

On some band decompositions of semigroups

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Abstract. M. S. PUTCHA in [5] described semigroups which are bands of r -Archimedean or t -Archimedean semigroups. L. N. ŠEVRIŇ, J. L. GALBIATI, M. L. VERONESI, S. BOGDANOVIĆ and M. ĆIRIĆ described rectangular bands of π -groups. In this paper we characterize some bands of r -Archimedean semigroups.

Let $\mathbb{N}$ be the set of all positive integers. A semigroup S is *right Archimedean* (or *r -Archimedean*) if, for every $a, b \in S$, there exists $n \in \mathbb{N}$ such that $a^n \in bS$. The dual of a right Archimedean semigroup is a *left Archimedean* (or *l -Archimedean*) one. A semigroup S is *t -Archimedean* if, for every $a, b \in S$, there exists $n \in \mathbb{N}$ for which $a^n \in bS \cap Sb$.

A semigroup B is a *band* if for each $x \in B$, $x^2 = x$ holds.

A semigroup S is a *band Y of semigroups S_α* if $S = \bigcup_{\alpha \in Y} S_\alpha$, Y is a band, $S_\alpha \cap S_\beta = \emptyset$ for $\alpha, \beta \in Y$ with $\alpha \neq \beta$ and $S_\alpha S_\beta \subseteq S_{\alpha\beta}$.

A congruence ϱ on S is called a *band congruence* if S/ϱ is a band.

Theorem 1 [5]. *A semigroup S is a band of r -Archimedean semigroups if and only if*

$$(1) \quad (\forall a \in S)(\forall x, y \in S^1)(\exists i, j \in \mathbb{N})(xay)^i \in xa^2yS, (xa^2y)^j \in xayS.$$

In this theorem, it is proved that if (1) holds then the relation ϱ defined on S by

$$(2) \quad a\varrho b \iff (\forall x, y \in S^1)(\exists i, j \in \mathbb{N})(xay)^i \in xbyS, (xby)^j \in xayS$$

is a band congruence on S .

For undefined notions and notations we refer to [1] and [3].

Recall that a band B is a *right regular band* if $ef = fef$ for every $e, f \in B$.

Theorem 2. *A semigroup S is a right regular band of r -Archimedean semigroups if and only if*

$$(3) \quad (\forall u, v \in S)(\exists n \in \mathbb{N}) (uv)^n \in vS.$$

PROOF. Let S be a right regular band Y of r -Archimedean semigroups S_α . If $u \in S_\alpha$, $v \in S_\beta$ then $uv \in S_{\alpha\beta} = S_{\beta\alpha\beta}$, $vuv \in S_{\beta\alpha\beta}$. Since $S_{\beta\alpha\beta}$ is r -Archimedean, then there exists $n \in \mathbb{N}$ such that $(uv)^n \in vuvS_{\beta\alpha\beta} \subseteq vS$, and so (3) holds.

Conversely, let statement (3) hold on a semigroup S and let $a \in S$, $x, y \in S^1$. Then, for $u = a$, $v = ayx$, there exists $n \in \mathbb{N}$ such that

$$(xa^2y)^{n+1} = (xaay)^{n+1} = x(aayx)^na^2y \in xayxSa^2y \subseteq xayS.$$

Also, for $u = yxayx$, $v = ayxa$ there exists $m \in \mathbb{N}$ such that

$$\begin{aligned} (xay)^{3(m+1)} &= (xayxayxay)^{m+1} = xa(yxayxayxa)^myxayxay \\ &\in xaayxaSyxayxay \subseteq xa^2yS. \end{aligned}$$

Now, by Theorem 1 we have that S is a band Y of r -Archimedean semigroups S_α .

Let $a, b \in S$. Then, by (3), $(ab)^n = bt$ for some $t \in S$ and $n \in \mathbb{N}$. If $a \in S_\alpha$, $b \in S_\beta$, $t \in S_\gamma$ then $\alpha\beta = \beta\gamma = \beta\beta\gamma = \beta\alpha\beta$. Hence Y is a right regular band. $\square$

Recall that a band B is a *right zero band* if $e = fe$ for every $e, f \in B$.

