

On several classes of additively non-regular C -semirings

By YIZHI CHEN (Huizhou), XIANZHONG ZHAO (Nanchang)
and XIAOJIANG GUO (Nanchang)

Abstract. In this paper, the authors study several classes of additively non-regular C -semirings whose additive idempotents are central, including the generalized C -rpp semirings, C -rpp semirings, generalized C -abundant semirings and C -abundant semirings. After introducing the concept of generalized C -rpp semirings, the authors obtain their equivalent characterizations, and show that a semiring is a generalized C -rpp semiring if and only if it is a strong b-lattice of additively left cancellative halfrings, and if and only if it is a subdirect product of a b-lattice and an additively left cancellative halfring. Also, the authors give the constructions of C -rpp semirings, generalized C -abundant semirings and C -abundant semirings. Consequently, the corresponding results of Clifford semirings and generalized Clifford semirings in [7] and [29] are extended and generalized.

1. Introduction

A semiring is an algebra $(R, +, \cdot)$ with two binary operations $+$ and $\cdot$ such that both $(R, +)$ and $(R, \cdot)$ are semigroups and such that the distributive laws

$$x(y + z) \approx xy + xz \quad \text{and} \quad (x + y)z \approx xz + yz$$

are satisfied.

The additive identity (if it exists) of a semiring R is called zero and denoted by 0. An additively commutative semiring R with a zero satisfying $0x = x0 = 0$

Mathematics Subject Classification: 20M10, 16Y60.

Key words and phrases: generalized C -rpp semirings, C -rpp semirings, generalized C -abundant semirings, C -abundant semirings, Clifford semirings, generalized Clifford semirings.

This paper is supported by National Natural Science Foundation of China (11226287, 113260446); Natural Science Foundation of Guangdong Province (S2012040007195, S2013040016970); Outstanding Young Innovative Talent Training Project in Guangdong Universities.

for all $x \in R$, is called a hemiring. A halfring is a hemiring whose additive reduct $(R, +)$ is a cancellative monoid, i.e., for any $a, b, c \in R$, $a + b = a + c$ or $b + a = c + a$ implies $b = c$. A skew-ring $(R, +, \cdot)$ [29] is a semiring whose additive reduct $(R, +)$ is a group, not necessarily an abelian group. An additively cancellative skew-halfring (additively left cancellative skew-halfring, respectively) is a semiring whose additive reduct is an additively cancellative monoid (left cancellative monoid, respectively), not necessarily to be additively commutative. Also, a semiring $(R, +, \cdot)$ is said to be a b-lattice [29] if its additive reduct $(R, +)$ is a semilattice and its multiplicative reduct $(R, \cdot)$ is a band.

The algebraic theory of semirings have some important applications in automation theory, optimization theory and models of discrete event networks etc. There are a series of papers in the literature considering semirings (for example, see [2], [7]–[10], [16]–[17], [20]–[21], [23]–[32]).

Since semirings are generalizations of distributive lattices, b-lattices, rings, skew-rings, skew-halfrings and left skew-halfrings, it is interesting to use those semirings to establish the constructions of some semirings. In [2], BANDELT and PETRICH introduced Bandelt–Petrich Construction in semirings and described the semirings with regular addition which is a subdirect products of a distributive lattice and a ring. In [7], GHOSH established the constructions of strong distributive lattice of semirings which include the Bandelt–Petrich Construction, and characterized all semirings which are subdirect products of a distributive lattice and a ring. In particular, the authors introduced Clifford semirings, and showed that a semiring is a Clifford semiring if and only if it is a strong distributive lattice of rings, and if and only if it is an inverse subdirect product of a distributive lattice and a ring. Later, Sen, MAITY and SHUM in [29] defined the Clifford semiring which is a completely regular and an additively inverse semiring such that the set of its additive idempotents is a distributive sublattice as well as a k-ideal (without assuming that its additive reduct is commutative) and verified that a semiring is a Clifford semiring if and only if it is a strong distributive lattice of skew-rings. Meanwhile, they introduced generalized Clifford semirings which are completely regular and inverse semirings such that its additive idempotent set is a k-ideal, and obtained that a semiring is a generalized Clifford semiring if and only if it is a strong b-lattice of skew-rings, and if and only if it is an additively inverse semiring and is a subdirect product of a b-lattice and a skew-ring. It is not hard to see that all the semirings studied in [2], [7] and [29] are additively regular.

On the other hand, as we know, in order to generalize regular semigroups, new Green's relations, namely, the Green's $*$ -relations on a semigroup have been

introduced as follows (see [3], [19], or [22]):

$$\mathcal{L}^* = \{(a, b) \in S \times S : (\forall x, y \in S^1) ax = ay \Leftrightarrow bx = by\},$$

$$\mathcal{R}^* = \{(a, b) \in S \times S : (\forall x, y \in S^1) xa = ya \Leftrightarrow xb = yb\},$$

$$\mathcal{H}^* = \mathcal{L}^* \cap \mathcal{R}^*,$$

$$\mathcal{D}^* = \mathcal{L}^* \vee \mathcal{R}^*.$$

It is clear that $\mathcal{L} \subseteq \mathcal{L}^*$, $\mathcal{R} \subseteq \mathcal{R}^*$, $\mathcal{H} \subseteq \mathcal{H}^*$, $\mathcal{D} \subseteq \mathcal{D}^*$. A semigroup S is abundant [3] if its each $\mathcal{L}^*$ -class and each $\mathcal{R}^*$ -class contains an idempotent, a semigroup S is an rpp semigroup (a lpp semigroup, respectively) if its each $\mathcal{L}^*$ -class ($\mathcal{R}^*$ -class, respectively) contains an idempotent (see [5]). A semigroup S is a C -rpp semigroup ([5]) if its every $\mathcal{L}^*$ -class contains an idempotent and $E(S)$ is central. Dually, we will get the definition of C -lpp semigroups. A semigroup S is said to be a C -abundant semigroup if it is abundant and $E(S)$ is central, i.e., it is both a C -lpp semigroup and a C -rpp semigroup. In general, abundant semigroups, C -rpp semigroups, C -lpp semigroups and C -abundant semigroups are not regular, so we will call them non-regular semigroups in the following. There are also a series of papers in the literature considering non-regular semigroups (for example, see [1], [3]–[5], [11]–[15], [18] etc.).

