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On the powers of integers and conductors of quadratic fields

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Abstract. We consider non-zero integers of the maximal order $\mathcal{O} = O_F$ of the quadratic field $F = \mathbb{Q}(\sqrt{d})$ where $d \in \mathbb{Z}$ is square-free. Let p be an odd prime and $0 \neq \alpha \in O_F$. Using the embedding into $\mathrm{GL}(2, \mathbb{R})$ we obtain bounds for the first $\nu \in \mathbb{N}$ such that $\alpha^\nu \equiv 1 \bmod p$. For a conductor f, we then study the smallest positive integer $n = n(f)$ such that $\alpha^n \in \mathcal{O}_f$. We obtain bounds for $n(f)$ and for $n(fp^k)$. The most interesting case is where α is the fundamental unit of $\mathbb{Q}(\sqrt{d})$.

1. Introduction

We consider quadratic fields $F = \mathbb{Q}(\sqrt{d})$ where $d \in \mathbb{Z}$ is square-free. We write $d = 4q + r$ with $r \in \{1, 2, 3\}$. The algebraic integers α of $\mathbb{Q}(\sqrt{d})$ are given by

$$\alpha = \begin{cases} a + b\sqrt{d},\ a, b \in \mathbb{Z} & \text{if } r = 2, 3 \\ \dfrac{1}{2}(a + b\sqrt{d}),\ a, b \in \mathbb{Z},\ a + b \in 2\mathbb{Z} & \text{if } r = 1. \end{cases} \tag{1.1}$$

Throughout the paper α denotes a non-zero integer of F. Let p be an odd prime. First we study the problem to find small exponents n such that $\alpha^n \equiv 1 \bmod p$. We will extensively use Legendre symbols.

We adapt the classical Chebyshev polynomials T_n and U_n (for detailed information see [9] Section 5.7, [1] Chapter 22) by defining

$$t_n(x) = t_n(x; s) = 2s^{n/2} T_n\left(\frac{x}{2\sqrt{s}}\right), \tag{1.2}$$

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$$u_n(x) = u_n(x;s) = s^{n/2} U_n\left(\frac{x}{2\sqrt{s}}\right) \tag{1.3}$$

for $n \in \mathbb{N}_0$ where s is the norm of a non-zero integer in the quadratic field F. These are unimodular polynomials with integer coefficients. For technical reasons we use this modification of Chebyshev polynomials for treating the cases $d \equiv 1 \bmod 4$ and $d \equiv 2, 3 \bmod 4$ simultaneously. In Section 6 we present all properties of these adapted polynomials which we use for proving our results. Then we specialize the results of the paper [2] about $\mathrm{GL}(2,\mathbb{Z})$ to quadratic fields. For previous works on this subject see e.g. [4], [5], [6].

In Section 2, we consider 2×2 matrices over the rational integers and show how the integers of any quadratic field $F = \mathbb{Q}(\sqrt{d}\,)$ can be embedded into $\mathrm{GL}(2,\mathbb{R})$. We also prove that $\alpha^n \equiv 1 \bmod p$ holds if and only if $A^n \equiv I \bmod p$ where the matrix A is the image of α. In the next sections we consider non-zero integers α of F and especially units α. In these sections we apply the results of [2] to the case of quadratic fields. Let f denote a conductor for F. In Section 5, we give upper estimates for

$$n(f) := \min\{\nu \in \mathbb{N} : \alpha^\nu \in \mathcal{O}_f\}$$

and also for $n(fp^k)$ where $k \in \mathbb{N}$ and p is an odd prime.

2. The embedding of algebraic integers of $\mathbb{Q}(\sqrt{d}\,)$ into $\mathrm{GL}(2,\mathbb{R})$

Let $A \in \mathrm{GL}(2,\mathbb{C})$, that is

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a,b,c,d \in \mathbb{C},\ ad - bc \neq 0. \tag{2.1}$$

