

Some remarks on Rizza–Kähler manifolds

By TADASHI AIKOU (Kagoshima)

Dedicated to the 90th birthday of Professor Lajos Tamássy

Abstract. In the present paper we prove that if an almost complex structure J on a Finsler manifold (M, L) is parallel with respect to the Berwald connection D of (M, L) , then (M, L) is a Berwald space. Furthermore, in this case, the Berwald connection D is induced from the Levi–Civita connection of a Kähler metric on M .

1. Introduction

Let M be an n -dimensional smooth manifold, and $\pi : TM \rightarrow M$ its tangent bundle. We denote by $V := \ker\{d\pi : TTM \rightarrow TM\}$ the *vertical subbundle* over TM . Since the quotient bundle TTM/V is isomorphic to the pull-back bundle π^*TM , we obtain the following short exact sequence of vector bundles:

$$\mathbb{O} \longrightarrow V \xrightarrow{\iota} TTM \xrightarrow{\widetilde{d\pi}} \pi^*TM \longrightarrow \mathbb{O}, \quad (1.1)$$

where $\iota : V \hookrightarrow TTM$ is the inclusion, and $\widetilde{d\pi} := (\pi, d\pi)$.

Let $y \in T_xM$ be a tangent vector at $x \in M$, where $T_xM = \pi^{-1}(x)$ is the tangent space at $x \in M$. Then the pair $v = (x, y)$ denotes a point in TM . Since every subspace of T_vTM complementary to the fibre V_v at $v \in TM$ is mapped isomorphically onto the tangent space $T_{\pi(v)}M$, there is no canonical choice of a subspace H_v complementary to V_v . Thus we shall fix a complementary subspace at each point $v \in TM$. An *Ehresmann connection* for TM is a subbundle $H \subset TTM$ complementary to V .

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Definition 1.1. An Ehresmann connection H of a vector bundle TM is called a *nonlinear connection* for TM if it satisfies the following conditions:

- (1) The distribution $H : TM \ni v \mapsto H_v \subset T_v TM$ is smooth on $TM \setminus \{0_M\}$ and is continuous on the whole of TM , where 0_M is the zero section of TM .
- (2) The distribution H is invariant under the action m of $\mathbb{R}$ on TM defined by $m_\lambda(v) := (x, \lambda \cdot y)$, i.e.,

$$dm_\lambda(H_v) = H_{m_\lambda(v)} \quad (1.2)$$

for any $\lambda \in \mathbb{R}$ and $v = (x, y) \in TM$.

Remark 1.1. If H is smooth on the whole of TM , then it is called *linear*.

Alternatively, a non-linear connection is defined as a V -valued 1-form θ on TM satisfying $\theta(\mathcal{Z}) = \mathcal{Z}$ for any section $\mathcal{Z}$ of V . Thus θ is a splitting of the exact sequence (1.1):

$$\mathbb{O} \longrightarrow V \xrightleftharpoons[\theta]{\iota} TTM \xrightarrow{\widetilde{d\pi}} \pi^* TM \longrightarrow \mathbb{O}.$$

The Ehresmann connection $H = \ker(\theta)$ is also called a *horizontal subbundle* of TTM .

The action m of $\mathbb{R}$ on TM induces the so-called *Liouville vector field* $\mathcal{E}$ by

$$\mathcal{E}_v(f) := \left. \frac{d}{d\lambda} \right|_{\lambda=0} f(m_{e^\lambda}(v)), \quad f \in C^\infty(TM). \quad (1.3)$$

Considering $\mathcal{E}$ as a section of V , we call it the *tautological section* of V .

Let θ be a nonlinear connection for TM . A vector field X in M is parallel along a regular curve $c : [a, b] \rightarrow M$ with respect to θ if it satisfies the ordinary differential equation

$$(X \circ c)^* \theta = 0, \quad (1.4)$$

or, equivalently, its velocity vector field $(X \circ c)'$ is always horizontal, i.e.,

$(X \circ c)'(t) \in H_{(X \circ c)(t)}$ for all $t \in [a, b]$. Equation (1.4) has a unique solution X_v for each initial value $v \in T_{c(a)}M$, on which it depends smoothly. The parallel transport $P_{c(t)} : T_{c(a)}M \rightarrow T_{c(t)}M$ defined by

$$P_{c(t)}(v) = X_v(t) \quad (1.5)$$

has the homogeneity property

$$P_{c(t)}(\lambda \cdot v) = \lambda \cdot P_{c(t)}(v) \quad (1.6)$$

for any $v \in TM_{c(a)}$ and $\lambda \in \mathbb{R}$, where we write $\lambda \cdot v := m_\lambda(v)$ for simplicity. The parallel transport $P_{c(t)}$ is a diffeomorphism between the fibres, but not a linear isomorphism in general. A nonlinear connection is linear if and only if $P_{c(t)}$ is a linear isomorphism between the fibres for every curve $c : [a, b] \rightarrow M$ and all $t \in [a, b]$.