Theorem 3. *A semigroup S is a right zero band of r -Archimedean semigroups if and only if*

$$(4) \quad (\forall u, v \in S)(\exists m, n \in \mathbb{N})(uv)^m \in vS, v^n \in uvS.$$

PROOF. Let S be a right zero band Y of r -Archimedean semigroups S_α , $\alpha \in Y$. If $u \in S_\alpha$, $v \in S_\beta$ then $uv \in S_{\alpha\beta} = S_\beta$. As S_β is r -Archimedean, statement (4) holds.

Conversely, let statement (4) hold on a semigroup S . Then, by Theorem 2, it follows that S is a right regular band Y of r -Archimedean semigroups S_α , $\alpha \in Y$. Let $a \in S_\alpha$, $b \in S_\beta$. Then by (4) there exists $t \in S_\gamma$ such that $b^n = abt$ whence $\beta = \alpha\beta\gamma = \alpha\beta\alpha\beta\gamma = \alpha\beta\beta = \alpha\beta$. Thus Y is a right zero band and so the semigroup S is a right zero band Y of r -Archimedean semigroups S_α , $\alpha \in Y$. $\square$

Recall that a band B is a *left normal band* if $efg = egf$ for every $e, f, g \in B$.

Theorem 4. *A semigroup S is a left normal band of r -Archimedean semigroups if and only if*

$$(5) \quad (\forall u, v, w \in S)(\exists n \in \mathbb{N}) (uvw)^n \in uvvS.$$

PROOF. Let $S = \bigcup_{\alpha \in Y} S_\alpha$ where Y is a left normal band and S_α are r -Archimedean semigroups for every $\alpha \in Y$. If $u \in S_\alpha$, $v \in S_\beta$, $w \in S_\gamma$ then $uvw \in S_{\alpha\beta\gamma} = S_{\alpha\gamma\beta}$. Since $uvw \in S_{\alpha\gamma\beta}$ and since $S_{\alpha\gamma\beta}$ is r -Archimedean we have that (5) holds.

Conversely, let statement (5) hold on a semigroup S . If $a \in S$ and $x, y \in S^1$ then by (5) for $u = xa$, $v = a$, $w = yxa^2y$ there exists $n \in \mathbb{N}$ such that

$$(xa^2y)^{2n} = (xaayxa^2y)^n \in xayxa^2yaS \subseteq xayS.$$

Also, for $u = xa$, $v = yxayx$, $w = ay$ there exists $m \in \mathbb{N}$ such that

$$(xay)^{3m} = (xayxayxay)^m \in xaayyxayxS \subseteq xa^2yS.$$

By Theorem 1 it follows that S is a band of r -Archimedean semigroups. Now we shall prove that the congruence ϱ defined by (2) is a left normal band congruence on S . Let $a, b, c \in S$ and $x, y \in S^1$. For $u = xa$, $v = b$, $w = cy$ by (5) there exists $n \in \mathbb{N}$ such that $(xabcy)^n \in xacybS$. Hence $(xabcy)^n = xacybs$ for some $s \in S$ and $(xabcy)^{n+1} = xacybsxabcy$. By (5) for $u = xac$, $v = ybsxa$, $w = bcy$ there exists $m \in \mathbb{N}$ such that $(xacybsxabcy)^m \in xacbcyybsxaS$. Now, $(xacybsxabcy)^m = xacbcyyt$ for some $t \in bsxaS$. By (5) for $u = xacb$, $v = cy$, $w = yt$ there exists $p \in \mathbb{N}$ such that $(xacbcyyt)^p \in xacbytcyS \subseteq xacbyS$. Hence

$$(xabcy)^{(n+1)mp} = (xacybsxabcy)^{mp} = (xacbcyyt)^p \in xacbyS.$$

Similarly we prove that there exist $q, r, l \in \mathbb{N}$ such that $(xacby)^{(q+1)rl} \in xabcyS$. Hence $abc\varrho acb$ and ϱ is a left normal band congruence on S . It follows that S is a left normal band of r -Archimedean semigroups. $\square$

Recall that a band B is a *right quasinormal band* if $efg = egfg$ for every $e, f, g \in B$.