We always write

$$x := \operatorname{tr} A = a + d, \quad s := \det A = ad - bc. \tag{2.2}$$

Proposition 2.1. *For $n \in \mathbb{N}$ we have*

$$A^n = u_{n-1}(x)A - s u_{n-2}(x) I, \tag{2.3}$$

$$A^n = \frac{1}{2} t_n(x) I + u_{n-1}(x)(A - \frac{1}{2} x I). \tag{2.4}$$

This proposition is known in various forms. For instance, (2.3) with $s = 1$ is Lemma 3.1.3 in [8] where $p_n = u_{n-1}$ and $q_n = u_{n-2}$. The last matrix in (2.4) has zero trace and it follows that

$$\operatorname{tr} A^n = t_n(x). \tag{2.5}$$

With the notation (2.1) we can write (2.4) as

$$A^n = \begin{pmatrix} \frac{1}{2}t_n(x) + \frac{1}{2}(a-d)u_{n-1}(x) & bu_{n-1}(x) \\ cu_{n-1}(x) & \frac{1}{2}t_n(x) - \frac{1}{2}(a-d)u_{n-1}(x) \end{pmatrix}. \tag{2.6}$$

Now, we consider algebraic integers α of $\mathbb{Q}(\sqrt{d}\,)$ in the notation (1.1). We define a homomorphism φ of the multiplicative semigroup of non-zero integers α into $\mathrm{GL}(2,\mathbb{R})$. For $r = 2, 3$ we set (see e.g. [3, p. 38])

$$\varphi(\alpha) := A = \begin{pmatrix} a & b \\ bd & a \end{pmatrix} \tag{2.7}$$

whereas for $r = 1$ we set

$$\varphi(\alpha) := A = \begin{pmatrix} \frac{1}{2}(a+b) & b \\ qb & \frac{1}{2}(a-b) \end{pmatrix}. \tag{2.8}$$

It can be checked that this indeed defines an injective homomorphism. We have

$$s = \det A = \mathrm{Norm}(\alpha) = \begin{cases} a^2 - b^2 d & \text{if } r = 2, 3 \\ \frac{1}{4}(a^2 - b^2 d) & \text{if } r = 1, \end{cases} \tag{2.9}$$

$$x = \mathrm{tr}\, A = \begin{cases} 2a & \text{if } r = 2, 3 \\ a & \text{if } r = 1. \end{cases} \tag{2.10}$$

Since $A^n = \varphi(\alpha^n)$ and φ is injective, it follows from (2.6) that

$$\alpha^n = \begin{cases} \frac{1}{2}t_n(2a) + u_{n-1}(2a)b\sqrt{d} & \text{if } r = 2, 3 \\ \frac{1}{2}t_n(a) + \frac{1}{2}u_{n-1}(a)b\sqrt{d} & \text{if } r = 1. \end{cases} \tag{2.11}$$

Proposition 2.2. *If p is an odd prime and α_k, α_m are integers of $\mathbb{Q}(\sqrt{d}\,)$ then $\alpha_k \equiv \alpha_m \bmod p$ if and only if $\varphi(\alpha_k) \equiv \varphi(\alpha_m) \bmod p$.*

Proof. We prove only the more complicated case $r = 1$ (see (1.1)). The statement can be proved in a similar way for $r = 2, 3$.

First we assume $\alpha_k \equiv \alpha_m \bmod p$ and we prove $\varphi(\alpha_k) \equiv \varphi(\alpha_m) \bmod p$. For $\alpha_k \equiv \alpha_m \bmod p$ with

$$\alpha_k = \frac{1}{2}\left(a_k + b_k\sqrt{d}\right), \quad \alpha_m = \frac{1}{2}\left(a_m + b_m\sqrt{d}\right).$$

we have $a_k \equiv a_m \bmod p$ and $b_k \equiv b_m \bmod p$. This implies $a_k+b_k \equiv a_m+b_m \bmod p$ and $a_k - b_k \equiv a_m - b_m \bmod p$. Since p is odd we obtain

$$\frac{1}{2}(a_k + b_k) \equiv \frac{1}{2}(a_m + b_m) \bmod p, \quad \frac{1}{2}(a_k - b_k) \equiv \frac{1}{2}(a_m - b_m) \bmod p.$$

Then (2.8) yields $\varphi(\alpha_k) \equiv \varphi(\alpha_m) \bmod p$.