In this paper, we will study several classes of additively non-regular C -semirings whose additive idempotents are central, including the generalized C -rpp semirings, C -rpp semirings, generalized C -abundant semirings and C -abundant semirings. Our purpose is to extend the results of Clifford semirings and generalized Clifford semirings in [29] and the semirings which are subdirect products of a distributive lattice and a ring in [7] to the non-regular C -semirings. We will show that a semiring is a generalized C -rpp semiring (C -rpp semiring, generalized C -abundant semiring, C -abundant semiring, respectively) if and only if it is a strong b-lattice (strong distributive lattice, strong b-lattice, strong distributive lattice, respectively) of additively left cancellative (left cancellative, cancellative, cancellative, respectively) halfrings, and if and only if it is a subdirect product of a b-lattice (distributive lattice, b-lattice, distributive lattice, respectively) and an additively left cancellative (left cancellative, cancellative, cancellative, respectively) halfring.

For notations and terminologies not mentioned in this paper, the readers are referred to [3], [8] or [29].

2. Generalized C -rpp semirings and C -rpp semirings

In this section, we will study the classes of generalized C -rpp semirings and C -rpp semirings, and show that a semiring is a generalized C -rpp semiring (C -rpp semiring, respectively) if and only if it is a strong b-lattice (strong distributive lattice, respectively) of additively left cancellative halfrings, and if and only if it is a subdirect product of a b-lattice (distributive lattice, respectively) and an additively left cancellative halfring. Also, we will give some other characterizations of such semirings.

Let $(R, +, \cdot)$ be a semiring. We denote the Green's relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$ on additive reduct $(R, +)$ by $\overset{+}{\mathcal{L}}$, $\overset{+}{\mathcal{R}}$, $\overset{+}{\mathcal{H}}$, respectively. These are also equivalence relations on semiring $(R, +, \cdot)$. Now, we introduce Green's $*$ -relations $\overset{+}{\mathcal{L}}^*$, $\overset{+}{\mathcal{R}}^*$, $\overset{+}{\mathcal{H}}^*$ on semiring R which are given by

$$\overset{+}{\mathcal{L}}^* = \{(a, b) \in R \times R : (\forall x, y \in R^1) a + x = a + y \Leftrightarrow b + x = b + y\},$$

$$\overset{+}{\mathcal{R}}^* = \{(a, b) \in R \times R : (\forall x, y \in R^1) x + a = y + a \Leftrightarrow x + b = y + b\},$$

$$\overset{+}{\mathcal{H}}^* = \overset{+}{\mathcal{L}}^* \cap \overset{+}{\mathcal{R}}^*.$$

It is clear that $\overset{+}{\mathcal{L}} \subseteq \overset{+}{\mathcal{L}}^*$, $\overset{+}{\mathcal{R}} \subseteq \overset{+}{\mathcal{R}}^*$, $\overset{+}{\mathcal{H}} \subseteq \overset{+}{\mathcal{H}}^*$ on $(R, +, \cdot)$. In particular, if R is an additively regular semiring, $\overset{+}{\mathcal{L}} = \overset{+}{\mathcal{L}}^*$, $\overset{+}{\mathcal{R}} = \overset{+}{\mathcal{R}}^*$, $\overset{+}{\mathcal{H}} = \overset{+}{\mathcal{H}}^*$ [4]. In general, Green's equivalence relations $\overset{+}{\mathcal{L}}^*$, $\overset{+}{\mathcal{R}}^*$ and $\overset{+}{\mathcal{H}}^*$ are not congruences on $(R, +, \cdot)$.

For a semiring R , we denote by $E^+(R)$ the set of all additive idempotents of R . For any $e, f \in E^+(R)$, we write $e \leq_+ f$ if $e + f = f = f + e$. Remark that $\leq_+$ is a partial order which is compatible with the multiplication.

In the following, we will introduce the concepts of strong b-lattice and strong distributive lattice of semirings.

Definition 1 (Definition 2.3 in [29]). Let T be a b-lattice and $\{R_\alpha : \alpha \in T\}$ be a family of pairwise disjoint semirings which are indexed by the elements of T . For each $\alpha \leq \beta$ in T , we now embed R_α in R_β via a semiring monomorphism $\phi_{\alpha, \beta}$ satisfying the following conditions:

$$(1.1) \quad \phi_{\alpha, \alpha} = I_{R_\alpha}, \text{ the identity mapping on } R_\alpha;$$

$$(1.2) \quad \phi_{\alpha, \beta} \phi_{\beta, \gamma} = \phi_{\alpha, \gamma} \text{ if } \alpha \leq \beta \leq \gamma;$$

$$(1.3) \quad R_\alpha \phi_{\alpha, \beta} R_\beta \phi_{\beta, \gamma} \subseteq R_{\alpha\beta} \phi_{\alpha\beta, \gamma} \text{ if } \alpha + \beta \leq \gamma, \text{ i.e., } \alpha + \beta + \alpha\beta \leq \gamma.$$

On $R = \cup_{\alpha \in Y} R_\alpha$, we define addition $+$ and multiplication $\cdot$ for $a \in R_\alpha$, $b \in R_\beta$, as follows:

$$(1.4) \quad a + b = a\phi_{\alpha, \alpha+\beta} + b\phi_{\beta, \alpha+\beta}$$

and

$$a \cdot b = c \in R_{\alpha\beta}$$

such that

$$c\phi_{\alpha\beta, \alpha+\beta} = a\phi_{\alpha, \alpha+\beta} \cdot b\phi_{\beta, \alpha+\beta}.$$

Same as the notation of strong semilattice of semigroups, we denote the above system by $R = \langle T, R_\alpha, \phi_{\alpha, \beta} \rangle$ and call it the strong b-lattice T of the semirings $R_\alpha, \alpha \in T$.

In an obvious way, we may replace b-lattice T in the above definition by distributive lattice D , $R = \langle D, R_\alpha, \phi_{\alpha, \beta} \rangle$ and call it strong distributive lattice D of the semirings $R_\alpha, \alpha \in D$.

Lemma 1 (Theorem 2.4 in [29]). *The system $R = \langle T, R_\alpha, \phi_{\alpha, \beta} \rangle$ defined above is a semiring.*

Lemma 2 ([5]). *A semigroup S is a C -rpp semigroup if and only if it is a strong semilattice of left cancellative monoids.*

By Lemma 2, we can dually obtain that a semigroup S is a C -lpp semigroup if and only if it is a strong semilattice of right cancellative monoids.

From [5], it is also known that a semigroup $(S, +)$ is called a [right, left, respectively]adequate semigroup if its idempotents commute and every $\mathcal{L}^*$ -class and $\mathcal{R}^*$ -class [$\mathcal{L}^*$ -class, $\mathcal{R}^*$ -class, respectively] contain a unique idempotent. For an element a of such a semigroup, the unique idempotent in the $\mathcal{L}^*$ -class [$\mathcal{R}^*$ -class, respectively] containing a is denoted by $a^*[a^+]$. A [right, left, respectively] adequate semigroup S is called [right, left, respectively] type A if $[e + a = a + (e + a)^*, a + e = (a + e)^+ + a]$ $e + a = a + (e + a)^*$ and $a + e = (a + e)^+ + a$ for $a \in S$ and $e \in E^+(S)$.