2. Canonical connection

Let in the sequel $A^k(F)$ be the space of all k -forms with values in a vector bundle F . Then, in particular, $A^0(F) = \Gamma(F)$ denotes the space of all smooth sections of F .

If a nonlinear connection θ is specified in TM , then there exists a *partial connection* $\delta : \Gamma(V) \rightarrow \Gamma(V \otimes H^*)$ along $H = \ker(\theta)$ in the bundle V , where H^* is the dual bundle of H . Moreover, any partial connection δ can be extended to a connection $D : \Gamma(V) \rightarrow \Gamma(V \otimes T^*TM)$ so that the diagram

$$\begin{array}{ccc} \Gamma(V) & \xrightarrow{D} & \Gamma(V \otimes T^*TM) \\ & \searrow \delta & \downarrow 1 \otimes p \\ & & \Gamma(V \otimes H^*) \end{array}$$

is commutative, where $p : T^*TM \rightarrow H^*$ is the natural projection and T^*TM is the dual bundle of TTM (see [Ba-Bo]).

We suppose that a nonlinear connection θ is given in TM . Since we do not assume the differentiability of θ over 0_M , the parallel translation P_c along any curve c in M is compatible only with the scalar multiplication, but not with the addition in general. Therefore we can not define a connection ∇ on TM from a nonlinear connection θ in general, however, we can show that any θ induces a connection D on the vertical subbundle V as the extension of a partial connection δ .

A *connection* D in the bundle V is usually defined to be a covariant derivative in V , i.e., as a homomorphism $D : \Gamma(V) \rightarrow A^1(V)$ satisfying the *Leibniz rule*

$$D(f \cdot Z) = df \otimes Z + fDZ$$

for all $f \in C^\infty(TM)$ and $Z \in \Gamma(V)$. We shall now introduce a connection D associated with a given nonlinear connection θ .

Since the vertical subbundle V is isomorphic to the induced bundle π^*TM , any vector field X in M is naturally lifted to a section $X^V \in \Gamma(V)$. The section X^V is defined as the vector field which is tangent to the curve $c(t) = (x, y + tX(x))$ in the fiber T_xM at $t = 0$. The map $T_xM \ni X(x) \mapsto X^V(v) \in V_v$ is an isomorphism. The vector field X^V is called the *vertical lift* of X . In the sequel, we use the superscript V for the vertical lifts of vector fields on M .

On the other hand, for any vector field X in M , there exists a section X^H of H such that $d\pi_v(X^H) = X_{\pi(v)}$ at any point $v \in TM$. The vector field X^H on

the total space TM is called the *horizontal lift* of X . In the sequel, we use the superscript H for the horizontal lifts of vector fields on M .

Since any vector field Y on M is a smooth map $Y : M \rightarrow TM$ such that $\pi \circ Y = \text{id}$, its derivative $dY_x : T_x M \rightarrow T_{Y(x)} TM$ satisfies

$$d\pi \left(dY \left(\frac{dc}{dt} \right) - \left(\frac{dc}{dt} \right)^H \right) = 0$$

for any regular curve c in M . Then it is easy to check that

$$dY \left(\frac{dc}{dt} \right) = \mathcal{L}_{(dc/dt)^H} Y^V + \left(\frac{dc}{dt} \right)^H$$

holds, where $\mathcal{L}_{X^H}$ denotes the Lie derivative by X^H . Then, since $H = \ker(\theta)$, we have

$$(Y \circ c)^* \theta \left(\frac{d}{dt} \right) = \theta \left(\mathcal{L}_{(dc/dt)^H} Y^V + \left(\frac{dc}{dt} \right)^H \right) = \theta \left(\mathcal{L}_{(dc/dt)^H} Y^V \right),$$

and thus Y is parallel vector field on M with respect to θ if and only if

$$\theta \left(\mathcal{L}_{X^H} Y^V \right) = 0$$

for all $X \in \Gamma(TM)$. Hence, it is natural to define a partial connection $\delta : \Gamma(V) \rightarrow \Gamma(V \otimes H^*)$ by

$$\delta_{\mathcal{X}} \mathcal{Z} := \theta \left(\mathcal{L}_{\mathcal{X}} \mathcal{Z} \right) = \theta([\mathcal{X}, \mathcal{Z}]) \quad (2.1)$$

for all $\mathcal{Z} \in \Gamma(V)$ and $\mathcal{X} \in \Gamma(H)$.

Since the vertical subbundle V is relatively flat, the partial connection δ may be extended to a connection D of V so that the covariant derivative along V is flat, i.e.,

$$D_{\mathcal{Z}} X^V = 0 \quad (2.2)$$

for all $\mathcal{Z} \in \Gamma(V)$ and $X \in \Gamma(TM)$.

Definition 2.1. The connection $D : \Gamma(V) \rightarrow \Gamma(V \otimes T^*TM) := A^1(V)$ defined by (2.1) and (2.2) is called the *canonical connection* on V associated with the given nonlinear connection θ .