Now we assume $\varphi(\alpha_k) \equiv \varphi(\alpha_m) \bmod p$ and prove $\alpha_k \equiv \alpha_m \bmod p$. Using the definition in (2.8) we can write

$$\varphi(\alpha_j) = \begin{pmatrix} \frac{1}{2}(a_j + b_j) & b_j \\ qb_j & \frac{1}{2}(a_j - b_j) \end{pmatrix}$$

for $j = k, m$. We immediately see that $b_k \equiv b_m \bmod p$, $\frac{1}{2}(a_k + b_k) \equiv \frac{1}{2}(a_m + b_m) \bmod p$ and $\frac{1}{2}(a_k - b_k) \equiv \frac{1}{2}(a_m - b_m) \bmod p$ and obtain $a_k \equiv a_m \bmod p$, hence $\alpha_k \equiv \alpha_m \bmod p$. □

Proposition 2.3. *If $p \nmid b$, $p \nmid d$ then $\alpha^n \equiv 1 \bmod p$ if and only if $A^n \equiv I \bmod p$.*

Proof. (a) First, we assume $\alpha^n \equiv 1 \bmod p$. For $r = 2, 3$,

$$\alpha^n = \frac{1}{2}t_n(x) + u_{n-1}(x)b\sqrt{d} \equiv 1 \bmod p$$

with $p \nmid b$, $p \nmid d$ and x was defined in (2.10). Since $u_{n-1}(x) \equiv 0 \bmod p$ by (2.11) we get $\frac{1}{2}t_n(x) \equiv 1 \bmod p$. Hence, $A^n = \frac{1}{2}t_n(x)I + u_{n-1}(x)(A - \frac{1}{2}xI) \equiv I \bmod p$. For $r = 1$, namely, $\alpha^n = \frac{1}{2}t_n(x) + \frac{1}{2}u_{n-1}(x)b\sqrt{d}$, the proof is similar.

(b We assume $A^n \equiv I \bmod p$. Then

$$A^n = \frac{1}{2}t_n(x)I + u_{n-1}(x)\left(A - \frac{1}{2}xI\right) \equiv I \bmod p$$

and we want to prove $\alpha^n = \frac{1}{2}t_n(x) + u_{n-1}(x)b\sqrt{d} \equiv 1 \bmod p$ for $r = 2, 3$. By (2.6) we have $bu_{n-1}(x) \equiv 0 \bmod p$. Because of $b \not\equiv 0 \bmod p$ we get $u_{n-1}(x)(A - \frac{1}{2}xI)v \equiv 0 \bmod p$ and $\mathrm{tr}(A - \frac{1}{2}xI) \equiv 0 \bmod p$, hence

$$u_{n-1}(x)\begin{pmatrix} * & b \\ bd & * \end{pmatrix} \equiv 0 \bmod p.$$

This implies $u_{n-1}(x)b \equiv 0 \bmod p$. From (2.6) we obtain $\frac{1}{2}t_n(x) \equiv 1 \bmod p$ for the cases $r = 2, 3$ and $r = 1$, hence $\alpha^n \equiv 1 \bmod p$. □

3. Non-zero integers α of F

In this section, we specialize the results of [2] to the case of quadratic fields using the embedding introduced in Section 2. We note that we allow d to be negative. Again we write $d = 4q + r$ and $s = \mathrm{Norm}(\alpha)$ for non-zero integers α of $F = \mathbb{Q}(\sqrt{d}\,)$ as in (1.1).

Let p be an odd prime. We assume that $p \nmid d$, $p \nmid b$ and that

$$a^2 - 4s \not\equiv 0 \bmod p \text{ for } r = 2, 3, \quad a^2 - s \not\equiv 0 \bmod p \text{ for } r = 1. \tag{3.1}$$

Throughout the rest of the paper let x be the trace and s be the norm of α as defined in (2.10) and (2.9). Since t_n and u_n are polynomials with integer coefficients the identities in Section 6 can be transferred into congruences. We let ℓ be the Legendre symbol

$$\ell := \left(\frac{x^2 - 4s}{p}\right). \tag{3.2}$$

Then $p - \ell$ becomes $= p \mp 1$ for $\ell = \pm 1$.

Theorem 3.1. *Let p be an odd prime with $p \nmid d$, $p \nmid b$ and $s = \mathrm{N}(\alpha) \neq 0$. Let ℓ be the Legendre symbol defined above. We set $\sigma = 1$ for $\ell = +1$ and $\sigma = s$ for $\ell = -1$. Then*

$$t_{p-\ell}(x) \equiv 2\sigma \bmod p, \quad u_{p-\ell-1}(x) \equiv 0 \bmod p.$$

We sum up the further results in the following table.