By the definition of C -rpp semigroups, it is not hard to see that a right type A semigroup is a C -rpp semigroup.

Lemma 3 (Corollary 2.8 in [4]). *Let $(S, +)$ be a right type A semigroup with semilattice of idempotents $E = E(S)$ and μ_L the largest congruence contained in $\mathcal{L}^*$. Then the following conditions are equivalent:*

- (1) $S/\mu_L \cong E$;

- (2) $\mu_L = \mathcal{L}^*$;
- (3) E is central in S ;
- (4) S is a strong semilattice of left cancellative monoids.

Lemma 4 (Proposition 2.9 in [4]). *Let $(S, +)$ be a adequate semigroup with semilattice of idempotents $E = E(S)$ and μ the largest congruence contained in $\mathcal{H}^*$. Then the following conditions are equivalent:*

- (1) $S/\mu \cong E$;
- (2) $\mu = \mathcal{H}^*$;
- (3) E is central in S ;
- (4) S is a strong semilattice of cancellative monoids.

From Lemma 2, it is known that a semigroup S is a C -rpp semigroup if and only if it is a strong semilattice of left cancellative monoids. Now, we will similarly give the definition of generalized C -rpp semirings and then investigate some of their equivalent characterizations and constructions.

Definition 2. A semiring R is said to be a generalized C -rpp semiring if it is a strong b-lattice of additively left cancellative skew-halfring.

In the following, for any $a \in R$, the unique idempotent in the $\mathcal{H}^+$ -class containing a is denoted by a^0 .

Theorem 1. *Assume that R is a generalized C -rpp semiring. Then the following conditions hold:*

- (GCR1) $(R, +)$ is a C -rpp semigroup;
- (GCR2) $E^+(R)$ is a b-lattice;
- (GCR3) for any $a, b \in S$, $(ab)^0 + a^0b^0 = a^0b^0$;
- (GCR4) if $a^0 = b^0$ and $a + e = b + e$ for $a, b \in R$ and some $e \in E^+(R)$, then $a = b$.

PROOF. Assume that R is a generalized C -rpp semiring, then it is a strong b-lattice of additively left cancellative skew-halfrings, say $R = \langle T, R_\alpha, \phi_{\alpha, \beta} \rangle$, where R_α are the additively left cancellative skew-halfrings in which the zero of additive reduct is denoted by 0_α and T is a b-lattice.

i) Since R is a strong b-lattice of left additively cancellative skew-halfrings R_α , $(R, +)$ is a strong semilattice of left cancellative monoids $(R_\alpha, +)$, by Lemma 2, $(R, +)$ is a C -rpp semigroup, and condition (GCR1) holds.

ii) Notice that $E^+(R) = \{0_\alpha \mid \alpha \in T\} \cong T$, where T is a b-lattice, then $(E^+(R), +, \cdot)$ is also a b-lattice, and condition (GCR2) holds.

iii) We will show that $a^0 = 0_\alpha$ for any $a \in R_\alpha$ at first. In fact, notice that $(R, +)$ is a strong semilattice of left cancellative monoids $(R_\alpha, +)$, by Lemma 3, we obtain that $a \mathcal{L}^* = R_\alpha^+$ for any $a \in R_\alpha$. And then, $a^0 = 0_\alpha$. Thus, for any $a \in R_\alpha$, $b \in R_\beta$, we have $(ab)^0 = 0_{\alpha\beta} = 0_\alpha 0_\beta = a^0 b^0$, $(ab)^0 + a^0 b^0 = a^0 b^0$. The condition (GCR3) holds.

iv) Assume that $a^0 = b^0$ and $a + e = b + e$ for $a, b \in R$ and some $e \in E^+(R)$, then there exist $\alpha, \beta \in T$, s.t., $a, b \in R_\alpha$ and $e \in E(R_\beta)$. Now let $f = a^0 + e$. Then

$$a + f = b + f,$$

i.e.,

$$a\varphi_{\alpha, \alpha+\beta} + f\varphi_{\alpha+\beta, \alpha+\beta} = b\varphi_{\alpha, \alpha+\beta} + f\varphi_{\alpha+\beta, \alpha+\beta}.$$

Since $\varphi_{\alpha+\beta, \alpha+\beta}$ is a monomorphism, we have

$$f\varphi_{\alpha+\beta, \alpha+\beta} = f,$$

where f is the additive identity of $S_{\alpha+\beta}$. And then, we have

$$a\varphi_{\alpha, \alpha+\beta} = b\varphi_{\alpha, \alpha+\beta}.$$

Also, notice that $\varphi_{\alpha, \alpha+\beta}$ is a monomorphism, we immediately get $a = b$. The condition (GCR4) holds. $\square$

Actually, the converse of the above theorem also holds. To show this, we need the following proposition.

Proposition 1. *If R satisfies the conditions (GCR1)–(GCR3), then the following conclusions hold:*

- (1) $\mathcal{L}^*$ is a semiring congruence;
- (2) $R/\mathcal{L}^* \cong E^+(R)$.

PROOF. (1) Assume that R satisfies the conditions (GCR1)–(GCR3), we will show that $\mathcal{L}^*$ is a semiring congruence.

Firstly, since (GCR1) holds, by Lemma 3, $(R, +)$ is a strong semilattice Y of left cancellative monoids $(R_\alpha, +)$, where $Y \cong (E^+(R), +)$. By Lemma 3 again, $(R, +)/\mathcal{L}^* \cong (E^+(R), +)$, we obtain that $\mathcal{L}^*$ is a semilattice congruence on $(R, +)$. To show that $\mathcal{L}^*$ is a semiring congruence, we only need to prove that $\mathcal{L}^*$ is a multiplicative congruence on $(R, \cdot)$, i.e., for any $a, b \in R$,

$$(ab)^0 = a^0 b = ab^0 = a^0 b^0.$$

In fact, for any $a, b \in R$, since $a^0b, ab^0, a^0b^0 \in E^+(R)$, by condition (GCR2), we have

$$(ab)^0 = [(a + a^0)(b + b^0)]^0 = (ab + a^0b + ab^0 + a^0b^0)^0 = (ab)^0 + a^0b + ab^0 + a^0b^0.$$

And then

$$(ab)^0 + a^0b^0 = (ab)^0.$$

Together with (GCR3), we have

$$(ab)^0 = a^0b^0.$$

Notice that $(ab)^0 + a^0b = (ab)^0$ and $(ab)^0 + ab^0 = (ab)^0$ also hold, we immediately get

$$(ab)^0 = a^0b = ab^0 = a^0b^0.$$

(2) Define a mapping

$$\phi : S/\mathcal{L}^* \rightarrow E^+(S), \quad a\mathcal{L}^* \mapsto a^0.$$

It is a routine way to check that ϕ is bijective, and

$$\begin{aligned} (a\mathcal{L}^* + b\mathcal{L}^*)\phi &= [(a + b)\mathcal{L}^*]\phi = (a + b)^0 = a^0 + b^0, \\ [(a\mathcal{L}^*)(b\mathcal{L}^*)]\phi &= [(ab)\mathcal{L}^*]\phi = (ab)^0 = a^0b^0. \end{aligned}$$

Thus, $S/\mathcal{L}^* \cong E^+(S)$. □

Now, we have the following theorem.

