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Groups of bounded automorphisms of a free group of countably infinite rank

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Abstract. We consider the subgroup $\mathcal{E}_X < \mathrm{Aut}(F_\infty)$ generated by elementary simultaneous Nielsen automorphisms, introduced by R. Cohen in 1973. The still open Solitar's conjecture states that $\mathcal{E}_X$ coincides with the subgroup of bounded automorphisms. We prove that $\mathcal{E}_X$ is generated by involutions, $\mathcal{E}_X$ is perfect, and it has trivial center. We conclude that a subgroup of bounded automorphisms of a relatively free nilpotent group of infinite rank is generated by involutions.

1. Introduction and main results

Let F be a free group of infinite, countable rank. A basis of F is a set of free generators of F. If X is a basis of F then $|w|$ denotes the length of a word $w \in F$ with respect to the basis X. When it is necessary to distinguish between the lengths in different bases X and Y we shall write $|w|_X$ for X-length and $|w|_Y$ for Y-length. We say that a group G is perfect if G coincides with its commutator subgroup G' and we say that G is complete if it has trivial center and every its automorphism is inner.

The aim of this paper is to investigate a subgroup generated by elementary simultaneous Nielsen automorphisms. We prove that this subgroup is generated by involutions and that it is perfect and has trivial center. And we conclude that the subgroup of bounded automorphisms of a relatively free nilpotent group of infinite rank is generated by involutions.

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Definition 1. Let X be a basis of F. An automorphism $\tau \in \mathrm{Aut}(F)$ is *an elementary simultaneous Nielsen automorphism* if it has one of the following forms:

(1) τ permutes the set X.

(2) τ inverts some elements of X and acts trivially on the rest of the elements of X.

(3) There exists the subset U of X such that $u^\tau = uv$ or vu for $u \in U$ and some v in $X \setminus U$ and $v^\tau = v$ for every $v \in X \setminus U$.

Following Bogopolski and Singhof [1] we call the automorphisms of type (1) and (2) *monomial automorphisms.*

We use the notation $\mathcal{M}_X$ for the subgroup of all monomial automorphisms, $\mathcal{E}_X$ for the subgroup generated by elementary simultaneous Nielsen automorphisms, and $\mathcal{M}$ or $\mathcal{E}$ for the subgroups generated by all $\mathcal{M}_Y$ and $\mathcal{E}_Y$, respectively where Y runs over the family of all bases of F.

The notion of elementary simultaneous Nielsen automorphisms was introduced by Cohen in [6].

Definition 2. Let X be a basis of F. We say that an automorphism α of F is bounded if there exists N such that for every $x \in X$ we have $|x^\alpha| \leq N$ and $|x^{\alpha^{-1}}| \leq N$.

The subgroup of all automorphisms of F that are bounded on a basis X will be denoted by $\mathcal{B}_X$. We shall denote by $\mathcal{B}$ the subgroup generated by all $\mathcal{B}_Y$, here Y runs over the family of all bases of F. It follows from Definition 1 that an automorphism α is X-monomial if and only if α is bounded on X by $N = 1$ and that the subgroup $\mathcal{E}_X$ is generated by all automorphisms bounded on X by $N \leq 2$.

Solitar's Conjecture (see [6]) states that $\mathcal{B}_X = \mathcal{E}_X$. This conjecture is still an open problem.

Solitar's Conjecture can be formulated for every relatively free group F/V all automorphisms of which are induced by automorphisms of F. If F/V is nilpotent then by [3] every its automorphism can be lifted to an automorphism of F. In [12] Macedońska proved that Solitar's conjecture is true in a free abelian group of infinite rank. Then Burns and Lian Pi in [5] obtained a positive answer for groups F/V where V is a characteristic subgroup of F containing $\gamma_{c+1}(F)$, the $(c+1)$st term of the lower central series of F. Hence if V is a characteristic subgroup of F containing $\gamma_{c+1}(F)$ then every bounded automorphism of F/V can be lifted to a bounded automorphism of the group F.

