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# Generalized near-derivations and their applications in Lie algebras

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**Abstract.** Let $L$ be a Lie algebra. The aim of this paper is to investigate generalized near-derivations of $L$, which is a generalization of near-derivation initiated by Brešar in 2008. As an application we determine all linear maps $f : L \to L$ with the property that $[\dots[[f,\delta_1],\delta_2],\dots,\delta_n]$ is a derivation whenever $\delta_1,\delta_2,\dots,\delta_n$ are derivations of $L$, where $n$ is a fixed positive integer.

## 1. Introduction

Let $n$ be a fixed positive integer. Let $R$ be a not necessarily associative algebra with multiplication $\cdot$ . Recall that a linear map $\delta : R \to R$ is said to be a derivation if $\delta(x\cdot y) = \delta(x)\cdot y + x\cdot\delta(y)$ for all $x,y \in R$. A linear map $f : R \to R$ is said to be a *generalized derivation* if there exist linear maps $g,h : R \to R$ such that

$$f(x)\cdot y = g(x\cdot y) + x\cdot h(y) \quad \text{for all } x,y \in R.$$

Leger and Luks [4] investigated generalized derivations of Lie algebras. As it is well known, $[\delta',\delta] = \delta'\delta - \delta\delta'$ is a derivation whenever $\delta$ and $\delta'$ are derivations. Is it possible to determine all linear maps $f : R \to R$ with the property that $[f,\delta]$ is a derivation whenever $\delta$ is a derivation? Brešar [2] discussed this question in Lie algebras. He answered this question by introducing an interesting concept of near-derivations in Lie algebras.

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Let $L$ be a Lie algebra over a field $\mathbb{F}$. By ad$x$ we denote the inner derivation induced by $x \in L$, i.e. $(\operatorname{ad} x)(y) = [x, y]$ for all $y \in L$. A linear map $f : L \to L$ is said to be a *near-derivation* of $L$ if there exists a linear map $g : L \to L$ such that

$$(\operatorname{ad} x)f - g(\operatorname{ad} x)$$

is a derivation for every $x \in L$ (see [2, Section 1]). Note that every generalized derivation is a near-derivation. Brešar gave a description of $f$ in certain Lie algebras that arise from associative ones. The typical result in [2] states that a near-derivation $f$ of $L$ is of the form $f = \delta + \gamma I + \tau$, where $\delta$ is a derivation, $\gamma$ is an element in the center $C$ of a certain associative algebra containing $L$, and $\tau$ is a linear map from $L$ into $C$.

It is natural for us to consider the following general question: How to determine all linear maps $f : L \to L$ with the property that $[\ldots[[f, \delta_1], \delta_2], \ldots, \delta_n]$ is a derivation whenever $\delta_1, \delta_2, \ldots, \delta_n$ are derivations of $L$?

We shall solve this problem on Lie algebras. For this purpose we shall extend the definition of near-derivations. Let $Q$ be a unital associative algebra, containing $L$ as its Lie subalgebra (the $Q$ always exists). First, we give a slight generalization of near-derivations as follows: A linear map $f : L \to L$ is said to be a *near-derivation* if there exists a linear map $g : L \to Q$ such that

$$(\operatorname{ad} x)f - g(\operatorname{ad} x)$$

is a derivation for every $x \in L$. It is easy to see that the typical result $\delta + \gamma I + \tau$ in [2] is a near derivation in the above sense. Now, we say that a linear map $f : L \to L$ is a *generalized near-derivation* if there exists a linear map $g : L \to Q$ such that

$$(\operatorname{ad} x)f - g(\operatorname{ad} x)$$

is a near-derivation for every $x \in L$. It is easy to see that every near-derivation is a generalized near-derivation.

We shall apply the powerful theory of functional identities [3] to the descriptions of generalized near-derivations in certain Lie algebras that arise from associative ones.

## 2. Functional identities preliminaries

Let $Q$ be a unital algebra and let $S$ be a $d$-free subset of $Q$ for some positive integer $d$. Denote by $P$ as the set of all quasi-polynomials. We refer the reader to the recent book [3] for the basic properties of $d$-free subsets and quasi-polynomials, upon which the present paper heavily rests.