Theorem 2. *A semiring R is a generalized C -rpp semiring if and only if it satisfies the conditions (GCR1)–(GCR4).*

PROOF. We only need to show the sufficiency. By Proposition 1, it is known that if S satisfies the conditions (GCR1)–(GCR4), then $\mathcal{L}^*$ is a semiring congruence on $(R, +, \cdot)$, and $R/\mathcal{L}^* \cong E^+(R)$ is a b-lattice. Also, for any $a \in R$, notice that $(L_a^*, +)$ is an additively left cancellative monoid, we obtain that R is a b-lattice of additively left cancellative skew-halfings.

For any $e, f \in E^+(R)$ with $e \leq_+ f$, define a mapping

$$\phi_{e,f} : L_e^* \rightarrow L_f^*, \quad a \mapsto a + f.$$

In the following, we will show that $R = \langle E^+(R), L_e^*, \phi_{e,f} \rangle$ is a strong b-lattice of the additively left cancellative skew-halfings $L_e^*, e \in E^+(R)$. That is, we will show that $\phi_{e,f}$ satisfies the conditions of strong b-lattice.

For any $a, b \in L_e^*$,

$$(a + b)\phi_{e,f} = a + b + f = a + (f + b + f) = (a + f) + (b + f) = a\phi_{e,f} + b\phi_{e,f}.$$

Also, since $af \in L_{ef}^* \cap E^+(R)$, we have $af = ef$. And then,

$$\begin{aligned} (ab)\phi_{e,f} &= ab + f = ab + e + f = ab + (e + f)^2 = ab + e + ef + fe + f \\ &= ab + af + fb + f = (a + f)(b + f) = a\phi_{e,f}b\phi_{e,f}. \end{aligned}$$

Hence, $\phi_{e,f}$ is a semiring morphism.

For any $a, b \in L_e^*$, if $a\phi_{e,f} = b\phi_{e,f}$, we have $a + f = b + f$. Notice that $(L_e^*, +)$ is an additively left cancellative monoid, we will get $a^0 = b^0 = e$. It follows from (GCR4) that $a = b$. Thus, $\phi_{e,f}$ is a semiring monomorphism.

Moreover, we can check that the monomorphism $\phi_{e,f}$ satisfies the conditions (1.1)–(1.4) of Definition 1.

(i) $\phi_{e,e}$ is clearly an identity morphism.

(ii) For any $e, f, g \in E^+(R)$ with $e \leq_+ f \leq_+ g$, we have

$$a\phi_{e,f}\phi_{f,g} = a + f + g = a + g = a\phi_{e,g}.$$

Hence, $\phi_{e,f}\phi_{f,g} = \phi_{e,g}$.

(iii) For any $e, f, g \in E^+(R)$, if $e + f \leq_+ g$, then for any $a \in L_e^*$, $b \in L_f^*$,

$$\begin{aligned} a\phi_{e,g}b\phi_{f,g} &= (a + g)(b + g) = ab + ag + gb + g = ab + eg + gf + g \\ &= ab + g = (ab)\phi_{ef,g}. \end{aligned}$$

(iv) For any $e, f \in E^+(R)$, $a \in L_e^*$, $b \in L_f^*$, we have

$$\begin{aligned} a\phi_{e,e+f} + b\phi_{f,e+f} &= (a + e + f) + (b + e + f) = a + f + b + e \\ &= a + b + e + f = a + b; \\ a\phi_{e,e+f}b\phi_{f,e+f} &= (a + e + f)(b + e + f) = ab + a(e + f) + (e + f)b + (e + f) \\ &= ab + (e + f) = (ab)\phi_{ef,e+f} \end{aligned}$$

Thus, we have shown that R is a strong b-lattice of additively left cancellative skew-halfrings. And then it is a generalized C -rpp semiring. $\square$

Example 1. Let $(A, +)$ and $(B, +)$ be the infinite cyclic monoids generated by a and b respectively. Let $M = A \cup B \cup \{0\}$ with additive identity 0 and addition $+$ defined by

$$ma + nb = (m + n)b, nb + ma = (n + m)a$$

for any $m, n \in N^+$. Also, we define the multiplication $\cdot$ of M as follows: $s_1 \cdot s_2 = 0$ for any $s_1, s_2 \in M$, it is a routine way to check that $(M, +, \cdot)$ is an additively left cancellative skew-halfring.

On the other hand, let $D = \{e, f\}$ such that $e + e = e \cdot e = e, f + f = e + f = f + e = f \cdot f = e \cdot f = f \cdot e = f$. Then $(D, +, \cdot)$ is a b-lattice.

Now, construct the direct product of D and M , and denote it by R , i.e., $R = D \times M$. Then, we can check that $E^+(R) = D \times \{e_M\}$, where e_M is the identity element of $(M, +)$. It is also not hard to check that $(R, +, \cdot)$ is a semiring which satisfies the following conditions:

- (i) $(R, +)$ is a C -rpp semigroup;
- (ii) $(E^+(R), +, \cdot)$ is a b-lattice;
- (iii) for any $a, b \in R$, $(ab)^0 + a^0b^0 = a^0b^0$;
- (iv) if $a^0 = b^0$ and $a + e = b + e$ for $a, b \in R$ and some $e \in E^+(R)$, then $a = b$.

By Theorem 2, $(R, +, \cdot)$ is just a generalized C -rpp semiring.

Next, we will give another construction of generalized C -rpp semirings. Recall that a subdirect product algebra T is a subalgebra of a direct product of algebras such that the projection mapping from the algebra T to each of its components is surjective.