If $G = F/V$ is a relatively free group freely generated by a set $\bar{X}$ then we use notations: $\bar{\mathcal{E}}_{\bar{X}}$, $\bar{\mathcal{M}}_{\bar{X}}$, $\bar{\mathcal{B}}_{\bar{X}}$ for subgroups generated by elementary simultaneous Nielsen automorphisms, monomial automorphisms, bounded automorphisms in G, respectively.

Now we introduce two subgroups of the group $\mathcal{E}_X$. The first subgroup is $\mathcal{S}_X$, which consists of automorphisms permuting the basis X. We shall use the symbol $\mathcal{S}$ for the subgroup generated by all $\mathcal{S}_X$, where X is an arbitrary basis of F. The second subgroup is $\mathcal{L}_X$. An automorphism belongs to $\mathcal{L}_X$ if it inverts some elements from X, and does not change remaining elements. As previously $\mathcal{L}$ is the subgroup generated by $\mathcal{L}_X$ for all bases X of F.

The subgroup $\mathcal{L}_X\mathcal{S}_X$ is the subgroup of all monomial automorphisms on the set X and it is a semidirect product $\mathcal{L}_X \rtimes \mathcal{S}_X$ of $\mathcal{L}_X$ by $\mathcal{S}_X$. If we look closely we will see that this subgroup is isomorphic to the unrestricted wreath product $Z_2 \operatorname{Wr}_{\mathbb{N}} S(\mathbb{N})$ of a cyclic group of order two and a symmetric group on the set of natural numbers. Elements of this wreath product have the form $u = (\sigma, (\varepsilon_1, \varepsilon_2, \ldots))$, where $\sigma \in S(\mathbb{N})$ and $(\varepsilon_1, \varepsilon_2, \ldots) \in Z_2^{\mathbb{N}}$ and u is associated with automorphism $\alpha \in \mathcal{S}_X\mathcal{L}_X$ acting on $X = \{x_1, x_2, \ldots\}$ as follows: $x_i^{\alpha} = x_{i^{\sigma}}^{(-1)^{\varepsilon_{i^{\sigma}}}}$.

The aim of this paper is to prove the following Theorem and Corollaries.

Theorem 1. *The subgroup $\mathcal{E}_X$ (generated by elementary simultaneous Nielsen automorphisms) is generated by all subgroups $\mathcal{S}_{X^{\tau}}$, where τ belongs to $\mathcal{E}_X$.*

Corollary 1. (i) *$\mathcal{E} = \mathcal{S}$ and $\mathcal{E}$ is a normal subgroup in* $\operatorname{Aut}(F)$.

(ii) *Every subgroup $\mathcal{E}_X$ and the subgroup $\mathcal{E}$ are generated by involutions.*

(iii) *Centralizers of $\mathcal{E}_X$ and $\mathcal{E}$ in* $\operatorname{Aut}(F)$ *are trivial. In particular $\mathcal{E}_X$ and $\mathcal{E}$ have trivial centers.*

(iv) *Every subgroup $\mathcal{E}_X$ and the subgroup $\mathcal{E}$ are perfect groups.*

Proof. (i) The inclusion $\mathcal{S} \subseteq \mathcal{E}$ is clear. By Theorem 1 $\mathcal{E}_X = \mathcal{S}_{X^{\tau}} \subseteq \mathcal{S}$, so it gives us the converse inclusion. Since $\eta^{-1}\mathcal{S}_X\eta = \mathcal{S}_{X^{\eta}}$, the subgroup $\mathcal{S} = \mathcal{E}$ is closed under the conjugation, so it is normal.

(ii) It follows from [8] (Lemma 8.1A, p. 256) that every permutation of a countable set is the product of two involutions. So $\mathcal{E}$ being generated by permutations of infinite sets is also generated by involutions.

(iii) For every $x \in X$ the inner automorphism $i_x(a) = x^{-1}ax$ is a product of two elementary simultaneous Nielsen automorphism. So i_x belongs to $\mathcal{E}_X$. The subgroup of inner automorphisms is generated by all i_x, where $x \in X$. So we get the inclusion $\operatorname{Inn}(F) \subseteq \mathcal{E}_X$.