	$r = 2, 3$	$r = 1$
$\left(\frac{s}{p}\right) = +1$	$t_{\frac{p-\ell}{2}}(2a)^2 \equiv 4\sigma \bmod p$, $u_{\frac{p-\ell}{2}-1}(2a) \equiv 0 \bmod p$	$t_{\frac{p-\ell}{2}}(a)^2 \equiv 4\sigma \bmod p$, $u_{\frac{p-\ell}{2}-1}(a) \equiv 0 \bmod p$
$\left(\frac{s}{p}\right) = -1$	$t_{\frac{p-\ell}{2}}(2a) \equiv 0 \bmod p$, $(a^2 - s)u_{\frac{p-\ell}{2}-1}(2a)^2 \equiv \sigma \bmod p$	$t_{\frac{p-\ell}{2}}(a) \equiv 0 \bmod p$, $(a^2 - 4s)u_{\frac{p-\ell}{2}-1}(a)^2 \equiv 4\sigma \bmod p$.

This is [2, Theorem 4.1] specialized to our present situation.

The proof in [2] uses Chebyshev polynomials. In the present context of quadratic fields, many of the previous formulas can be proved by other methods, see for instance [3], [7, Theorem 1.7].

4. Units of F

First we consider the case $s = \mathrm{Norm}(\alpha) = +1$. Again we let ℓ be the Legendre symbol defined in (3.2), and x is defined in (2.10).

The following results are obtained by specializing the results in Sections 5 and 6 of [2]. The Legendre polynomials t_n and u_{n-1} depend only on x and s as defined in (2.9) and (2.10); the specific form (1.1) of α is not important.

Proposition 4.1. *Let $k \in \mathbb{N}$ divide $p-\ell$ and we assume that $\ell = \left(\frac{x^2-4s}{p}\right) \neq 0$. If $x \equiv t_k(y) \bmod p$ for some $y \in \mathbb{Z}$ then, with $n = \frac{p-\ell}{k}$,*

$$t_n(x) \equiv 2 \bmod p, \quad u_{n-1}(x) \equiv 0 \bmod p, \quad \alpha^n \equiv 1 \bmod p. \tag{4.1}$$

For a proof compare [2, Theorem 5.1].

For the special case that $k = 2^j$ we can say much more. We construct $x_0, \ldots, x_m$ recursively by the following rule. Let $x_0 = x$. For $\left(\frac{x+2}{p}\right) = -1$ we set $m = 0$ and stop. Now let $\left(\frac{x+2}{p}\right) = +1$ and suppose that $x_0, \ldots, x_k$ have already been constructed such that $2^k \mid (p-\ell)$ and

$$x_{\nu-1} \equiv t_2(x_\nu) \bmod p, \ ((x_\nu^2 - 4)/p) = \ell \quad \text{for } 1 \leq \nu \leq k. \tag{4.2}$$

For $2^{k+1} \nmid (p-\ell)$ or $\left(\frac{x_k+2}{p}\right) = -1$ we set $m = k$ and stop. Otherwise we have $2^{k+1} \mid (p-\ell)$ and $\left(\frac{x_k+2}{p}\right) = +1$. Then there exists x_{k+1} subject to $x_k + 2 \equiv x_{k+1}^2 \bmod p$ and thus $x_k = t_2(x_{k+1})$. It follows from (4.2) that

$$((x_k - 2)/p) = ((x_k + 2)/p)((x_k - 2)/p) = ((x_k^2 - 4)/p) = \ell$$

and therefore $((x_{k+1}^2 - 4)/p) = ((x_k - 2)/p) = \ell$. This completes our construction. We note that $2^m \mid (p-\ell)$.

Theorem 4.2. *Let $\mathrm{N}(\alpha) = 1$, $\ell = \left(\frac{x^2-4}{p}\right) \neq 0$ and $x_0, \ldots, x_m$ be constructed as above. Then*

$$t_{(p-\ell)/2^k}(x) \equiv 2 \bmod p \quad \text{for } k = 0, \ldots, m, \tag{4.3}$$

$$t_{(p-\ell)/2^{m+1}}(x) \equiv -2 \bmod p \quad \text{or } 2^{m+1} \nmid (p-\ell). \tag{4.4}$$

The proof is analogous to that of [2, Theorem 5.4].