Theorem 3. *A semiring R is a generalized C -rpp semiring if and only if it is a subdirect product of a b-lattice and an additively left cancellative skew-halfring.*

PROOF. ($\Leftarrow$) Suppose that R is a subdirect product of a b-lattice T and an additively left cancellative skew-halfring M . Consider $R \subseteq T \times M$. For each $\alpha \in T$, let $R_\alpha = (\{\alpha\} \times M) \cap R$. Then R_α is an additively left cancellative skew-halfring for each $\alpha \in T$ and $R = \cup_{\alpha \in T} R_\alpha$. Now for each pair $\alpha, \beta \in T$ with $\alpha \leq_+ \beta$, define a mapping

$$\phi_{\alpha, \beta} : R_\alpha \rightarrow R_\beta, (\alpha, r)\phi_{\alpha, \beta} = (\beta, r).$$

Then $\phi_{\alpha, \beta}$ is clearly a monomorphism satisfying the conditions $\phi_{\alpha, \alpha} = I_{R_\alpha}$ and $\phi_{\alpha, \beta}\phi_{\beta, \gamma} = \phi_{\alpha, \gamma}$ if $\alpha \leq_+ \beta \leq_+ \gamma$ for $\alpha, \beta, \gamma \in T$.

Let $\alpha, \beta, \gamma \in T$ be such that $\alpha + \beta \leq_+ \gamma$. Denote $a = (\alpha, r) \in R_\alpha$, $b = (\beta, r') \in R_\beta$. And then

$$a + b = (\alpha, r) + (\beta, r') = (\alpha + \beta, r + r') \in R_{\alpha + \beta}$$

and

$$ab = (\alpha, r)(\beta, r') = (\alpha\beta, rr') \in R_{\alpha\beta}.$$

Now, we have

$$(a\phi_{\alpha,\gamma})(b\phi_{\beta,\gamma}) = (\alpha\beta, rr')\phi_{\alpha\beta,\gamma} = (ab)\phi_{\alpha\beta,\gamma}.$$

Also, since

$$a+b = (\alpha, r) + (\beta, r') = (\alpha+\beta, r+r') = (\alpha+\beta, r) + (\alpha+\beta, r') = a\phi_{\alpha,\alpha+\beta} + b\phi_{\beta,\alpha+\beta}$$

and

$$\begin{aligned} (a\phi_{\alpha,\alpha+\beta})(b\phi_{\beta,\alpha+\beta}) &= (\alpha+\beta, r)(\alpha+\beta, r') = (\alpha+\beta, rr') \\ &= (\alpha\beta, rr')\phi_{\alpha\beta,\alpha+\beta} = (ab)\phi_{\alpha\beta,\alpha+\beta}, \end{aligned}$$

R is a strong b-lattice of additively left cancellative shew-halftrings. Hence, R is a generalized C -rpp semiring.

($\Rightarrow$) Assume that R is a generalized C -rpp semiring. We will show that it is a subdirect product of a b-lattice and an additively left cancellative shew-halftring by the following steps.

Firstly, from Proposition 1, $\mathcal{L}^*$ is a semilattice congruence on $(R, +)$ and a semiring congruence on R . Also, since R is a generalized C -rpp semiring, we have $aa^0 = a$ for any $a \in R$, and then $a = aa^0\mathcal{L}^*a^2$. Hence, $R/\mathcal{L}^*$ is an idempotent semiring with the semilattice additive reduct and band multiplicative reduct, i.e., $R/\mathcal{L}^*$ is a b-lattice.

Secondly, define a binary relation

$$\theta = \{(a, b) \mid (\exists e \in E^+(R)) a + e = b + e\}.$$

It can be easily seen that θ is an equivalence relation on R . Moreover, θ is the minimum additively left cancellative shew-halftring congruence on $(R, +, \cdot)$. In fact, by the Proposition 1.7 in [18], θ is a minimum left cancellative monoid congruence on the additive reduct $(R, +)$. Also, if $a\theta b$ for some $a, b \in R$, there exists $e \in E^+(R)$ such that $a + e = b + e$. Now, for any $c \in R$, we have

$$ac + ec = bc + ec, ca + ce = cb + ce.$$

Notice that $ce, ec \in E^+(R)$. We immediately obtain that

$$ac\theta bc, ca\theta cb.$$

Thus, we have shown that θ is the minimum additively left cancellative shew-halftring congruence on $(R, +, \cdot)$. This also shows that R/θ is an additively left cancellative shew-halftring.

Finally, define a mapping

$$\Phi : R \rightarrow R/\theta \times R/\mathcal{L}^*, \quad a \mapsto (a\theta, a\mathcal{L}^*).$$

It is a routine way to check that R can be embedded into $R/\theta \times R/\mathcal{L}^*$, and the projection mapping from R into each of its components is surjective. Consequently, R is a subdirect product of a b-lattice and an additively left cancellative shew-halftring. $\square$

So we have obtained some constructions and characterizations of generalized C -rpp semirings. In the following, we will investigate another class of additive non-regular C -semirings, called C -rpp semirings.

Definition 3. A semiring R is said to be a C -rpp semiring if it is a strong distributive lattice of additively left cancellative shew-halftrings.

Theorem 4. Assume that R is a C -rpp semiring. Then the following conditions hold:

- (CR1) $(R, +)$ is a C -rpp semigroup;
- (CR2) $(E^+(R), +, \cdot)$ is a distributive lattice;
- (CR3) for any $a, b \in R$, $(ab)^0 + a^0b^0 = a^0b^0$;
- (CR4) if $a^0 = b^0$ and $a + e = b + e$ for $a, b \in R$ and some $e \in E^+(R)$, then $a = b$.

Conversely, if a semiring R satisfies the conditions (CR1)–(CR4), then it is a C -rpp semiring.

PROOF. ($\Rightarrow$) From Definition 2 and Definition 3, it is known that a C -rpp semiring is a generalized C -rpp semiring. Thus, by Theorem 2, (CR1), (CR3), (CR4) hold. We only need to prove that (CR2) holds.

Assume that S is a C -rpp semiring. Then it is a strong distributive lattice of additively left cancellative skew-halftrings, say $R = \langle D, R_\alpha, \phi_{\alpha, \beta} \rangle$, where each R_α is an additively left cancellative skew-halftring in which the additive identity is denoted by 0_α and T is a distributive lattice. Notice that $E^+(R) = \{0_\alpha \mid \alpha \in T\} \cong T$, we immediately obtain that $(E^+(R), +, \cdot)$ is also a distributive lattice. Hence, (GC2) holds.

($\Leftarrow$) Assume that the semiring R satisfies the conditions (CR1)–(CR4), then by Theorem 2, it is clearly a generalized C -rpp semiring. Also, note that (CR2) holds. By analogy with the discussions of Theorem 1, R is a C -rpp semiring. $\square$

Example 2. Let $(M = A \cup B \cup \{0\}, +, \cdot)$ be an additively left cancellative skew-halftring as defined in Example 1. Let $D = \{e, f\}$ be such that $e + e = e \cdot e =$

$e + f = f + e = e, f + f = f \cdot f = e \cdot f = f \cdot e = f$. Then $(D, +, \cdot)$ is a distributive lattice.