If α belongs to a centralizer of $\mathcal{E}_X$ then α commutes with every inner automorphism of F. So for every $a \in F$ we have $i_a\alpha = \alpha i_a$. Hence, for every $w \in F$ we get $g^{-1}w^\alpha g = g^{-\alpha}w^\alpha g^\alpha$. Thus $g^\alpha g^{-1}w^\alpha = w^\alpha g^\alpha g^{-1}$, and it means that $g^\alpha g^{-1}$ commutes with every element of F, so it has to be trivial. Thus, we have $g^\alpha = g$, and since it is satisfied for every g, the automorphism α is an identity map.

(iv) In [13] Ore proved that every permutation on infinite set is a commutator. By Theorem 1 every element of $\mathcal{E}_X$ and $\mathcal{E}$ is a product of automorphisms permuting some bases. Since every automorphism permuting a basis can be identified with a permutation of the set $\mathbb{N}$ by Ore Theorem it is a commutator. Hence every automorphism from $\mathcal{E}_X$ or $\mathcal{E}$ is a product of commutators. Thus subgroups $\mathcal{E}_X$ and $\mathcal{E}$ are perfect. □

Corollary 2. *Let $\bar{\mathcal{B}}_{\bar{X}}$ be the subgroup of bounded automorphisms on the basis $\bar{X}$ of the free abelian group $\bar{F} = F/F'$. Then*

(i) $\bar{\mathcal{B}} = \bar{\mathcal{S}}$.

(ii) *Every subgroup $\bar{\mathcal{B}}_{\bar{X}}$ and the subgroup $\bar{\mathcal{B}}$ are generated by involutions.*

(iii) *Every subgroup $\bar{\mathcal{B}}_{\bar{X}}$ and the subgroup $\bar{\mathcal{B}}$ are perfect groups.*

Proof. It follows from [5] that the equality $\bar{\mathcal{B}}_{\bar{X}} = \bar{\mathcal{E}}_{\bar{X}}$ holds. The Corollary follows from (i) and Corollary 1. □

We also show some properties and examples of groups of bounded automorphisms of the group F.

There are some papers describing certain classes of automorphisms of the free group of countable rank and properties of the group $\mathrm{Aut}(F)$. For example, in [6] and [10] one can find a description and properties of bounded, triangular and permutational automorphisms. In [1] Bogopolski and Singhof show that the group $\mathrm{Aut}(F)$ has a countable presentation in a big free group sense. In [2] Bryant and Evans show that the group $\mathrm{Aut}(F)$ has the small index property. In [15] Tostykh shows that the automorphism group of a free group of countable rank is complete. Earlier, the same for free groups of finite rank was shown by Dyer and Formanek in [9]. Also Tostykh in [16] shows that $\mathrm{Aut}(F)$ has the Bergman property, that is if S generates $\mathrm{Aut}(F)$ then there exists a natural number k such that every automorphism is a product of no more than k generators from S.

The List of subgroups mentioned here and connections between them:

- $\mathcal{L}_X$ – the subgroup of all automorphisms inverting some elements of X and leaving the remainder fixed,
- $\mathcal{S}_X$ – the subgroup of all automorphisms permuting elements of X,

- $\mathcal{M}_X$ – the subgroup of monomial automorphisms,
- $\mathcal{E}_X$ – the subgroup generated by all elementary simultaneous Nielsen automorphisms,
- $\mathcal{B}_X$ – the subgroup of all bounded automorphisms,
- $\mathcal{M}_X = \mathcal{L}_X\mathcal{S}_X < \mathcal{E}_X < \mathcal{B}_X$,
- for subgroups $\mathcal{H}_X$ of automorphisms depending on X we denote by $\mathcal{H}$ the subgroup generated by all $\mathcal{H}_X$, where X is an arbitrary basis of F,
- $\mathcal{M} = \mathcal{LS} < \mathcal{E} < \mathcal{B}$.

2. Proof of Theorem 1

In this section we prove Theorem 1.