The following result is a slight generalization of [2, Lemma 2].

**Lemma 2.1.** *Let $Q$ be a unital algebra and let $S$ be a subset of $Q$. Let $B : S \times S \to Q$ be a skew-symmetric map. Suppose that*

$$[B(x,y),z] + [B(z,x),y] + [B(y,z),x] \in P$$

*for all $x,y,z \in S$. If $S$ is a 4-free subset of $Q$, then there exist $\lambda \in C$, the center of $Q$, a linear map $\mu : S \to C$, and a skew-symmetric map $\nu : S^2 \to C$ such that*

$$B(x,y) = \lambda[x,y] + \mu(x)y - \mu(y)x + \nu(x,y) \quad \textit{for all } x,y \in S.$$

Proof. Using [3, Theorem 4.13] it follows that $B$ is a quasi-polynomial. This means that there exist $\lambda_1, \lambda_2 \in C$ and maps $\mu_1, \mu_2 : S \to C$, $\nu : S^2 \to C$ such that

$$B(x,y) = \lambda_1 xy + \lambda_2 yx + \mu_1(x)y + \mu_2(y)x + \nu(x,y).$$

Since $B(x,y) = -B(y,x)$ it follows that

$$(\lambda_1 + \lambda_2)(xy + yx) + (\mu_1 + \mu_2)(x)y + (\mu_1 + \mu_2)(y)x + \nu(x,y) + \mu(y,x) = 0.$$

But then $\lambda_1 = -\lambda_2$, $\mu_1 = -\mu_2$ and $\nu$ is skew-symmetric [3, Lemma 4.4]. Setting $\lambda = \lambda_1$ and $\mu = \mu_1$ we thus have

$$B(x,y) = \lambda[x,y] + \mu(x)y - \mu(y)x + \nu(x,y) \quad \text{for all } x,y \in S. \qquad \square$$

Let $A$ be a prime associative algebra. By $Q_{ml}(A)$ we denote the maximal left ring of quotients of $A$ (see [1, Chapter 2] or [3, Appendix A]). The center $C$ of $Q_{ml}(A)$ is a field called the extended centroid of $A$. By $\deg(x)$ we denote the degree of the algebraicity of $x \in A$ over $C$. If $x$ is not algebraic, then we write $\deg(x) = \infty$. Further, we set

$$\deg(A) = \sup\{\deg(x) \mid x \in A\}.$$

The condition that $\deg(A) = \infty$ is equivalent to the condition that $A$ is not a PI-algebra, while the condition that $\deg(A) = n < \infty$ is equivalent to the condition that $A$ is a PI-algebra satisfying the standard polynomial identity of degree $2n$, but not satisfying a polynomial identity of degree $< 2n$. If $A$ is a central simple algebra, then $\deg(A) = \infty$ is the same as saying that $A$ is infinite-dimensional over $\mathbb{F}$, while $\deg(A) = n < \infty$ is equivalent to $\dim_{\mathbb{F}} A = n^2$ [3, Appendix C]. If $\deg(A) \geq d+1$, then every noncommutative Lie ideal $L$ of $A$ is a $d$-free subset of $Q_{ml}(A)$ [3, Corollary 5.16]. If $\deg(A) \geq 2d+3$, $A$ has an involution and $K$ is the set of skew elements in $A$, then every noncentral Lie ideal $L$ of $K$ is a $d$-free subset of $Q_{ml}(A)$ [3, Corollary 5.19].

## 3. Generalized near-derivations on Lie algebras

We begin with the following crucial result.