Corollary 4.3. *Let $s = \mathrm{N}(\alpha) = 1$, $\ell = \left(\frac{x^2-4}{p}\right) \neq 0$ and let $x_0, \ldots, x_m$ be constructed as above. Setting $n = (p-\ell)/2^m$ we have*

$$u_{n-1}(x) \equiv 0 \bmod p, \quad \alpha^n \equiv 1 \bmod p. \tag{4.5}$$

For $2^{m+1} \mid (p-\ell)$ we additionally get

$$u_{\frac{n}{2}-1}(x) \equiv 0 \bmod p, \quad \alpha^{n/2} \equiv -1 \bmod p. \tag{4.6}$$

These bounds are best possible: $2^{m+2} \mid (p-\ell)$ implies $u_{\frac{n}{2}-1}(x) \not\equiv 0 \bmod p$.

PROOF. Because of $s = 1$ and $x^2 - 4 \not\equiv 0 \bmod p$ it follows from (6.1) and (4.3) that $u_{n-1} \equiv 0 \bmod p$ and therefore $A^n \equiv I \bmod p$ by (2.4). By Proposition 2.3 we have $\alpha^n \equiv 1 \bmod p$. This proves (4.5). For $2^{m+1} \mid (p - \ell)$ the congruences (4.6) follow from (4.4) analogously. Finally, we let $2^{m+2} \mid (p - \ell)$. Then it follows from (4.4) that $t_{n/2}(x) \equiv -2 \bmod p$ so that $t_{n/4}(x) \equiv 0 \bmod p$ by the recursion formula for $t_n(x)$ which is similar to that for $u_n(x)$ in Section 6. Hence, $u_{\frac{n}{4}-1}(x) \not\equiv 0 \bmod p$. □

Now we consider the more complicated case of units with norm -1, i.e.q $t_n(x) = t_n(x; -1)$. As before we set $\ell := \left(\frac{x^2-4s}{p}\right)$ and assume that (3.1) with $s = -1$ holds. We set $n = \frac{p-\ell}{2}$. Because of $(-1/p) = (-1)^{(p-1)/2}$ Theorem 3.1 (with $\sigma = \ell$) yields

$$t_{2n}(x) \equiv 2\ell \bmod p, \quad t_n(x)^2 \equiv 4\ell \bmod p, \quad u_{n-1}(x) \equiv 0 \bmod p \qquad \text{for } p \equiv 1 \bmod 4, \tag{4.7}$$

$$t_{2n}(x) \equiv 2\ell \bmod p, \quad t_n(x) \equiv 0 \bmod p, \quad u_{n-1}(x) \not\equiv 0 \bmod p \qquad \text{for } p \equiv 3 \bmod 4. \tag{4.8}$$

Then (6.3) implies that

$$t_{2(p-\ell)}(x) \equiv 2 \bmod p. \tag{4.9}$$

Hence, $t_n(x) \equiv \pm 2 \bmod p$ if and only if $p \equiv 1 \bmod 4$ and $\ell = +1$. Assuming the latter we obtain from (6.7) with $t_2(x; -1) = x^2 + 2$ that

$$t_{2n}(x; -1) = t_n(x^2 + 2; 1) \quad \text{for } n \in \mathbb{N}. \tag{4.10}$$

Because of $\left(\frac{-1}{p}\right) = +1$ there exists $j \in \mathbb{Z}$ with $j^2 \equiv -1 \bmod p$. We now assume that $x \not\equiv 0 \bmod p$ and $x \not\equiv \pm 2j \bmod p$. This implies

$$(x^2 + 2)^2 - 4 = x^2(x^2 + 4) \not\equiv 0 \bmod p. \tag{4.11}$$

Similar to Section 4, we construct numbers $y_0, \dots, y_m$ subject to the initial condition $y_0 = x^2 + 2$ instead of $x_0 = x$. It follows from (4.11) that also $((y_0^2 - 4)/p) = \ell$. We have $y_0 + 2 = x^2 + 4$ and therefore $((y_0 + 2)/p) = \ell = +1$. Hence, the first step of our construction can always be carried out resulting in $m \geq 1$. The construction stops if $((y_m + 2)/p) = -1$ or $2^{m+1} \nmid (p - 1)$.