Theorem 5. *A semigroup S is a right quasinormal band of r -Archimedean semigroups if and only if*

$$(6) \quad (\forall u, v, w \in S)(\exists n \in \mathbb{N}) (uvw)^n \in uvvwS.$$

PROOF. Let $S = \bigcup_{\alpha \in Y} S_\alpha$ where Y is a right quasinormal band and S_α are r -Archimedean semigroups for each $\alpha \in Y$. If $u \in S_\alpha$, $v \in S_\beta$,

$w \in S_\gamma$, then we have $uvw \in S_{\alpha\beta\gamma} = S_{\alpha\gamma\beta\gamma}$. Since $uwvw \in S_{\alpha\gamma\beta\gamma}$ we have that statement (6) holds.

Conversely, let statement (6) hold on a semigroup S . If $a \in S$, $x, y \in S^1$ then for $u = xa$, $v = a$, $w = yxa^2y$ there exists $n \in \mathbb{N}$ such that

$$(xa^2y)^{2n} = (xaayxa^2y)^n \in xayxa^2yayxa^2yS \subseteq xayS.$$

Also, for $u = xa$, $v = yxayx$, $w = ay$ there exists $m \in \mathbb{N}$ such that

$$(xay)^{3m} = (xayxayxay)^m \in xaayyxayxayS \subseteq xa^2yS.$$

Hence, by Theorem 1 the semigroup S is a band of r -Archimedean semigroups.

Let $a, b, c \in S$, $x, y \in S^1$. By (6), for $u = xa$, $v = b$, $w = cy$, there exists $n \in \mathbb{N}$ such that $(xabcy)^n \in xacybcyS$ and so $(xabcy)^n = xacybcyt$ for some $t \in S$.

Using (6) for $u = xac$, $v = ybcytxacy$, $w = bcyt$, there exists $m \in \mathbb{N}$ such that $(xacybcyt)^{2m} = (xacybcytxacybcyt)^m \in xacbcytybcytxacybcytS \subseteq xacbcyS$. Thus

$$(xabcy)^{2nm} = (xacybcyt)^{2m} \in xacbcyS.$$

Similarly, by (6) for $u = xa$, $v = c$, $w = bcy$ there exists $p \in \mathbb{N}$ such that

$$(xacbcy)^p \in xacbcybcyS \subseteq xacbcyS.$$

Hence, by (2) it follows that $abc\varrho acbc$ whence ϱ is a right quasinormal band congruence on S and S is a right quasinormal band of r -Archimedean semigroups. $\square$

Recall that a band B is a *right seminormal band* if $efg = egefg$ for every $e, f, g \in B$.

Theorem 6. *A semigroup S is a right seminormal band of r -Archimedean semigroups if and only if*

$$(7) \quad (\forall u, v, w \in S)(\exists n \in \mathbb{N})(uvw)^n \in uwS.$$

PROOF. Let $S = \bigcup_{\alpha \in Y} S_\alpha$ where Y is a right seminormal band and S_α is an r -Archimedean semigroup for every $\alpha \in Y$. Then, for $u \in S_\alpha$, $v \in S_\beta$, $w \in S_\gamma$ we have $uvw \in S_{\alpha\beta\gamma} = S_{\alpha\gamma\alpha\beta\gamma}$. Since $uwuvw \in S_{\alpha\gamma\alpha\beta\gamma}$, there exists $n \in \mathbb{N}$ such that

$$(uvw)^n \in uwuvwS \subseteq uwS$$

and so (7) holds.