**Lemma 3.1.** *Let $L$ be a Lie algebra with $\mathrm{char}(\mathbb{F}) \neq 2, 3$ and let $Q$ be a unital associative algebra, containing $L$ as its Lie subalgebra. Let $f$ be a generalized near-derivation of $L$. Suppose $L$ is a 4-free subset of $Q$. Then there exist $\gamma \in C$, the center of $Q$, and a skew-symmetric bilinear map $\beta : L^2 \to C$ such that*
$$(f + \gamma I)([x, y]) = [f(x), y] + [x, f(y)] + \beta(x, y) \quad \text{for all } x, y \in L.$$

Proof. Our assumption is that there exists $g : L \to Q$ such that $(\mathrm{ad}\, x)f - g(\mathrm{ad}\, x)$ is a near-derivation for every $x \in L$. In view of [2, Lemma 3.1] we have that there exists $\lambda_x \in C$ (depending on $x$) such that

$$((\mathrm{ad}\, x)f - g(\mathrm{ad}\, x) + \lambda_x I)([y, z]) - [((\mathrm{ad}\, x)f - g(\mathrm{ad}\, x))(y), z] - [y, ((\mathrm{ad}\, x)f - g(\mathrm{ad}\, x))(z)] \in C.$$

That is,

$$[x, f([y, z])] - g([x, [y, z]]) + \lambda_x [y, z] - [[x, f(y)] - g([x, y]), z] - [y, [x, f(z)] - g([x, z])] \in C \quad (1)$$

for all $x, y, z \in L$. Now define $\theta : L \to C$ by the rule

$$\theta(x) = \lambda_x \quad \text{for all } x \in L.$$

We claim that $\theta$ is well-defined. It is enough to show that $x = 0$ implies $\lambda_x = 0$. Picking $x = 0$ in (1) we obtain $\lambda_0 [y, z] \in C$ for all $y, z \in L$. Applying [3, Lemma 4.4] we get $\lambda_0 = 0$ as desired. Hence, the identity (1) can be rewritten as

$$[x, f([y, z])] - g([x, [y, z]]) + \theta(x)[y, z] - [[x, f(y)] - g([x, y]), z] - [y, [x, f(z)] - g([x, z])] \in C \quad (2)$$

for all $x, y, z \in L$. In view of the Jacobi identity we have

$$g([x, [y, z]]) + g([z, [x, y]]) + g([y, [z, x]]) = 0;$$

according to (2) this can be rewritten as

$$\begin{aligned}[x, f([y, z])] &- [[x, f(y)], z] + [g([x, y]), z] - [y, [x, f(z)]] + [y, g([x, z])] \\ &+ [z, f([x, y])] - [[z, f(x)], y] + [g([z, x]), y] - [x, [z, f(y)]] \\ &+ [x, g([z, y])] + [y, f([z, x])] - [[y, f(z)], x] + [g([y, z]), x] \\ &- [z, [y, f(x)]] + [z, g([y, x])] + \theta(x)[y, z] \\ &+ \theta(z)[x, y] + \theta(y)[z, x] \in C.\end{aligned}$$

Rearranging the terms we get

$$\begin{aligned}&[(2g-f)([y,z])-[f(y),z]-[y,f(z)],x]\\&\qquad+[(2g-f)([x,y])-[f(x),y]-[x,f(y)],z]\\&\qquad+[(2g-f)([z,x])-[f(z),x]-[z,f(x)],y]\\&\qquad+\theta(x)[y,z]+\theta(z)[x,y]+\theta(y)[z,x]\in C\end{aligned}\tag{3}$$

for all $x,y,z\in L$. That is,

$$\begin{aligned}&[(2g-f)([y,z])-[f(y),z]-[y,f(z)],x]\\&\qquad+[(2g-f)([x,y])-[f(x),y]-[x,f(y)],z]\\&\qquad+[(2g-f)([z,x])-[f(z),x]-[z,f(x)],y]\in P\end{aligned}$$

for all $x,y,z\in L$. We are now in a position to apply Lemma 2.1. Thus there exist $\lambda\in C$, $\mu_1:L\to C$, and a skew-symmetric map $\nu:L^2\to C$ such that

$$(2g-f)([x,y])-[f(x),y]-[x,f(y)]=\lambda[x,y]+\mu_1(x)y-\mu_1(y)x+\nu(x,y).\tag{4}$$

Substituting (4) into (3) we obtain

$$(2\mu_1(x)+\theta(x))[y,z]+(2\mu_1(z)+\theta(z))[x,y]+(2\mu_1(y)+\theta(y))[z,x]\in C.$$

