $\alpha = \alpha_1 = \alpha_3$, $\alpha' = \alpha_2 = \alpha_5 \leq 1$ and $\alpha_4 = \alpha + h$ with $h \in \{0, 1\}$, and assume $h = 0$. Then, in the virtue of Lemma 4 there does no solution exist. Note that we may apply Lemma 4 only if $\beta_2 > 0$, but $\beta_2 = 0$ means $s_2 \leq p \leq s_1$. Similarly, we may also exclude the case $h = 1$ and $\alpha' = 1$. Hence we are reduced to the possibility $h = 1$ and $\alpha' = 0$. According to Lemma 5, we obtain

$$pa^2 = q^{\beta_2 - 1} - r$$

with $0 < r < 2p$. On the other hand, $a|s_3 - s_1$ implies $a|q - 1$ (Lemma 3), hence $pq^2 > pa^2 + 2p > q^{\beta_2-1}$. Since $\beta_2 > 1$ we deduce $\beta_2 = 2, 3$. Applying the second part of Lemma 5, after canceling common factors, we get

$$p^{\alpha+1} - 2pq^{\beta_2-2} < q^{\beta_2-1} < p^{\alpha+1}.$$

Note that $p^\delta q^\epsilon = p = \frac{s_4}{s_3} > \frac{s_4}{s_2}$. In case of $\beta_2 = 2$ we see from $c|s_4 - s_2$ that $c|p^{\alpha+1} - q$ (Lemma 3), and from the inequality above that $c \leq p^{\alpha+1} - q < 2p$. Therefore $p^{\alpha+1}q = bc + 1 < 4p^2$, subsequently $\alpha = 0$ and $\alpha_3 = \alpha_5$ and $s_3|s_5$.

Suppose now that $\beta_2 = 3$ and $p^{\alpha+1} - 2pq < q^2 < p^{\alpha+1}$. Evaluating

$$\begin{aligned} bd + 1 = \frac{(s_3 - 1)(s_4 - 1)}{s_2 - 1} &= \frac{(p^{\alpha+1}q - 1)(p^\alpha q - 1)}{q^3 - 1} + 1 = \frac{p^{2\alpha+1}}{q} + L\left(\frac{2p^{\alpha+1}}{q^2}\right) \\ &= \frac{p^{2\alpha+1}}{q} + L\left(2\frac{q^2}{q^2} + 4\frac{p}{q}\right) = \frac{p^{2\alpha+1}}{q} + L(6) < \frac{(q^2 + 2pq)^2}{pq} + 6 < q^3, \end{aligned}$$

it leads to a contradiction by $\beta_2 = \beta_5 < 3$.

Now let us consider case III. Here we write $\beta' = \beta_1 = \beta_6 \leq 1$, $\beta = \beta_3 = \beta_4 = \beta_2 - 1$, $\alpha = \alpha_1 = \alpha_3$, $\alpha_4 = \alpha + h$ with $h \in \{0, 1\}$ and $\alpha' = \alpha_2 = \alpha_5 \leq 1$. Unless $h = 1$ and $\alpha' = \beta' = 0$ we can apply Lemma 4. Since $p^\delta q^\epsilon = p = \frac{s_4}{s_3} < \frac{s_4}{s_2}$ we can use the second part of Lemma 5 in the remaining case, and we obtain

$$p^{\alpha+1} - \frac{2p}{q^{\beta-1}} < q < p^{\alpha+1}.$$

But it contradicts the assumption q is odd unless $\beta = \beta_3 = \beta_4 \leq 1$. But this case has been treated above.

6.2. The case when $\alpha_1 = \alpha_4$ is minimal. Observe, that only the cases II and III may hold under this assumption. By $\beta_1 < \beta_4$ we have $\beta_1 = \beta_3$ or $\beta_1 = \beta_6$. But the first equality is not possible in the cases II and III. Therefore we may assume $\beta_1 = \beta_6$. Since $\alpha_6 < \alpha_3$ would imply $s_3 > s_6$, we have $\alpha_1 = \alpha_4 \leq \alpha_3 < \alpha_6$, and now Proposition 1 yields $\alpha_1 = \alpha_4 = \alpha_3 - 1$. Note that $\alpha_1 = \alpha_3$ has already been investigated above.

In case II we write $\alpha = \alpha_1 = \alpha_4 = \alpha_3 - 1$ and $\alpha' = \alpha_2 = \alpha_5 \leq 1$ and we have $\beta_1 = \beta_6 = 0$ and $\beta_3 = \beta_4 = 1$. Therefore Lemma 4 settles this case.

Case III is analogous. Let $\alpha = \alpha_1 = \alpha_4 = \alpha_3 - 1$ and $\alpha' = \alpha_2 = \alpha_5 \leq 1$. Moreover, we have $\beta' = \beta_1 = \beta_6 \leq 1$ and $\beta = \beta_3 = \beta_4 = \beta_2 - 1$. We apply Lemma 4 again.