Theorem 4.4. *Let* $\mathrm{N}(\alpha) = -1$, $p \equiv 1 \bmod 4$, $a^2 + 4 \not\equiv 0 \bmod p$, $\ell = +1$ *and let* $y_0, \dots, y_m$ *be constructed as above. Then* $m \geq 1$ *and*

$$t_{(p-1)/2^k}(x) \equiv 2 \bmod p \quad \textit{for} \ \ k = 0, \dots, m-1, \tag{4.12}$$

$$t_{(p-1)/2^m}(x) \equiv \begin{cases} -2 \bmod p & \textit{for} \;\; 2^{m+1} \mid (p-l), \\ 0 \bmod p & \textit{for} \;\; 2^{m+1} \nmid (p-\ell). \end{cases} \tag{4.13}$$

See [2, Theorem 6.1] for the proof. The next result is not a surprise because of $\mathrm{N}(\alpha^2) = 1$. The proof is similar to that of Corollary 4.3, so we omit it.

Corollary 4.5. *Under the assumptions of Theorem 4.4, we now write $n = (p-\ell)/2^{m-1}$. Then (4.5) holds, and in case $2^{m+1} \mid (p-\ell)$ then (4.6) is also fulfilled. These bounds are best possible: For $2^{m+1} \mid (p-\ell)$ we have $u_{\frac{n}{4}-1}(x) \not\equiv 0 \bmod p$.*

Theorem 4.6. *Let $\mathrm{N}(\alpha) = -1$ and k be odd with $k \mid (p-\ell)$. We put $n = (p-\ell)/k$. If $x^2+4 \not\equiv 0 \bmod p$ and $x \equiv t_k(y;-1) \bmod p$ for some $y \in \mathbb{Z}$ then*

$$t_{2n}(x) \equiv 2 \bmod p, \quad t_n(x) \equiv 2\ell \bmod p, \quad \alpha^n \equiv \ell \bmod p. \tag{4.14}$$

Proof. This was shown more generally in [2]. □

5. Estimates for conductors

We continue to study the quadratic field $F = \mathbb{Q}(\sqrt{d}\,)$ with $d > 0$ and $r \in \{1,2,3\}$. The order with conductor $f \in \mathbb{N}$ is

$$\mathcal{O}_f = \begin{cases} \{a' + b' f\sqrt{d} : a', b' \in \mathbb{Z}\} & \text{if } r = 2,3, \\ \left\{ \dfrac{1}{2}(a' + (f-1)b') + \frac{1}{2} b' f \sqrt{d} : a', b' \in \mathbb{Z},\ 2 \mid a' + b' \right\} & \text{if } r = 1. \end{cases} \tag{5.1}$$

We fix an integer α of $\mathbb{Q}(\sqrt{d}\,)$ with $s = \mathrm{N}(\alpha) \neq 0$. Let x be given by (2.10). Again we use the notation in (1.1). The most interesting case is that α is the fundamental unit of $\mathbb{Q}(\sqrt{d}\,)$. Following Halter–Koch we define

$$n(f) = n(f,\alpha) := \min\{\nu \in \mathbb{N} : \alpha^\nu \in \mathcal{O}_f\}. \tag{5.2}$$

Lemma 5.1. *Let $b \neq 0$ be given by (1.1) and s, x by (2.9). We write*

$$c := \gcd(b,f), b_0 := b/c, f_0 = f/c. \tag{5.3}$$

Then we have

$$n(f) = n(f_0) = \min\{\nu \in \mathbb{N} : u_{\nu-1}(x;s) \equiv 0 \bmod f_0\}. \tag{5.4}$$

Proof. By (2.9) and (2.10) we have

$$\alpha^\nu \in \mathcal{O}_f \Leftrightarrow bu_{\nu-1}(x) \equiv 0 \bmod f.$$

Since $\gcd(b_0, f_0) = 1$ it follows by (5.3) that

$$\alpha^\nu \in \mathcal{O}_f \Leftrightarrow b_0 u_{\nu-1}(x) \equiv 0 \bmod f_0 \Leftrightarrow u_{\nu-1}(x) \equiv 0 \bmod f_0.$$

We note that b has not been replaced by b_0. Therefore we still have $u_{\nu-1}(x) = u_{\nu-1}(x; s)$ with x and s unchanged. $\square$

Let $g \in \mathbb{N}$ and $\gcd(b, g) = \gcd(f, g) = 1$. Then it follows from (5.4) and (6.5) that $u_{n(f)n(g)-1}(x; s) \equiv 0 \bmod \operatorname{lcm}(f, g)f$. Hence, we get

$$n(fg) \leq n(f)n(g) \quad \text{for } \gcd(f, g) = 1. \tag{5.5}$$

For an odd prime p we define

$$q(p) = q(p; \alpha) := \min\{\nu \in \mathbb{N} : u_{\nu-1}(x; s) \equiv 0 \bmod p\}. \tag{5.6}$$

The results of Sections 3 and 4 provide upper estimates for $q(p)$. These results depend explicitly on x and s, and implicitly on a, b and d in (1.1).