Applying [3, Lemma 4.4] it follows that $2\mu_1(x)+\theta(x)=0$ for all $x\in L$ and hence $\mu_1=-\frac{1}{2}\theta$. Thus, the expression (4) becomes

$$(2g-f)([x,y])-[f(x),y]-[x,f(y)]=\lambda[x,y]-\frac{1}{2}\theta(x)y+\frac{1}{2}\theta(y)x+\nu(x,y).\tag{5}$$

Set $h=2g-f-\lambda I$. Then

$$h([x,y])-[f(x),y]-[x,f(y)]=-\frac{1}{2}\theta(x)y+\frac{1}{2}\theta(y)x+\nu(x,y).\tag{6}$$

Since

$$h([x,[y,z]])+h([z,[x,y]])+h([y,[z,x]])=0$$

for all $x,y,z\in L$, we get from (6) that

$$\begin{aligned}&[f(x),[y,z]]+[x,f([y,z])]+[f(z),[x,y]]+[z,f([x,y])]+[f(y),[z,x]]\\&\qquad+[y,f([z,x])]-\frac{1}{2}\theta(x)[y,z]+\frac{1}{2}\theta([y,z])x-\frac{1}{2}\theta(z)[x,y]\\&\qquad+\frac{1}{2}\theta([x,y])z-\frac{1}{2}\theta(y)[z,x]+\frac{1}{2}\theta([z,x])y\\&\qquad+\nu(x,[y,z])+\nu(z,[x,y])+\nu(y,[z,x])=0.\end{aligned}$$

We rewrite this as

$$
\begin{aligned}
&[f([y,z]) - [f(y),z] - [y,f(z)],x] + [f([x,y]) - [f(x),y] - [x,f(y)],z] \\
&\quad + [f([z,x]) - [f(z),x] - [z,f(x)],y] - \frac{1}{2}\theta(x)[y,z] + \frac{1}{2}\theta([y,z])x \\
&\quad - \frac{1}{2}\theta(z)[x,y] + \frac{1}{2}\theta([x,y])z - \frac{1}{2}\theta(y)[z,x] + \frac{1}{2}\theta([z,x])y \\
&\quad + \nu(x,[y,z]) + \nu(z,[x,y]) + \nu(y,[z,x]) = 0. \qquad (7)
\end{aligned}
$$

That is,

$$
\begin{aligned}
&[f([y,z]) - [f(y),z] - [y,f(z)],x] + [f([x,y]) - [f(x),y] - [x,f(y)],z] \\
&\qquad + [f([z,x]) - [f(z),x] - [z,f(x)],y] \in P.
\end{aligned}
$$

Applying Lemma 2.1 again we get

$$
f([x,y]) - [f(x),y] - [x,f(y)] = \alpha[x,y] + \mu_2(x)y - \mu_2(y)x + \beta(x,y) \qquad (8)
$$

for some $\alpha \in C$, $\mu_2 : L \to C$, and skew-symmetric map $\beta : L^2 \to C$. It is clear that the linearity of $f$ implies the bilinearity of $\beta$.

Substituting (8) into (7) we obtain

$$
\begin{aligned}
&(2\mu_2(x) - \frac{1}{2}\theta(x))[y,z] + (2\mu_2(z) - \frac{1}{2}\theta(z))[x,y] + (2\mu_2(y) - \frac{1}{2}\theta(y))[z,x] \\
&\qquad + \frac{1}{2}\theta([y,z])x + \frac{1}{2}\theta([x,y])z + \frac{1}{2}\theta([z,x])y \in C.
\end{aligned}
$$

Applying [3, Lemma 4.4] it follows that $2\mu_2(x) - \frac{1}{2}\theta(x) = 0$ for all $x \in L$ and hence $\mu_2 = \frac{1}{4}\theta$. Thus, the expression (8) becomes

$$
f([x,y]) - [f(x),y] - [x,f(y)] = \alpha[x,y] + \frac{1}{4}\theta(x)y - \frac{1}{4}\theta(y)x + \beta(x,y) \qquad (9)
$$

We claim that $\theta = 0$. Indeed, subtracting (9) from (5) we get

$$
2(g-f)([x,y]) = (\lambda - \alpha)[x,y] - \frac{3}{4}\theta(x)y + \frac{3}{4}\theta(y)x + \nu(x,y) - \beta(x,y).
$$