Now, Construct the direct product of D and M and denote it by R , i.e., $R = D \times M$. Clearly, $E^+(R) = D \times \{e_M\}$, where e_M is the identity element of $(M, +)$. It is also not hard to check that $(R, +, \cdot)$ is a semiring which satisfies the following conditions:

- (i) $(R, +)$ is a C -rpp semigroup;
- (ii) $(E^+(R), +, \cdot)$ is a distributive lattice;
- (iii) for any $a, b \in R$, $(ab)^0 + a^0b^0 = a^0b^0$;
- (iv) if $a^0 = b^0$ and $a + e = b + e$ for $a, b \in R$ and some $e \in E^+(R)$, then $a = b$.

Thus, by Theorem 4, $(R, +, \cdot)$ is a C -rpp semiring.

Further, by analogy with the discussions of the subdirect decompositions of generalized C -rpp semirings, we have the following theorem.

Theorem 5. *A semiring R is a C -rpp semiring if and only if it is a subdirect product of a distributive lattice and an additively left cancellative skew-halfring.*

3. Generalized C -abundant semirings and C -abundant semirings

In this section, we will study generalized C -abundant semirings and C -abundant semirings, and will show that a semiring is a generalized C -abundant semiring (C -abundant semiring, respectively) if and only if it is a strong b -lattice (strong distributive lattice, respectively) of additively cancellative skew-halfrings, and if and only if it is a subdirect product of a b -lattice (distributive lattice, respectively) and an additively cancellative skew-halfring. Also, we will give some characterizations of such semirings.

Firstly, by Lemma 2 and its dual, we immediately have

Lemma 5. *A semigroup S is a C -a(or C -abundant) semigroup if and only if it is a strong semilattice of cancellative monoids.*

Definition 4. A semiring R is said to be a generalized C -abundant semiring if it is a strong b -lattice of additively cancellative skew-halfrings.

Theorem 6. *Assume that R is a generalized C -abundant semiring, then the following conditions hold:*

- (GCA1) $(R, +)$ is a C -abundant semigroup;
- (GCA2) $(E^+(R), +, \cdot)$ is a b -lattice;

(GCA3) for any $a, b \in R$, $(ab)^0 + a^0b^0 = a^0b^0$;

(GCA4) if $a^0 = b^0$ and $a + e = b + e$ for $a, b \in S$ and some $e \in E^+(R)$, then $a = b$.

PROOF. From Definition 2 and Definition 4, it is known that, a generalized C -abundant semiring is a generalized C -rpp semiring, then the conditions (GCA2)–(GCA4) hold. We only need to show that condition (GCA1) holds.

Actually, if R is a generalized C -abundant semiring, then it is a strong b-lattice of additively cancellative skew-halfrings, say $R = \langle T, R_\alpha, \phi_{\alpha,\beta} \rangle$, where each R_α is an additively cancellative skew-halfring and T is a b-lattice. It follows that $(R, +)$ is strong semilattice of cancellative monoids $(R_\alpha, +)$, i.e., $(R, +)$ is a C -abundant semigroup. Thus, the condition (GCA1) holds. $\square$

Proposition 2. Assume that a semiring R satisfies the conditions (GCA1)–(GCA3). Then the following conclusions hold:

- (1) $\mathcal{H}^*$ is a semiring congruence;
- (2) $R/\mathcal{H}^* \cong E^+(R)$.

PROOF. (1) Assume that semiring R satisfies the conditions (GCA1)–(GCA3). We will show that $\mathcal{H}^*$ is a semiring congruence.

Firstly, since condition (GCA1) holds, by Lemma 4 or Lemma 5, $(R, +)$ is a strong semilattice Y of cancellative monoids R_α , where $Y \cong (E^+(R), +)$. By Lemma 4 again, we obtain that $\mathcal{H}^* = \mathcal{L}^* = \mathcal{R}^*$ is a semilattice congruence on $(R, +)$. And then, by analogy with the discussions of Proposition 1, we can get $\mathcal{H}^* = \mathcal{L}^* = \mathcal{R}^*$ is a semiring congruence.

- (2) Define a mapping

$$\phi : R/\mathcal{H}^* \rightarrow E^+(R), a\mathcal{H}^* \mapsto a^0.$$

It is not hard to check that ϕ is bijective, and

$$\begin{aligned} (a\mathcal{H}^* + b\mathcal{H}^*)\phi &= [(a+b)\mathcal{H}^*]\phi = (a+b)^0 = a^0 + b^0, \\ [(a\mathcal{H}^*)(b\mathcal{H}^*)]\phi &= [(ab)\mathcal{H}^*]\phi = (ab)^0 = a^0b^0. \end{aligned}$$

Thus, $R/\mathcal{H}^* \cong E^+(R)$. $\square$

Theorem 7. A semiring R is a generalized C -abundant semiring if and only if it satisfies the conditions (GCA1)–(GCA4).

PROOF. We only need to show the sufficiency. By Proposition 2, it is known that if S satisfies the conditions (GCA1)–(GCA4), then $\mathcal{H}^*$ is a semiring congruence on $(R, +, \cdot)$, and $R/\mathcal{H}^* \cong E^+(R)$ is a b-lattice. Also, notice that $(H_a^*, +)$ is an additively cancellative monoid, we obtain that R is a b-lattice of additively cancellative skew-halfrings.

For any $e, f \in E^+(R)$ with $e \leq_+ f$, define mapping

$$\phi_{e,f} : H_e^* \rightarrow H_f^*, a \mapsto a + f.$$

In the following, we begin to show that $R = \langle E^+(R), R_e, \phi_{e,f} \rangle$ is a strong b-lattice of the semirings $R_e, e \in E^+(R)$.

For any $a, b \in H_e^*$,

$$(a + b)\phi_{e,f} = a + b + f = a + (f + b + f) = (a + f) + (b + f) = a\phi_{e,f} + b\phi_{e,f}.$$

Also, since $af \in H_{ef}^* \cap E^+(R)$, we have $af = ef$. And then,

$$\begin{aligned} (ab)\phi_{e,f} &= ab + f = ab + e + f = ab + (e + f)^2 = ab + e + ef + fe + f \\ &= ab + af + fb + f = (a + f)(b + f) = a\phi_{e,f}b\phi_{e,f}. \end{aligned}$$

Hence, $\phi_{e,f}$ is a semiring morphism.

For any $a, b \in H_e^*$, if $a\phi_{e,f} = b\phi_{e,f}$, we have $a + f = b + f$. Notice that $(H_e^*, +)$ is an additively cancellative monoid, we have $a^0 = b^0 = e$. By condition (GCA4), we have $a = b$. Thus, $\phi_{e,f}$ is a semiring monomorphism.

Moreover, we can check that the monomorphism $\phi_{e,f}$ satisfies the conditions (1.1)–(1.4) of Definition 1.