6.3. The case when $\alpha_1 = \alpha_6$ is minimal. Obviously, only case I may hold. Therefore we have $\alpha_1 = \alpha_6 = 0$, $\alpha_3 = \alpha_4 = 1$ and $\beta' = \beta_2 = \beta_5 \leq 1$. Moreover, $\beta_3 < \beta_1$ or $\beta_4 < \beta_1$ would yield $s_3 < s_1$ or $s_4 < s_1$, and we obtain either $\beta_1 = \beta_3$ or $\beta_1 = \beta_4$. In case of $\beta_1 = \beta_4$, the application of Lemma 4 gives a contradiction. Therefore Proposition 1 implies $\beta := \beta_1 = \beta_3 = \beta_4 - 1$. Considering now $d | s_6 - s_3$ and $c | s_6 - s_4$, we obtain (by Lemma 3) $d < q^{\beta_6 - \beta}$ and $c < q^{\beta_6 - \beta - 1}$. Thus $q^{2\beta_6 - 2\beta - 1} > cd + 1 > q^{\beta_6}$, i.e. $\beta_6 > 2\beta + 1$. On the other hand, $ad + 1 = pq^\beta$ and therefore $c, d < pq^\beta$ and $q^{\beta_6} = cd + 1 < p^2 q^{2\beta} < q^{2\beta+2}$ follow, which contradicts the bound for β_6 found before.

6.4. The case when $\alpha_3 = \alpha_4$ is minimal. From $\alpha = \alpha_3 = \alpha_4 \leq \alpha_1, \alpha_6$ we deduce that $\beta_1 < \beta_3, \beta_4$ hence $\beta' = \beta_1 = \beta_6 < \beta_3, \beta_4$. Note that only the cases II and III may hold, hence $\beta = \beta_3 = \beta_4$, $\beta' \leq 1$ and $\alpha' = \alpha_2 = \alpha_5 \leq 1$. We may exclude the case $\beta_2 < \beta_4$ since otherwise case II would be fulfilled, and $\beta_1 = \beta_6 = \beta_2 = 0$ and $\alpha_1 < \alpha_2 \leq 1$ would yield a contradiction by $ab + 1 = 1$. Therefore we suppose $\beta_4 \leq \beta_2$ and apply Lemma 4.

6.5. The case when $\alpha_4 = \alpha_6$ is minimal. Clearly, under this assumption only the cases II and III may hold. Thus $\alpha_4 = \alpha_6 \leq \alpha_1$, and we obtain $\beta_1 < \beta_4, \beta_6$, hence $\beta_1 = \beta_3$ in the virtue of Proposition 1. But, this contradicts $\beta_1 = \beta_6$, since we obtain $s_3 | s_6$.

7. Open problems and questions

Let $s(k)$ respectively $S(k)$ be the smallest m such that there is no respectively only finitely many sets of primes S with $|S| = k$, such that an S-Diophantine m-tuple exists. If no such k exists set $s(k) = \infty$ respectively $S(k) = \infty$. Therefore the first question arises:

Question 1. Are $s(k)$ and $S(k)$ for any positive integer k finite?

In case of $k = 1$ Corollary 2 tells us that $s(1) = S(1) = 3$. In case of $k = 2$ the statement of Conjecture 1 may be read as $s(2) \leq 4$ and $S(2) \leq 4$ respectively. By a simple computer search we obtain the following result:

Lemma 9. *Let $1 \leq a < b < c < d \leq 1000$, then there exist only three S-Diophantine quadruples with $|S| = 3$ and no quadruples with $|S| = 2$, i.e. $s(3) > 4$.*

Let $1 \leq a < b < c < d < e \leq 300$, then there exist 32 S-Diophantine quintuples with $|S| = 5$ and no quintuples with $|S| = 4$, i.e. $s(4) \geq 5$ and $s(5) > 5$.

In view of the lemma above we guess that $s(3) = 5$ and $S(3) = 4$.

Question 2. Are $s(2) = 4$, $S(4) = 2$, $s(3) = 5$ and $S(3) = 4$ correct? What are the values of $s(4), S(4)$ and more generally what are the values of $s(k)$ and $S(k)$ for $k \geq 5$?