Proposition 1. *Let X and Y be bases of F. Then*

(i) $\mathcal{B}_X = \eta\mathcal{B}_Y\eta^{-1}$ *and* $\mathcal{E}_X = \eta\mathcal{E}_Y\eta^{-1}$, *where η is an automorphism induced by an arbitrary bijection $\eta : X \to Y$.*

(ii) *If η is bounded on X then the equation $\mathcal{B}_X = \mathcal{B}_Y$ holds.*

(iii) *If η belongs to $\mathcal{E}_X$ then the equation $\mathcal{E}_X = \mathcal{E}_Y$ holds.*

Proof. (i) Let $X = \{x_1, x_2, \ldots\}$ and $Y = \{y_1, y_2, \ldots\}$ and let η map every x_i onto y_i for every $i \geq 1$. If α belongs to $\mathcal{B}_X$ then there exists a natural number N, such that for every $x_i \in X$ we have $|x_i^\alpha|_X \leq N$. If $x_i^\alpha = w_i(x_1, \ldots)$ then $y_i^{\eta^{-1}\alpha\eta} = x_i^{\alpha\eta} = w_i(x_1, \ldots)^\eta = w_i(y_1, \ldots)$, so $|y_i^{\eta^{-1}\alpha\eta}|_Y = |x_i^\alpha|_X$ and $\eta^{-1}\alpha\eta \in \mathcal{B}_Y$. It gives us the inclusion $\mathcal{B}_X \subseteq \eta\mathcal{B}_Y\eta^{-1}$. The proof of the converse inclusion is analogous.

If τ is an elementary simultaneous Nielsen automorphism on basis X then $\eta^{-1}\tau\eta$ is an elementary simultaneous Nielsen automorphism on basis Y. Thus, we get the equation $\mathcal{E}_X = \eta\mathcal{E}_Y\eta^{-1}$.

(ii) If η is bounded on X then it follows from *(i)* that $\mathcal{B}_Y = \eta^{-1}\mathcal{B}_X\eta = \mathcal{B}_X$.

(iii) If η belongs to $\mathcal{E}_X$ then by *(i)* we get $\mathcal{E}_Y = \eta^{-1}\mathcal{E}_X\eta = \mathcal{E}_X$. □

Proposition 2. $\mathcal{E}_X = \langle \mathcal{M}_{X^\tau},\ \tau \in \mathcal{E}_X \rangle$.

Proof. Let τ belong to $\mathcal{E}_X$. Then by Proposition 1 (iii) $\mathcal{E}_{X^\tau} = \mathcal{E}_X$. Thus, subgroups $\mathcal{S}_{X^\tau}$, $\mathcal{L}_{X^\tau}$ are contained in $\mathcal{E}_{X^\tau} = \mathcal{E}_X$. This gives us the inclusion $\langle \mathcal{S}_{X^\tau}, \mathcal{L}_{X^\tau},\ \tau \in \mathcal{E}_X \rangle = \langle \mathcal{M}_{X^\tau},\ \tau \in \mathcal{E}_X \rangle \subseteq \mathcal{E}_X$.

It remains to prove the reverse inclusion. It is enough to prove that elementary simultaneous Nielsen transformations of the type (3) (see Definition 1) belong to $\langle \mathcal{M}_{X^\tau},\ \tau \in \mathcal{E}_X \rangle$.