- (i) $\phi_{e,e}$ is clearly an identity morphism.
- (ii) For any $e, f, g \in E^+(R)$ with $e \leq_+ f \leq_+ g$, we have

$$a\phi_{e,f}\phi_{f,g} = a + f + g = a + g = a\phi_{e,g}.$$

Hence, $\phi_{e,f}\phi_{f,g} = \phi_{e,g}$.

- (iii) For any $e, f, g \in E^+(S)$, if $e + f \leq_+ g$, then for any $a \in H_e^*, b \in H_f^*$,

$$a\phi_{e,g}b\phi_{f,g} = (a + g)(b + g) = ab + ag + gb + g = ab + eg + gf + g = (ab)\phi_{ef,g},$$

i.e.,

$$\phi_{e,g}b\phi_{f,g} = \phi_{ef,g}.$$

(iv) For any $e, f \in E^+(R)$, $a \in H_e^*$, $b \in H_f^*$, we have

$$\begin{aligned} a\phi_{e,e+f} + b\phi_{f,e+f} &= (a + e + f) + (b + e + f) = a + f + b + e \\ &= a + b + (e + f) = a + b; \\ a\phi_{e,e+f}b\phi_{f,e+f} &= (a + e + f)(b + e + f) = ab + a(e + f) + (e + f)b + (e + f) \\ &= ab + (e + f) = (ab)\phi_{ef,e+f}. \end{aligned}$$

Thus, we have shown that R is a strong b-lattice of additively cancellative skew-halfrings, and then it is a generalized C -abundant semiring. $\square$

Example 3. Let T be a b-lattice and M an additively cancellative skew-halfring. Construct the direct product of T and R , and denote it by M , i.e., $R = T \times M$. Then, we can check that $E^+(R) = T \times \{e_M\}$, where e_M is the identity element of $(M, +)$. We can also check that $(R, +, \cdot)$ is a semiring which satisfies the conditions (GCA1)–(GCA4). Thus, by Theorem 7, $(R, +, \cdot)$ is really a generalized C -abundant semiring.

Theorem 8. *A semiring R is a generalized C -abundant semiring if and only if it is a subdirect product of a b-lattice and an additively cancellative skew-halfring.*

PROOF. ($\Leftarrow$) By Theorem 3 and its dual, the sufficiency is clear.

($\Rightarrow$) Assume that R is a generalized C -rpp semiring, we will show that it is a subdirect product of a b-lattice and an additively cancellative skew-halfring by the following steps.

Firstly, from Proposition 2, $\mathcal{H}^*$ is a semilattice congruence on $(R, +)$ and a semiring congruence on R . Also, since R is a generalized C -a semiring, we have $aa^0 = a = a^0a$ for any $a \in R$, and then $a = aa^0\mathcal{H}^*a^2$. Hence, $R/\mathcal{H}^*$ is an idempotent semiring with the semilattice additive reduct and band multiplicative reduct, i.e., $R/\mathcal{H}^*$ is a b-lattice.

Secondly, define a binary relation

$$\theta = \{(a, b) \mid (\exists e \in E^+(R)) a + e = b + e\}.$$

It can be easily seen that θ is an equivalence relation on $(R, +, \cdot)$. Moreover, θ is the minimum additively cancellative skew-halfring congruence on $(R, +, \cdot)$. In fact, by the Proposition 1.7 in [18] and its dual, θ is a minimum cancellative monoid congruence on the additive reduct $(R, +)$. Also, if $a\theta b$ for some $a, b \in R$, there exists $e \in E^+(R)$ such that $a + e = b + e$. Now, for any $c \in R$, we have

$$ac + ec = bc + ec, ca + ce = cb + ce.$$

Notice that $ce, ec \in E^+(R)$, we immediately obtain that

$$ac\theta bc, ca\theta cb.$$

Thus, we have shown that θ is the minimum additively cancellative skew-halfring congruence on $(R, +, \cdot)$. This also shows that R/θ is an additively cancellative skew-halfring.

Finally, define a mapping

$$\Phi : R \rightarrow R/\theta \times R/\mathcal{H}^*, \quad a \mapsto (a\theta, a\mathcal{H}^*).$$

It is a routine way to check that R can be embed into $R/\theta \times R/\mathcal{H}^*$, and the projection mapping from R into each of its components is surjective. Consequently, R is a subdirect product of a b-lattice and an additively cancellative skew-halfring. $\square$

Remark 1. From Theorem 8, we can see that the class of generalized C -abundant semirings is actually a general extension of the class of generalized Clifford semirings studied in [29].

At the end of this section, we will study C -abundant semirings.

Definition 5. A semiring R is said to be a C -abundant semiring if it is a strong distributive lattice of additively cancellative skew-halfrings.

Some characterizations of such semirings are also given below.

Theorem 9. *If R is a C -abundant semiring, then the following conditions hold:*

- (CA1) $(R, +)$ is a C -abundant semigroup;
- (CA2) $(E^+(R), +, \cdot)$ is a distributive lattice;
- (CA3) for any $a, b \in R, (ab)^0 + a^0b^0 = a^0b^0$;
- (CA4) if $a^0 = b^0$ and $a + e = b + e$ for $a, b \in S$ and some $e \in E^+(R)$, then $a = b$.

Conversely, if a semiring R satisfies the conditions (CA1)–(CA4), then it is a C -abundant semiring.

PROOF. $(\Rightarrow)$ By Definition 4 and Definition 5, a C -a semiring is clearly a generalized C -abundant semiring. Thus, by Theorem 6, condition (CA1), (CA3), (CA4) hold. We only need to prove that condition (CA2) holds.

Assume that R is a C -abundant semiring. Then it is a strong distributive lattice of additively cancellative skew-halfrings, say $R = \langle D, R_\alpha, \phi_{\alpha, \beta} \rangle$, where each R_α is an additively cancellative skew-halfring in which the additive identity is denoted by 0_α and D is a distributive lattice. Notice that $E^+(R) = \{0_\alpha \mid \alpha \in$

$D\} \cong D$, we immediately obtain that $(E^+(R), +, \cdot)$ is also a distributive lattice. (CA2) holds.

($\Leftarrow$) Assume that the semiring R satisfies the conditions (CA1)–(CA4), then by Theorem 7, it is clearly a generalized C -abundant semiring. Also, note that (CA2) holds, by analogy with the discussions of Theorem 1, together with Definition 5, R is a C -abundant semiring. $\square$

Example 4. Let D be a distributive lattice and M an additively cancellative skew-halfring. Construct the direct product of D and M , and denote it by R , i.e., $R = D \times M$. Then, $E^+(R) = D \times \{e_M\}$, where e_M is the identity element of $(M, +)$. We can also check that $(R, +, \cdot)$ is a semiring which satisfies the conditions (CA1)–(CA2). Thus, by Theorem 9, $(R, +, \cdot)$ is a C -abundant semiring.