Set $k = 2g - 2f - (\lambda - \alpha)I$. Then

$$
k([x,y]) = -\frac{3}{4}\theta(x)y + \frac{3}{4}\theta(y)x + \nu(x,y) - \beta(x,y). \qquad (10)
$$

Since

$$k([x,[y,z]]) + k([z,[x,y]]) + k([y,[z,x]]) = 0$$

by the Jacobi identity, we get from (10) that

$$-\theta(x)[y,z] + \theta([y,z])x - \theta(z)[x,y] + \theta([x,y])z - \theta(y)[z,x] + \theta([z,x])y \in C.$$

Applying [3, Lemma 4.4] it follows that $\theta(x) = 0$ for all $x \in L$.

Setting $\gamma = -\alpha$ we can get from (9) that

$$(f + \gamma I)([x,y]) = [f(x),y] + [x,f(y)] + \beta(x,y) \quad \text{for all } x,y \in L.$$

This proves the lemma. □

We denote by $\mathrm{H}_2(L,\mathbb{F})$ the second cohomology group of $L$. Applying Lemma 3.1 we have the following:

**Theorem 3.1.** *Let $L$ be a Lie algebra with $\mathrm{char}(\mathbb{F}) \neq 2,3$ and $H_2(L,\mathbb{F}) = 0$. Let $Q$ be a unital associative algebra, containing $L$ as its Lie subalgebra. Let $f$ be a generalized near-derivation of $L$. Suppose $L$ is a 4-free subset of $Q$. Then there exist $\gamma \in C$, the center of $Q$, a derivation $\delta : L \to Q$ and a linear map $\tau : L \to C$ such that $f = \delta + \gamma I + \tau$.*

Proof. By Lemma 3.1 the map $d = f - \gamma I : L \to CL \subseteq Q$ satisfies

$$d([x,y]) - [d(x),y] - [x,d(y)] = \beta(x,y) \in C$$

for all $x,y \in L$. Consequently,

$$\beta(x,[y,z]) = d([x,[y,z]]) - [d(x),[y,z]] - [x,[d(y),z]] - [x,[y,d(z)]],$$

since $[x,\beta(y,z)] = 0$. Using the Jacobi identity it readily follows that

$$\beta(x,[y,z]) + \beta(z,[x,y]) + \beta(y,[z,x]) = 0.$$

Since $\mathrm{H}^2(L,\mathbb{F}) = 0$ then exists a linear map $\tau : L \to C$ such that $\beta(x,y) = \tau([x,y])$ for all $x,y \in L$ (see [2, P. 3769]). That is,

$$d([x,y]) - [d(x),y] - [x,d(y)] = \tau([x,y]).$$

It follows that $\delta = d - \tau$ is a derivation from $L$ into $Q$. □

Let $L$ be a Lie algebra over $\mathbb{F}$. We denote $\mathrm{End}(L)$ by the $\mathbb{F}$-linear space of all $\mathbb{F}$-linear transformations on $L$. Set

$$\mathrm{Cent}(L) = \{\chi \in \mathrm{End}(L) \mid \chi([x, y]) = [\chi(x), y] = [x, \chi(y)] \quad \text{for all } x, y \in L\}.$$

We call $\mathrm{Cent}(L)$ the *centroid* of $L$. Applying Lemma 3.1 we also have the following:

**Theorem 3.2.** *Let $L$ be a Lie algebra with $\mathrm{char}(\mathbb{F}) \neq 2, 3$. Let $Q$ be a unital associative algebra, containing $L$ as its Lie subalgebra. Assume further that $L$ has trivial center and $[L, L] = L$. Let $f$ be a generalized near-derivation of $L$. Suppose that $L$ is a 4-free subset of $Q$. Then there exist a derivation $\delta : L \to L$ and $\zeta \in \mathrm{Cent}(L)$, the centroid of $L$, such that $f = \delta + \zeta$.*