Now let us consider m-tuples $(a_1, \ldots, a_m) \in \mathbb{Z}^m$ such that they are S-Diophantine with $|S|$ being small. In order to do so we choose a prime p and put $a_1 = 1$ and $a_i = p^{\alpha_i} - 1$ for $i = 2, \ldots, m$ and some fixed integers $\alpha_2 < \alpha_3 < \cdots < \alpha_m$ such that all the polynomials

$$P_{ij}(x) = x^{\alpha_i + \alpha_j} - x^{\alpha_i} - x^{\alpha_j} + 2$$

are irreducible. Note that we can always find such α's due to SCHINZEL [17][Theorem 5]. It is widely believed that irreducible polynomials take infinitely many prime values simultaneously (cf. [18][Hypotheses H] and [1]). Seeing the primes as a model of a randomly distributed sequence of density $1/\log n$ strongly suggests

that for infinitely many primes p all the quantities $a_i a_j + 1$ with $1 < i < j \leq m$ are primes again. In our context this means that there are infinitely many sets S with $|S| = \binom{m}{2} - m + 1 = \binom{m-1}{2}$ such that there is at least one S-Diophantine m-tuple. This motivates the next question:

Question 3. Is the statement

$$S(k) \geq \left\lfloor \frac{3 + \sqrt{8k+1}}{2} \right\rfloor$$

actually true? Are there better asymptotic estimates? What can be said about upper bounds?

References

[1] P. T. Bateman and R. A. Horn, Primes represented by irreducible polynomials in one variable, In: Proc. Sympos. Pure Math., Vol. VIII, *Amer. Math. Soc., Providence, R.I.*, 1965, 119–132.

[2] Y. Bugeaud, On the greatest prime factor of $(ab+1)(bc+1)(ca+1)$, *Acta Arith.* **86**(1) (1998), 45–49.

[3] Y. Bugeaud and A. Dujella, On a problem of Diophantus for higher powers, *Math. Proc. Cambridge Philos. Soc.* **135**(1) (2003), 1–10.

[4] Y. Bugeaud and F. Luca, A quantitative lower bound for the greatest prime factor of $(ab+1)(bc+1)(ca+1)$, *Acta Arith.* **114**(3) (2004), 275–294.

[5] H. Cohen, Number theory, Vol. I. Tools and Diophantine equations, volume 239 of Graduate Texts in Mathematics, *Springer, New York*, 2007.

[6] P. Corvaja and U. Zannier, On the greatest prime factor of $(ab+1)(ac+1)$, *Proc. Amer. Math. Soc.* **131**(6) (2003), 1705–1709, (electronic).

[7] A. Dujella, Diophantine m-tuples, available at `http://web.math.hr/~duje/dtuples.html`.

[8] A. Dujella, There are only finitely many Diophantine quintuples, *J. Reine Angew. Math.* **566** (2004), 83–214.

[9] A. Dujella and C. Fuchs, Complete solution of the polynomial version of a problem of Diophantus, *J. Number Theory* **106**(2) (2004), 326–344.

[10] P. Erdős and P. Turán, On a problem in the elementary theory of numbers, *Amer. Math. Monthly* **41**(10) (1934), 608–611.

[11] C. Fuchs, F. Luca and L. Szalay, Diophantine triples with values in binary recurrences, *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (5), **7**(4) (2008), 579–608.

[12] K. Győry, A. Sárközy and C. L. Stewart, On the number of prime factors of integers of the form $ab+1$, *Acta Arith.* **74**(4) (1996), 365–385.

[13] S. Hernández and F. Luca, On the largest prime factor of $(ab+1)(ac+1)(bc+1)$, *Bol. Soc. Mat. Mexicana* (3), **9**(2) (2003), 235–244.

[14] C. Heuberger, A. Pethő and R. F. Tichy, Thomas' family of Thue equations over imaginary quadratic fields, *J. Symbolic Comput.* **34**(5) (2002), 437–449.

[15] M. Laurent, M. Mignotte and Y. Nesterenko, Formes linéaires en deux logarithmes et déterminants d'interpolation, *J. Number Theory* **55**(2) (1995), 285–321.

[16] E. M. Matveev, An explicit lower bound for a homogeneous rational linear form in logarithms of algebraic numbers, II, *Izv. Ross. Akad. Nauk Ser. Mat.* **64**(6) (2000), 125–180.

[17] A. Schinzel, On the reducibility of polynomials and in particular of trinomials, *Acta Arith.* **11** (1965), 1–34.

[18] A. Schinzel and W. Sierpiński, Sur certaines hypothèses concernant les nombres premiers, *Acta Arith.* **4** (1958), 185–208, erratum, 5:259.

[19] W. M. Schmidt, Diophantine approximations and Diophantine equations, Vol. volume 1467, Lecture Notes in Mathematics, *Springer-Verlag, Berlin*, 1991.

[20] C. L. Stewart and R. Tijdeman, On the greatest prime factor of $(ab+1)(ac+1)(bc+1)$, *Acta Arith.* **79**(1) (1997), 93–101.

László Szalay
Institute of Mathematics
University of West Hungary
Ady E. utca 5.
H-9400 Sopron
Hungary

E-mail: laszalay@emk.nyme.hu

Volker Ziegler
Johann Radon Institute for Computational
and Applied Mathematics (RICAM)
Austrian Academy of Sciences
Altenbergerstrasse 69
A-4040 Linz
Austria

E-mail: volker.ziegler@ricam.oeaw.ac.at

(Received April 16, 2012; revised October 31, 2012)