By analogy with the discussions of the subdirect decompositions of the generalized C -a semirings, we will have the following theorem.

Theorem 10. *A semiring R is a C -abundant semiring if and only if it is a subdirect product of a distributive lattice and an additively cancellative skew-halfring.*

Remark 2. From Theorem 10, we can see that the class of C -abundant semirings is actually a general extension of the one of Clifford semirings studied in [7] and [29].

References

- [1] S. ARMSTRONG, The structure of type A semigroups, *Semigroup Forum* **29** (1984), 319–366.
- [2] H. J. BANDELT and M. PETRICH, Subdirect products of rings and distributive lattice, *Proc. Edinburgh Math. Soc.* **25** **1982**, 155–171.
- [3] J. B. FOUNTAIN, Abundant semigroups, *Proc. London Math. Soc.* **344** **1982**, 103–129.
- [4] J. B. FOUNTAIN, Adequate semigroups, *Proc. Edinburgh Math. Soc.* **22** **1979**, 113–125.
- [5] J. B. FOUNTAIN, Right pp monoids with central idempotents, *Semigroup Forum* **13** (1977), 229–237.
- [6] J. M. HOWIE, An Introduction to Semigroup Theory, *Academic Press, London*, 1976.
- [7] S. GHOSH, A characterization of semirings which are subdirect products of a distributive lattice and a ring, *Semigroup Forum* **59**, no. 1 1999, 106–120.
- [8] J. S. GOLAN, Semirings and their Applications, *Kluwer, Dordrecht*, 1999.
- [9] M. P. GRILLET, Green's relations in a semiring, *Portugal. Math.* **29** (1970), 181–195.
- [10] M. P. GRILLET, Semirings with a completely simple additive semigroup, *J. Austral. Math. Soc. Ser. A* (1975), 257–267.
- [11] X.-J. GUO, K. P. SHUM and Y.-Q. GUO, Perfect rpp semigroups, *Comm. Algebra* **29** **2001**, 2447–2459.

- [12] X.-J. GUO and K. P. SHUM, On translational hulls of type A semigroups, *J. Algebra* **269** (2003), 240–249.
- [13] X.-J. GUO and Y.-Q. GUO, The translational hull of a strongly right type A semigroup, *Sci. China Ser. A* **43**, no. 1 (2000), 6–12.
- [14] Y.-Q. GUO, Structure of weakly left C semigroups, *Chinese Sci. Bull.* **41** **1996**, 462–467.
- [15] Y.-Q. GUO and K. P. SHUM, The structure of left C -rpp semigroups, *Semigroup Forum* **50** (1995), 9–23.
- [16] Y.-Q. GUO, K. P. SHUM and M. K. SEN, The semigroup structure of left Clifford semirings, *Acta Math. Sin. (Engl. Ser.)* **19**, no. 4 (2003), 783–792.
- [17] K. T. KANG, S. Z. SONG and Y. O. YANG, Structures of idempotent matrices over chain semirings, *Bull. Korean Math. Soc.* **44**, no. 4 (2007), 721–729.
- [18] M. V. LAWSON, The structure of type A semigroups, *Quart. J. Math. Oxford Ser. (2)* **37** (1986), 279–298.
- [19] D. B. McALISTER, One-to-one partial right translations of a right cancellative semigroup, *J. Algebra* **43** (1976), 231–251.
- [20] P. MUKHOPADHYAY, Subdirect products of semirings, *Int. J. Math. Math. Sci.* **26**, no. 9 (2001), 539–545.
- [21] M. OREL, Nonbijective idempotents preservers over semirings, *J. Korean Math. Soc.* **47**, no. 4 (2010), 805–818.
- [22] F. PASTIJN, A representation of a semigroup by a semigroup of matrices over a group with zero, *Semigroup Forum* **10** (1975), 238–249.
- [23] F. PASTIJN and Y.-Q. GUO, The lattice of idempotent distributive semiring varieties, *Sci. China Ser. A* **48**, no. 8 (1999), 785–804.
- [24] F. PASTIJN and X.-Z. ZHAO, Varieties of idempotent semirings with commutative addition, *Algebra Universalis* **54**, no. 3 (2005), 301–321.
- [25] F. PASTIJN and X.-Z. ZHAO, Green's $\mathcal{D}$ -relation for the multiplicative reduct of an idempotent semiring, *Arch. Math. (Brno)* **36** (2000), 77–93.
- [26] F. PASTIJN and Y.-Q. GUO, Semirings which are unions of rings, *Sci. China Ser. A* **45**, no. 2 (2002), 172–195.
- [27] O. A. PSHENITSYNA, Mappings that preserve the invertibility of matrices over semirings, *Russian Math. Surveys* **64**, no. 1 (2009), 162–164.
- [28] M. K. SEN, Y.-Q. GUO and K. P. SHUM, A class of idempotent semirings, *Semigroup Forum* **60** (2000), 351–367.
- [29] M. K. SEN, S. K. MAITY and K. P. SHUM, Clifford semirings and generalized Clifford semirings, *Taiwanese J. Math.* **19**, no. 3 (2005), 433–444.
- [30] S. Z. SONG, K. T. KANG, L. B. BEASLEY and N. S. SZE, Regular matrices and their strong preservers over semirings, *Linear Algebra Appl.* **429** (2008), 209–223.
- [31] X.-Z. ZHAO, K. P. SHUM and Y.-Q. GUO, $\mathcal{L}$ -subvarieties of the variety of idempotent semirings, *Algebra Universalis* **46** (2001), 75–96.

- [32] X.-Z. ZHAO, Y.-Q. GUO and K. P. SHUM, Sturdy frame of type $(2, 2)$ algebras and their application to semirings, *Fund. Math.* **179** (2003), 69–81.

YIZHI CHEN
DEPARTMENT OF MATHEMATICS
HUIZHOU UNIVERSITY
HUIZHOU, GUANGDONG, 516007
P. R. CHINA

E-mail: yizhichen1980@126.com

XIANZHONG ZHAO
INSTITUTE OF MATHEMATICS
AND INFORMATION SCIENCE
JIANGXI NORMAL UNIVERSITY
NANCHANG, JIANGXI, 330027
P. R. CHINA

E-mail: E-mail: xianzhongzhao@263.net

XIAOJIANG GUO
INSTITUTE OF MATHEMATICS
AND INFORMATION SCIENCE
JIANGXI NORMAL UNIVERSITY
NANCHANG, JIANGXI, 330027
P. R. CHINA

E-mail: E-mail: xjguo@jxnu.edu.cn

(Received September 6, 2011; revised April 22, 2013)