We partition X into three pairwise disjoint subsets $X = U \cup V \cup W$. Subset W is "nonactive", that is if $w \in W$ then $w^\tau = w$. Also for $v \in V$ we have $v^\tau = v$ but elements from this set act on elements from U. Let $V = \{v_1, v_2, v_3, \dots\}$. We partition U into subsets $U = U_1 \cup U_2 \cup \dots$. If $u \in U_i$ then $u^\tau = uv_i$. Let U_i consist of elements $u_{i1}, u_{i2}, \dots$ for $i = 1, 2, \dots$. Now we define the new basis Y which is the union $U' \cup W$, where $U' = U_1' \cup U_2' \cup \dots$ and $U_i' = \{u_{i1}, u_{i1}v_i, u_{i2}u_{i1}^{-1}, u_{i3}u_{i1}^{-1}, \dots\}$. One can check that Y is a free generating set of F and that the automorphism σ defined by any mapping $X \to Y$ belongs to $\mathcal{E}_X$. Now we define an automorphism η. The automorphism η interchanges u_{i1} and $u_{i1}v_i$ for $i = 1, 2, \dots$ and acts identically on other elements. This automorphism belongs to $\mathcal{S}_Y \subseteq \mathcal{E}_Y$. How does η work on elements of the basis X? Let us calculate $v_i^\eta = (u_{i1}^{-1}u_{i1}v_i)^\eta = (u_{i1}v_i)^{-1}u_{i1} = v_i^{-1}$, $u_{i1}^\eta = u_{i1}v_i$, and for $k > 1$ $u_{ik}^\eta = [(u_{ik}u_{i1}^{-1})u_{i1}]^\eta = u_{ik}u_{i1}^{-1}u_{i1}v_i = u_{ik}v_i$. Thus $\tau = \vartheta\eta$, where ϑ inverses all v_i and acts trivially on other elements. So $\vartheta \in \mathcal{L}_X$ and $\tau \in \langle \mathcal{M}_X, \mathcal{M}_Y \rangle$. For other variants of the mapping τ the reasoning is analogous. This completes the proof. □

Proposition 3. *$\mathcal{L}_X$ is generated by automorphisms δ for which there is a partition of the set $X = X_1 \cup X_2$ into two infinite, disjoint subsets such that δ acts trivially on X_1 and inverts all elements in X_2. Moreover, every element from $\mathcal{L}_X$ is a product of at most two such elements.*

Proof. The proof is left to the reader. □

Proof of Theorem 1. By Proposition 2, $\mathcal{E}_X$ is generated by all subgroups $\mathcal{M}_{X^\tau}$ for $\tau \in \mathcal{E}_X$ and we know that $\mathcal{M}_{X^\tau} = \mathcal{L}_{X^\tau}\mathcal{S}_{X^\tau}$. Now it is enough to prove that $\mathcal{L}_X$ is contained in $\langle \mathcal{S}_{X^\tau},\ \tau \in \mathcal{E}_X \rangle$.

By Proposition 3, the subgroup $\mathcal{L}_X$ is generated by all automorphisms δ for which there exist two infinite sets X_1 and X_2, such that $X = X_1 \cup X_2$ and δ acts trivially on X_1 and inverts all elements from X_2. We show that such automorphisms lie in $\langle \mathcal{S}_{X^\tau},\ \tau \in \mathcal{B}_X \rangle$. Let X_2 consist of elements $a_1, a_2, \dots, b_1, b_2, \dots$. Then $Y = X_1 \cup X_2' = X_1 \cup \{a_1, a_2, \dots, b_1', b_2', \dots\}$, where $b_i' = b_i^{-1}$ for $i = 1, 2, \dots$ is a basis of F and any bijection $X \to Y$ defines an automorphism belonging to $\mathcal{E}_X$. We define two automorphisms $\alpha \in \mathcal{S}_X$ and $\beta \in \mathcal{S}_Y$. Both of them act trivially on X_1. The first (α) is defined on X_2 by the rules $\alpha : a_i \leftrightarrow b_i$ for $i = 1, 2, \dots$ and the second (β) is defined on X_2 as follows: $\beta : a_i \leftrightarrow b_i'$. One can check that $\delta = \alpha\beta$. So δ belongs to $\langle \mathcal{S}_X, \mathcal{S}_Y \rangle \subseteq \langle \mathcal{S}_{X^\tau},\ \tau \in \mathcal{E}_X \rangle$, and the theorem is proved. □

3. Further properties of $\mathcal{B}_X$

It is not known whether the condition in point *(ii)* of Proposition 1 is not only necessary but also sufficient. In other words, is it possible that for unbounded η the equality $\mathcal{B}_X = \eta^{-1}\mathcal{B}_X\eta$ holds? Or we can ask about normalizer of $\mathcal{B}_X$ in $\mathrm{Aut}(F)$. Is it true that $\mathrm{N}_{\mathrm{Aut}(F)}(\mathcal{B}_X) = \mathcal{B}_X$? The series of examples below shows that for certain bases X and Y the equality $\mathcal{B}_X = \mathcal{B}_Y$ does not hold.