Proof. Lemma 3.1 implies that

$$\gamma[[x, y], z] = [[f(x), y], z] + [[x, f(y)], z] - [f([x, y]), z] \in L$$

for all $x, y, z \in L$. Since $[L, L] = L$, and hence also $[[L, L], L] = L$, it follows that $\gamma L \subseteq L$. That is, the map $\zeta : x \mapsto \gamma x$ maps $L$ into $L$ and so $\zeta \in \mathrm{Cent}(L)$. Further,

$$\beta(x, y) = f([x, y]) + \gamma[x, y] - [f(x), y] - [x, f(y)]$$

then lies in $L \cap C$ which is zero since $L$ has trivial center. But then $\delta = f - \zeta$ is a derivation of $L$. $\square$

Applying Lemma 3.1 and using the same arguments as in the corresponding results in [2, Section 3], we can obtain the following results. We omit their proofs for brevity.

**Corollary 3.1.** *Let $A$ be a central simple algebra with $\mathrm{char}(\mathbb{F}) \neq 2, 3$. Suppose that one of the following conditions is satisfied:*

(i) $\dim_{\mathbb{F}} A \geq 25$ *and* $1 \in L = [A, A]$;

(ii) *Suppose that $A$ has an involution of the first kind and $\dim_{\mathbb{F}} A \geq 121$. Let $K$ be the set of skew elements in $A$, and set $L = [K, K]$.*

*Then every generalized near-derivation $f$ of $L$ is of the form $f = \delta + \gamma I$ where $\delta$ is a derivation of $L$ and $\gamma \in \mathbb{F}$;*

**Corollary 3.2.** *Let $A$ be a prime algebra with $\mathrm{char}(\mathbb{F}) \neq 2, 3$, let $C$ be the extended centroid of $A$. Suppose that one of the following conditions is satisfied:*

(i) *Let $L$ be a noncommutative Lie ideal of $A$. Suppose that $\deg(A) \geq 5$;*

(ii) *Suppose that* $A$ *has an involution and* $\deg(A) \geq 11$. *Let* $K$ *be the skew elements of* $A$, *and let* $L$ *be a noncentral Lie ideal of* $K$.

*If* $f$ *is a generalized near-derivation of* $L$, *then there exist an (associative) derivation* $\delta : \langle L\rangle \to \langle L\rangle C + C$, $\gamma \in C$, *and a linear map* $\tau : L \to C$ *such that* $f = \delta + \gamma I + \tau$.

## 4. An application of generalized near-derivations

As an application of generalized near-derivations we shall answer the question posed in Section 1. More precisely, we have the following result.

**Theorem 4.1.** *Let* $L$ *be a Lie algebra with* $\mathrm{char}(\mathbb{F}) \neq 2, 3$ *and* $H_2(L, \mathbb{F}) = 0$. *Let* $f$ *be a linear maps of* $L$ *with the property that*

$$[\dots[[f, \delta_1], \delta_2], \dots, \delta_n]$$

*is a derivation whenever* $\delta_1, \delta_2, \dots, \delta_n$ *are derivations of* $L$. *Suppose there exists a unital associative algebra* $Q$, *containing* $L$ *as its Lie subalgebra, such that* $L$ *is a* 4*-free subset of* $Q$. *Then there exist* $\gamma \in C$, *the center of* $Q$, *a derivation* $\delta : L \to Q$ *and a linear map* $\tau : L \to C$ *such that* $f = \delta + \gamma I + \tau$.

Proof. For every $x_1, x_2, \dots, x_n \in L$ we get from our assumption that

$$[\dots[[f, \mathrm{ad}\, x_1], \mathrm{ad}\, x_2], \dots, \mathrm{ad}\, x_n]$$

is a derivation and so a near derivation. That is

$$[\dots[[f, \mathrm{ad}\, x_1], \mathrm{ad}\, x_2], \dots, \mathrm{ad}\, x_{n-1}]$$

is a generalized near-derivation. By Theorem 3.1 we get that

$$[\dots[[f, \mathrm{ad}\, x_1], \mathrm{ad}\, x_2], \dots, \mathrm{ad}\, x_{n-1}]$$

is also a near derivation. Following the same process we finally obtain that $f$ is a near-derivation. Then the result follows from Theorem 3.1. $\square$

Similarly, applying the corresponding results in the above section we can obtain the following results. We omit their proofs for brevity.