First let $\ell = \left(\frac{x^2-4s}{p}\right) \neq 0$. For $s = 1$ it follows from Corollary 4.3 that

$$q(p) \leq \frac{p-\ell}{2^m}, \quad \text{and} \quad q(p) \leq \frac{p-\ell}{2^{m+1}} \quad \text{for } 2^{m+1} \mid (p-\ell).$$

If $s = -1$, $p \equiv 1 \bmod 4$ and $\ell = +1$ then it follows from Corollary 4.5 that

$$q(p) \leq \frac{p-\ell}{2^{m-1}} \quad \text{and} \quad q(p) \leq \frac{p-\ell}{2^m} \quad \text{for } 2^m \mid (p-\ell).$$

Now let $x^2 - 4s \equiv 0 \bmod p$. Then for all $\nu \in \mathbb{N}$ it follows from (6.1) that $2^{\nu-1}u_{\nu-1}(x; s) \equiv \nu x^{\nu-1} \bmod p$. We conclude that $q(p) = p$ for $p \nmid s$ and $q(p) = 2$ for $p \mid s$.

Theorem 5.2. *For $\gcd(f, b) = 1$ and $p \nmid f$ we have*

$$n(p^k f) \leq q(p)p^{k-1}n(f) \quad \text{for all } k \geq 1. \tag{5.7}$$

Proof. We use induction on k. By (5.4) and (6.5) we have $u_{q(p)n(f)-1}(x;s) \equiv 0 \bmod f$. By (5.6) and (6.5) this congruence also holds modulo p. Since $\gcd(f,p)=1$ it follows that the congruence is true also modulo pf. Hence (5.7) holds for $k=1$ in view of (5.4).

Now let (5.7) hold for k. We write $\nu = q(p)p^{k-1}n(f)$ and have, by (5.7),

$$u_{\nu-1}(x;s) \equiv 0 \bmod p^k f. \tag{5.8}$$

We apply (6.1) with $n=p$ and with s^ν instead of s. The binomial coefficients in the sum are divisible by the prime p. Because of $2^{p-1} \equiv 1 \bmod p$ we get for $z \in \mathbb{Z}$

$$u_{p-1}(z;s^\nu) \equiv (z^2-4s^\nu)^{(p-1)/2} \bmod p.$$

For $z = t_\nu(x;s)$ we obtain by (6.2) that

$$u_{p-1}(t_\nu(x;s);s^\nu) \equiv \left[(x^2-4s)u_{\nu-1}(x;s)\right]^{\frac{p-1}{2}} \equiv 0 \bmod p. \tag{5.9}$$

Here we used (5.8) for $k \geq 1$. Now we apply (6.4) with $m=p$ and $n=\nu$. By (5.8) and (5.9) we obtain

$$u_{q(p)p^k-1}(x;s) = u_{p\nu-1}(x;s) \equiv 0 \bmod p^{k+1}f.$$

Hence, it follows from (5.4) that $n(p^{k+1}f) \leq q(p)p^k$. $\square$

Theorem 5.3. *Let $f \in \mathbb{N}$ be odd and let f_0 be defined as in* (5.3). *We write*

$$f_0 = \prod_{\nu=1}^{\mu} {p_\nu}^{k_\nu} \quad (k_\nu \in \mathbb{N}) \tag{5.10}$$

with different primes p_ν. Then

$$n(f) \leq \prod_{\nu=1}^{\mu} \left(q(p_\nu){p_\nu}^{k_\nu-1}\right). \tag{5.11}$$

Proof. Let $g_0 = 1$ and for $1 \leq \lambda \leq \mu$

$$g_\lambda = \prod_{\nu=1}^{\lambda} {p_\nu}^{k_\nu} \quad (1 \leq \lambda \leq \mu).$$