Example 1. Let $X = \{x_1, x_2, \ldots\}$ be an arbitrary basis of F. We define elements $y_i = x_1^{-i}x_ix_1^i$ for $i \geq 1$. Then $Y = \{y_1, y_2, \ldots\}$ is a basis of F and the automorphism $\eta : y_i \to x_i$ is unbounded on X and $\mathcal{B}_X \neq \mathcal{B}_Y$.

Proof. The automorphism η maps every x_i to $x_1^i x_i x_1^{-i}$. Thus, it is unbounded on X. Let σ be any permutation of the set of natural numbers such that $1^\sigma = 1$ and let $\bar{\sigma}$ be the corresponding automorphism defined on Y in the following way $y_i^{\bar{\sigma}} = y_{i^\sigma}$. Then $\bar{\sigma}$ is bounded on Y. Let us see how $\bar{\sigma}$ works on basis X. We have $x_i^{\bar{\sigma}} = y_1^i y_{i^\sigma} y_1^{-i} = x_1^{i-i^\sigma} x_{i^\sigma} x_1^{i^\sigma - i}$ and it depends on σ whether an automorphism $\bar{\sigma}$ is bounded on X or not. If for example σ changes only a finite number of numbers (i.e. is a finitary permutation) then $\bar{\sigma}$ is bounded on X, and for σ having an infinite cycle that maps every 2^i to 2^{i+1} for $i \geq 1$, $\bar{\sigma}$ is unbounded on X. □

Example 2. Let $X = \{x_1, x_2, \ldots\}$ be an arbitrary basis of F and let $\beta_{x_1x_2}$ be an elementary Nielsen automorphism (see [11], Section 4) which maps x_1 onto x_1x_2 and acts trivially on the rest of elements of X. Then there exists a basis Y such that $\beta_{x_1x_2}$ is not bounded on Y.

Proof. Let $y_i = x_1^{-i}x_ix_1^i$. Then $x_i = y_1^i y_i y_1^{-i}$, and $\beta_{x_1x_2}$ maps y_i for $i > 1$ to $(x_1x_2)^{-i}x_i(x_1x_2)^i = (y_1^3y_2y_1^{-2})^{-i}y_1^iy_iy_1^{-i}(y_1^3y_2y_1^{-2})^i$ and the Y-lengths of these images form a strictly increasing sequence. So, $\beta_{x_1x_2}$ is not bounded on Y. □

Example 3. Let $X = \{x_1, x_2, \ldots\}$ be an arbitrary basis of F and let σ be the Nielsen automorphism, which interchanges x_1 and x_2 and acts trivially on the rest of elements of X. Then there exists a basis Y such that σ is not bounded on Y.

Proof. The automorphism σ is not bounded on the basis $Y = \{y_i\}$ where $y_i = x_1^{-i}x_ix_1^i$ for $i \geq 1$. □

Example 4. Let $X = \{x_1, x_2, \ldots\}$ be an arbitrary basis of F and let us define $y_1 = x_1$ and $y_i = x_{i-1}^{-1}x_i$ for $i > 1$. Then $Y = \{y_1, y_2, \ldots\}$ is a basis of F and the automorphism $\eta : y_i \to x_i$ is unbounded on X. We associate an infinite sequence

$\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots)$, where $\varepsilon_i = \pm 1$ with the automorphism $\bar{\varepsilon}$ acting on basis Y as follows: $y_i^{\bar{\varepsilon}} = y_i^{\varepsilon_i}$. Then $\bar{\varepsilon}$ is bounded on basis Y and is bounded on basis X if and only if the number of -1 in the sequence ε is finite.