Conversely, let statement (7) hold on a semigroup S . Let $a \in S$, $x, y \in S^1$. Then for $u = xa$, $v = xayxay$ and $w = ay$ there exists $n \in \mathbb{N}$ such that

$$(xay)^{3n} = (xayxayxay)^n \in xaayS = xa^2yS.$$

Also, by (7) for $u = xa$, $v = a$ and $w = yxa^2y$ there exists $m \in \mathbb{N}$ such that $(xa^2y)^{2m} = (xaayxa^2y)^m \in xayxa^2yS \subseteq xayS$. By Theorem 1 we have that S is a band of r -Archimedean semigroups. Hence, $S = \bigcup_{\alpha \in Y} S_\alpha$, Y is a band and S_α is an r -Archimedean semigroup for all $\alpha \in S$. If $a \in S_\alpha$, $b \in S_\beta$, $c \in S_\gamma$, then $abc \in S_{\alpha\beta\gamma}$ and by (7) there exists $n \in \mathbb{N}$ such that $(abc)^n \in acS$. Now, there exists $t \in S$ such that $(abc)^n = act$. If $t \in S_\delta$ then $\alpha\beta\gamma = \alpha\gamma\delta = \alpha\gamma\alpha\gamma\delta = \alpha\gamma\alpha\beta\gamma$. Hence, Y is a right seminormal band. $\square$

Recall that a band B is a *rectangular band* if $efg = eg$ for every $e, f, g \in B$.

Theorem 7. *A semigroup S is a rectangular band of r -Archimedean semigroups if and only if*

$$(8) \quad (\forall x, y, z \in S)(\exists n \in \mathbb{N}) (xyz)^n \in xzS, (xz)^n \in xyzS.$$

PROOF. Let Y be a rectangular band, $S = \bigcup_{\alpha \in Y} S_\alpha$ and S_α an r -Archimedean semigroup for every $\alpha \in Y$. Then for $x, y, z \in S$ there exists $\alpha, \beta, \gamma \in Y$ such that $x \in S_\alpha$, $y \in S_\beta$, $z \in S_\gamma$ and $xyz \in S_\alpha S_\beta S_\gamma \subseteq S_{\alpha\beta\gamma} = S_{\alpha\gamma}$, $xz \in S_\alpha S_\gamma \subseteq S_{\alpha\gamma}$. Since $S_{\alpha\gamma}$ is an r -Archimedean semigroup, we have that (8) holds.

Conversely, let statement (8) hold on a semigroup S . Let η be the relation on S defined by

$$(9) \quad a\eta b \iff (\exists n \in \mathbb{N}) \quad a^n \in bS, b^n \in aS.$$

From $a^2 \in aS$ it follows that η is a reflexive relation. Clearly, η is a symmetric relation.

Let $a, b, c \in S$ and

$$a\eta b \iff (\exists n \in \mathbb{N}) a^n \in bS, b^n \in aS,$$

$$b\eta c \iff (\exists m \in \mathbb{N}) b^m \in cS, c^m \in bS.$$

For $k = \max\{n, m\}$ we have $a^k \in bS$, $b^k \in aS \cap cS$, $c^k \in bS$. Hence, there exist $u, v, w \in S$ such that $a^k = bu$, $b^k = cv$, $c^k = bw$. Now, by (8) for $x = b$, $z = u$, $y = b^k$ there exists $p \in \mathbb{N}$ such that

$$(10) \quad (a^k)^p = (bu)^p \in bb^k uS \subseteq b^k S \subseteq cS.$$

Similarly, by (8) for $x = b$, $z = w$, $y = b^k$ there exists $q \in \mathbb{N}$ such that

$$(11) \quad (c^k)^q = (bw)^q \in bb^k wS \subseteq b^k S \subseteq aS.$$

From (10) and (11) for $r = \max\{p, q\}$ we have $a^{kr} \in cS$, $c^{kr} \in aS$ and so $a\eta c$. Hence, η is a transitive relation and it follows that η is an equivalence relation.