Then $g_\lambda = p^{k_\lambda} g_{\lambda-1}$ and $p_\lambda \nmid g_{\lambda-1}$. Hence we obtain from Theorem 5.2 applied to f_0 that

$$n(f_\lambda) \leq q(p_\lambda)p^{k_\lambda-1}n(f_{\lambda-1}).$$

Hence, (5.11) with f replaced by f_0 follows by induction. Finally, we use that Lemma 5.1 implies $n(f) = n(f_0)$. $\square$

6. Addendum: useful formulas for Chebyshev polynomials

We present several formulas which we need in proving our results. We put our emphasis on the polynomials u_n defined in (1.3) (see [9, Section 5.7] and [2]). For odd n and $x, s \in \mathbb{C}$, we have

$$u_{n-1}(x;s) = \frac{1}{2^{n-1}} \sum_{k=0}^{(n-3)/2} \binom{n}{2k+1} x^{n-2k-1}(x^2-4s)^k + \frac{1}{2^{n-1}}(x^2-4s)^{\frac{n-1}{2}}. \quad (6.1)$$

The recursion formula $u_{n+1}(x) = xu_n(x) - su_{n-1}(x)$ shows that

$$u_0(x) = 1, \quad u_1(x) = x, \quad u_2(x) = x^2 - s, \quad u_3(x) = x^3 - 2sx,$$
$$u_4(x) = x^4 - 3sx^2 + s^2, \quad u_5(x) = x^5 - 4sx^3 + 2s^2x.$$

Furthermore, $t_n(x;s)$ and $u_n(x;s)$ are polynomials in $\mathbb{Z}[x,s]$. For $n \in \mathbb{N}$ we have

$$(x^2 - 4s)u_{n-1}(x;s)^2 = t_n(x;s)^2 - 4s^n \quad (6.2)$$

$$t_n(x;s)^2 = t_{2n}(x;s) + 2s^n. \quad (6.3)$$

We need a relation for products which involves different parameters.

$$u_{mn-1}(x;s) = u_{m-1}(t_n(x;s);s^n)\ u_{n-1}(x;s) \quad (m,n \in \mathbb{N}). \quad (6.4)$$

It follows that for $\mu \in \mathbb{N}$ and $x, s \in \mathbb{Z}$

$$u_{n-1}(x;s) \equiv 0 \bmod \mu \Rightarrow u_{mn-1}(x;s) \equiv 0 \bmod \mu. \quad (6.5)$$

To prove (6.4) it is sufficient to consider $\frac{x}{2\sqrt{s}} = \cos\theta$ with real θ. Then it follows from (1.2), (1.3) and the properties [9, p. 257] of the T_n and U_n that

$$t_n(x;s) = 2s^{\frac{n}{2}}\cos(n\theta), \quad u_{m-1}(x;s) = s^{\frac{m-1}{2}}\frac{\sin(m\theta)}{\sin\theta}. \quad (6.6)$$

By (1.3) and (1.2) we therefore have

$$\begin{aligned} u_{m-1}(t_n(x;s);s^n) &= s^{n\frac{m-1}{2}} U_{m-1}\left(\frac{1}{2s^{n/2}t_n(x;s)}\right) \\ &= s^{n\frac{m-1}{2}} U_{m-1}(\cos(n\theta)) = s^{\frac{mn-n}{2}}\frac{\sin(mn\theta)}{\sin n\theta}. \end{aligned}$$

Now we multiply by $u_{n-1}(x;s)$. Using (6.6) we obtain

$$u_{m-1}(t_n(x;s);s^n)u_{n-1}(x;s) = s^{\frac{mn-1}{2}}\frac{\sin(mn\theta)}{\sin n\theta} = u_{mn-1}(x;s)$$

using (6.6) again.

In Section 4 we use the following relation between the polynomials $t_n(x;s)$ with different parameters s. If $s \neq 0$ and $m,n \in \mathbb{N}$ then

$$t_{mn}(x;s) = t_n(t_m(x;s);s^m). \tag{6.7}$$

Indeed, (1.2) and the composition property $T_{mn} = T_n \circ T_m$ imply that

$$t_{mn}(x;s) = 2(s^m)^{n/2}T_n\left(T_m\left(\frac{x}{2\sqrt{s}}\right)\right) = t_n\left(\frac{1}{2\sqrt{s}^m}T_m\left(\frac{x}{2\sqrt{s}}\right);s^m\right)$$

from which (6.7) follows using (1.2).

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