PROOF. First let us notice that $x_i = y_1 y_2 \ldots y_i$ for every $i \geq 1$. So, Y is a basis of F and η is unbounded on X. The automorphism $\bar{\varepsilon}$ acts on X as follows: $x_i \to y_1^{\varepsilon_1} y_2^{\varepsilon_2} \ldots y_i^{\varepsilon_i} = x_1^{\varepsilon_1} (x_1^{-1} x_2)^{\varepsilon_2} \ldots (x_{i-1}^{-1} x_i)^{\varepsilon_i}$. If ε has only a finite number of -1's then there exists a number n such that $\varepsilon_i = 1$ for every $i > n$. Thus, for $i > n$ we have

$$x_i \to y_1^{\varepsilon_1} y_2^{\varepsilon_2} \ldots y_i^{\varepsilon_i} = y_1^{\varepsilon_1} y_2^{\varepsilon_2} \ldots y_n^{\varepsilon_n} y_{n+1} \ldots y_i = y_1^{\varepsilon_1} y_2^{\varepsilon_2} \ldots y_n^{\varepsilon_n} x_n^{-1} x_i.$$

So in this case $\bar{\varepsilon}$ is bounded on X.

If $\varepsilon_i = -1$ then $|y_1^{\varepsilon_1} y_2^{\varepsilon_2} \ldots y_i^{\varepsilon_i}|_X = |y_1^{\varepsilon_1} y_2^{\varepsilon_2} \ldots y_{i-1}^{\varepsilon_{i-1}}|_X + 2$ and $|y_1^{\varepsilon_1} y_2^{\varepsilon_2} \ldots y_k^{\varepsilon_k}|_X > |y_1^{\varepsilon_1} y_2^{\varepsilon_2} \ldots y_i^{\varepsilon_i}|_X$ for every $k > i$. So if ε has an infinite number of -1's then $\bar{\varepsilon}$ is unbounded on X. □

As we showed in Corollary 1, $\mathcal{E}$ is normal in $\operatorname{Aut} F$. Similarly by Proposition 1 (i) $\mathcal{B}$ also is normal in $\operatorname{Aut} F$. The question is whether they are proper. By Theorem C from [4] every proper normal subgroup in $\operatorname{Aut} F$ has index $2^{\aleph_0}$. So these subgroups have indexes 1 or $2^{\aleph_0}$. Both subgroups $\mathcal{E}$ and $\mathcal{B}$ have cardinality $2^{\aleph_0}$ and we do not know whether these subgroups coincide.

Theorem C from [4] shows that $\operatorname{Aut} F$ is perfect. We proved in Corollary 1 that $\mathcal{E}_X$ and $\mathcal{E}$ are perfect. Now we show that the subgroup $\mathcal{M}_X$ is also perfect. First we prove a lemma which is largely based on the result of Ore who showed in [13] that certain groups are perfect.

Lemma 1. *The symmetric group $S(\mathbb{N})$ and a wreath product $A \operatorname{Wr}_{\mathbb{N}} S(\mathbb{N})$ where A is an arbitrary cyclic group are perfect.*

PROOF. In [13] Ore proved that every element of $S(\mathbb{N})$ is a commutator, so this group is perfect. DARK in [7] showed that for any group G the wreath product $G \operatorname{Wr}_{\mathbb{N}} S(\mathbb{N})$ is a perfect group. This proof is based on the theorem on the structure of normal subgroups in $G \operatorname{Wr}_{\mathbb{N}} S(\mathbb{N})$ (see Theorem 11.3.4 in [14]).

The proof below shows that every element of $A \operatorname{Wr}_{\mathbb{N}} S(\mathbb{N})$ is the product of at most two commutators, so that the group $A \operatorname{Wr}_{\mathbb{N}} S(\mathbb{N})$ is perfect. We use additive notation for a cyclic group A. Every element of the wreath product $A \operatorname{Wr}_{\mathbb{N}} S(\mathbb{N})$ has the form $(\sigma, (a_1, a_2, \ldots))$, where σ belongs to $S(\mathbb{N})$, and $(a_1, a_2, \ldots)$ is the infinite sequence of elements of A. The multiplication and the inverse map are as follows:

$$(\sigma, (a_1, a_2, \ldots))(\delta, (b_1, b_2, \ldots)) = (\sigma\delta, (a_{1^\delta} + b_1, a_{2^\delta} + b_2, \ldots))$$

$$(\sigma, (a_1, a_2, \ldots))^{-1} = (\sigma^{-1}, (-a_{1^{\sigma^{-1}}}, -a_{2^{\sigma^{-1}}}, \ldots)).$$