To show that η is right compatible, let $a, b, c \in S$ be arbitrary elements such that

$$a\eta b \iff (\exists n \in \mathbb{N}) a^n \in bS, b^n \in aS$$

For $x = a$, $z = c$ and $y = a^n$ there exists (by (8)) $p \in \mathbb{N}$ such that $(ac)^p \in aa^n cS$ and so $(ac)^p = aa^n cu$ for some $u \in S$. Now, since $a^n = bv$ for some $v \in S$ we have

$$(12) \quad (ac)^p = aa^n cu = a^n acu = bvacu.$$

From (12) for $x = b$, $y = va$, $z = cu$ there exists $q \in \mathbb{N}$ such that

$$(13) \quad (ac)^{pq} = (bvacu)^q \in bcuS \subseteq bcS.$$

Similarly, for $x = b$, $y = b^n$ and $z = c$ there exists $r \in \mathbb{N}$ such that $(bc)^r \in bb^n cS$ and so $(bc)^r = bb^n cv = b^n bcv$ for some $v \in S$. Now, from $b^n = aw$, for some $w \in S$ we have

$$(14) \quad (bc)^r = b^n bcv = awbcv.$$

From (8) and (14) for $x = a$, $y = wb$ and $z = cv$ there exists $j \in \mathbb{N}$ such that

$$(15) \quad (bc)^{rj} = (awbcv)^j \in acvS \subseteq acS.$$

From (13) and (15) for $i = \max\{pq, rj\}$ we have $a\eta b\eta c$ and so η is right compatible.

From $a\eta b$ we have $a^n = bs$ for some $s \in S$, and by (8) for $x = c$, $y = a^n$, $z = a$ and some $m \in \mathbb{N}$ it follows that $(ca)^m \in ca^n aS = cbsaS \subseteq cbS$. Similarly, $(cb)^k \in caS$ for some $k \in \mathbb{N}$. For $r = \max\{m, k\}$ it follows that $a\eta cb$ and so η is left compatible.

Hence, η is a congruence relation.

From $(a^2)^4 \in aS$ and $a^4 \in a^2 S$ we have $a\eta a^2$ and so η is a band congruence relation.

By (8) we conclude that $abc\eta ac$ for every $a, b, c \in S$. Hence it follows that η is a rectangular band congruence on S .

Let $S = \bigcup_{\alpha \in Y} S_\alpha$, where Y is a rectangular band and S_α are η -classes. If $a, b \in S_\alpha$, then $b^2 \in S_\alpha$ and by (8) there exists $n \in \mathbb{N}$ such that

$a^n \in b^2S$. Now, $a^n = b^2u$ for some $u \in S$. If $u \in S_\beta$, then $a^{n+1} = bbua \in bS_\alpha S_\beta S_\alpha \subseteq bS_{\alpha\beta\alpha} = bS_\alpha$. Hence, S_α is an r -Archimedean semigroup and so the semigroup S is a rectangular band Y of r -Archimedean semigroups S_α . $\square$

Similarly, the semigroup S is a rectangular band of l -Archimedean semigroups if and only if for every $x, y, z \in S$ there exists $n \in \mathbb{N}$ such that $(xyz)^n \in Sxz$, $(xz)^n \in Sxyz$. Now, the semigroup S is a rectangular band of t -Archimedean semigroups if and only if for every $x, y, z \in S$ there exists $n \in \mathbb{N}$ such that $(xyz)^n \in xzS \cap Sxz$, $(xz)^n \in xyzS \cap Sxyx$.

We remark that Theorem 7 can be proved by Theorem 1. It is easy to see that $\eta \subseteq \varrho$ where ϱ is defined by (2) on S^1 and η is a congruence on S .

Example 1. Let S be a semigroup defined by the following Cayley table:

	a	e	f	g	h
a	e	e	f	e	f
e	e	e	f	e	f
f	e	e	f	e	f
g	g	g	h	g	h
h	g	g	h	g	h

Then $S = S_\alpha \cup S_\beta$ where $S_\alpha = \{a, e, f\}$, $S_\beta = \{g, h\}$, $S_\alpha S_\beta S_\alpha \subseteq S_\alpha$, $S_\beta S_\alpha S_\beta \subseteq S_\beta$ and S_α and S_β are r -Archimedean semigroups. In this example the semigroup S is a left zero band of r -Archimedean semigroups.

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