**Theorem 4.2.** *Let* $L$ *be a Lie algebra with* $\mathrm{char}(\mathbb{F}) \neq 2, 3$ *such that* $L$ *has trivial center and* $[L, L] = L$. *Let* $f$ *be a linear maps of* $L$ *with the property that*

$$[\dots[[f, \delta_1], \delta_2], \dots, \delta_n]$$

*is a derivation whenever* $\delta_1, \delta_2, \dots, \delta_n$ *are derivations of* $L$. *Suppose that there* *exists a unital associative algebra $Q$ , containing $L$ as its Lie subalgebra, such that $L$ is a 4-free subset of $Q$. Then there exist a derivation $\delta : L \to L$ and $\zeta \in \mathrm{Cent}(L)$, the centroid of $L$, such that $f = \delta + \zeta$.*

**Corollary 4.1.** *Let $A$ be a central simple algebra with $\mathrm{char}(\mathbb{F}) \neq 2, 3$. Suppose that one of the following conditions is satisfied:*

(i) $\dim_{\mathbb{F}} A \geq 25$ *and* $1 \in L = [A, A]$*;*

(ii) *Suppose that $A$ has an involution of the first kind and $\dim_{\mathbb{F}} A \geq 121$. Let $K$ be the set of skew elements in $A$, and $L = [K, K]$.*

*If $f$ is a linear map of $L$ with the property that*

$$[\dots[[f, \delta_1], \delta_2], \dots, \delta_n]$$

*is a derivation whenever $\delta_1, \delta_2, \dots, \delta_n$ are derivations of $L$, then $f = \delta + \gamma I$ where $\delta$ is a derivation of $L$ and $\gamma \in \mathbb{F}$.*

**Corollary 4.2.** *Let $A$ be a prime algebra with $\mathrm{char}(\mathbb{F}) \neq 2, 3$, let $C$ be the extended centroid of $A$. Suppose that one of the following conditions is satisfied:*

(i) *Suppose that $L$ is a noncommutative Lie ideal of $A$ and $\deg(A) \geq 5$;*

(ii) *Suppose that $A$ has an involution with $\deg(A) \geq 11$, $K$ is the skew elements of $A$ and let $L$ be a noncentral Lie ideal of $K$.*

*If $f$ is a linear map of $L$ with the property that*

$$[\dots[[f, \delta_1], \delta_2], \dots, \delta_n]$$

*is a derivation whenever $\delta_1, \delta_2, \dots, \delta_n$ are derivations of $L$, then there exist an (associative) derivation $\delta : \langle L\rangle \to \langle L\rangle C + C$, $\gamma \in C$, and a linear map $\tau : L \to C$ such that $f = \delta + \gamma I + \tau$.*

Let us show that there exists a generalized near-derivation that is not a near-derivation.

Let $L$ be the usual Lie algebra of all $4 \times 4$ strict upper triangular matrices over $\mathbb{F}$, i.e.,

$$L = \Big\{ \sum_{1\leq i<j\leq 4} a_{ij}e_{ij} \mid a_{ij} \in \mathbb{F} \Big\}.$$

Let $\varphi$ be a linear functional on $L$ such that $\varphi(e_{14}) = 1$. Now define $f : L \to L$ by $f(y) = \varphi(y)e_{24}$. We claim that $f$ is not a near-derivation of $L$. Indeed, for every linear map $g : L \to L$ we have

$$((\mathrm{ad}\, e_{12})f - g(\mathrm{ad}\, e_{12}))([e_{13}, e_{34}]) = e_{14}.$$

On the other hand, we have

$$[((\operatorname{ad} e_{12})f - g(\operatorname{ad} e_{12}))(e_{13}), e_{34}] + [e_{13}, ((\operatorname{ad} e_{12}) - g(\operatorname{ad} e_{12}))(e_{34})] = 0.$$

This implies that $(\operatorname{ad} e_{12})f - g(\operatorname{ad} e_{12})$ is not a derivation of $L$. However, since $(\operatorname{ad} x)(\operatorname{ad} y)f = 0$ for all $x, y \in L$, we get that $(\operatorname{ad} y)f$ is a near-derivation for every $y \in L$ and so $f$ is a generalized near-derivation.

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