Every element $(\sigma, (a_1, a_2, \ldots))$ in $A \operatorname{Wr}_{\mathbb{N}} S(\mathbb{N})$ is equal $(\sigma, (0, 0, \ldots)) \cdot (\mathrm{id}, (a_1, a_2, \ldots))$. The element $(\sigma, (0, 0, \ldots))$ is a commutator by Ore theorem. So it is enough to show that every element $(\mathrm{id}, (a_1, a_2, \ldots))$ is a commutator. Here a commutator $[a, b]$ of elements a, b is equal to $a^{-1}b^{-1}ab$. Let us calculate:

$$[(\sigma, (w_1, w_2, \ldots)), (\mathrm{id}, (w_1, w_2, \ldots))] = (\mathrm{id}, (w_1 - w_{1^\sigma}, w_2 - w_{2^\sigma})).$$

Let σ be a permutation acting on $\mathbb{N}$ as follows:

$$n^\sigma = \begin{cases} 2 & \text{if } \ n = 1 \\ 2k + 2 & \text{if } \ n = 2k \\ 2k - 1 & \text{if } \ n = 2k + 1 \text{ and } \ k > 0 \end{cases}$$

We define $(w_i)_{i \in \mathbb{N}}$ as follows:

$$w_1 = 0, \quad w_{2k} = -a_1 - \sum_{j=1}^{k-1} a_{2j}, \quad w_{2k+1} = \sum_{j=1}^{k} a_{2j+1} \quad \text{for } k \geq 1$$

and we get

$$[(\sigma, (w_1, w_2, \ldots)), (i, (w_1, w_2, \ldots))] = (i, (w_1 - w_{1^\sigma}, w_2 - w_{2^\sigma})) = (i, (a_1, a_2, \ldots)),$$

as required. □

Proposition 4. *For every basis X of the group F, the group $\mathcal{M}_X$ is perfect.*

Proof. The group $\mathcal{M}_X$ is isomorphic to the group $Z_2 \operatorname{Wr}_{\mathbb{N}} S(\mathbb{N})$, so by Lemma 1 it is perfect. □

Another problem is the question of the completeness of certain subgroups. We say that a group G is complete if it has trivial center and every automorphism of G is inner. We can also say that a centerless group is complete if its automorphism tower has length 1. Dyer and Formanek in [9] showed that the automorphism group of a free group of finite rank ≥ 2 is complete, and Tolstykh in [15] showed the same for a free group of countable rank. We do not know whether $\mathcal{E}_X$ or $\mathcal{E}$ are complete, but it follows from Corollary 1 that they have trivial centers.

Let us denote by $\mathcal{K}$ the intersection $\bigcap_X \mathcal{B}_X$, where X ranges over all bases of F. Then $\mathcal{K}$ is not trivial, since it contains the subgroup of inner automorphisms. By Proposition 1 the equality $\mathcal{K} = \bigcap_X \mathcal{B}_X = \bigcap_{\eta \in \mathrm{Aut}(F)} \eta^{-1} \mathcal{B}_X \eta$ holds, where X is an arbitrary basis of F. Hence, $\mathcal{K}$ is nontrivial, and by Examples 1-3 it is

a proper normal subgroup of $\mathrm{Aut}(F)$. Thus, by Bryant and Roman'kov ([4] Theorem C), $\mathcal{K}$ has index $2^{\aleph_0}$ in $\mathrm{Aut}(G)$. Since $\mathcal{K}$ contains $\mathrm{Inn}(F)$ it is centerless.

If we want to understand which automorphisms belong to $\mathcal{K}$ we should remember that by Examples 1–4 many of elementary simultaneous Nielsen transformations do not belong to $\mathcal{K}$. Even the so-called finitary automorphisms (see [11], Proposition 4.1) do not lie in $\mathcal{K}$. So, there arises the question whether $\mathcal{K}$ consists only of inner automorphisms.

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