From the definition of $\mathcal{E}$ and (2.2), we have $D_{\mathcal{Z}} \mathcal{E} = \mathcal{Z}$ for all $\mathcal{Z} \in \Gamma(V)$, and the homogeneity condition (1.2) implies $D_{\mathcal{X}} \mathcal{E} = \theta(\mathcal{L}_{\mathcal{X}} \mathcal{E}) = \theta(-\mathcal{L}_{\mathcal{E}} \mathcal{X}) = 0$ for all $\mathcal{X} \in \Gamma(H)$. Therefore the given nonlinear connection θ is recovered by D .

Proposition 2.1. *The canonical connection D associated with θ satisfies*

$$D\mathcal{E} = \theta \quad (2.3)$$

for the tautological section $\mathcal{E}$ of V .

3. Finsler manifolds and Berwald connections

In the sequel of this paper, we use the chart $(\pi^{-1}(U), (x^i, y^i)_{1 \leq i \leq n})$ in TM induced by a chart $(U, (x^i))_{1 \leq i \leq n}$ in M , where $y^1, \dots, y^n$ are the fibre coordinates in each $T_p M$, $p \in U$.

Definition 3.1. A function $L : TM \rightarrow \mathbb{R}$ is called a (real) *Finsler metric* if it satisfies

- (1) L is continuous on the total space TM , and is smooth on the slit tangent bundle $TM \setminus \{0_M\}$,
- (2) $L(v) \geq 0$ for every $v \in TM$, and the equality holds if and only if $v = 0$,
- (3) $L(\lambda \cdot v) = \lambda L(v)$ for every $v \in TM$ and $\lambda \in \mathbb{R}^+$,
- (4) L is strongly convex, i.e., the Hessian (G_{ij}) defined by

$$G_{ij} = \frac{1}{2} \frac{\partial^2 L^2}{\partial y^i \partial y^j} \quad (3.1)$$

is positive definite at each point of $\pi^{-1}(U)$.

Then the pair (M, L) is called a *Finsler manifold*. The *Minkowski norm* of $v \in TM$ is measured by $\|v\| = L(v)$.

The equation of a geodesic in (M, L) is given by

$$\frac{d^2 x^i}{ds^2} + 2G^i \left(x, \frac{dx}{ds} \right) = 0, \quad (3.2)$$

where s is the arc-length with respect to the Finsler metric L , and

$$G^i = \frac{1}{4} \sum G^{im} \left(\frac{\partial G_{jm}}{\partial x^k} + \frac{\partial G_{mk}}{\partial x^j} - \frac{\partial G_{jk}}{\partial x^m} \right) y^j y^k.$$

Then the velocity vector field of the natural lift $\tilde{c} = (c, c')$ of a geodesic c is given by

$$\tilde{c}' = \sum y^j \left[\frac{\partial}{\partial x^j} - \sum \frac{\partial G^i}{\partial y^j} \left(x, \frac{dx}{ds} \right) \frac{\partial}{\partial y^i} \right] (c, c').$$

We define a nonlinear connection θ so that $\tilde{c}'$ is horizontal, i.e., by

$$\theta = \sum \frac{\partial}{\partial y^i} \otimes \theta^i := \sum \frac{\partial}{\partial y^i} \otimes \left(dy^i + \sum N_j^i(x, y) dx^j \right), \quad (3.3)$$

where the coefficients N_j^i are given by

$$N_j^i := \frac{\partial G^i}{\partial y^j} = \frac{1}{2} \sum G^{im} \left(\frac{\partial G_{jm}}{\partial x^k} + \frac{\partial G_{mk}}{\partial x^j} - \frac{\partial G_{jk}}{\partial x^m} \right) y^k. \quad (3.4)$$

Definition 3.2. The nonlinear connection θ defined by (3.3) is called the *Berwald nonlinear connection* of (M, L) . The canonical connection D associated with the Berwald nonlinear connection θ is called the *Berwald connection* of (M, L) .

The connection forms ω_j^i of D with respect to the local frame field $(\partial/\partial y^i)_{1 \leq i \leq m}$ of V are given by $\omega_j^i = \sum \Gamma_{jk}^i dx^k$ with the coefficients

$$\Gamma_{jk}^i = \frac{\partial N_k^i}{\partial y^j}. \quad (3.5)$$

In fact, from (2.1)

$$D_{(\partial/\partial x^i)^H} \frac{\partial}{\partial y^j} = \theta \left[\left(\frac{\partial}{\partial x^i} \right)^H, \frac{\partial}{\partial y^j} \right] = \sum \frac{\partial N_i^h}{\partial y^j} \frac{\partial}{\partial y^h}.$$

In the case of TM , both V and H are isomorphic to the bundle π^*TM induced from TM via π . Therefore the derivative $d\pi$ of π can also be considered as the projection from $T(TM)$ onto V with $\ker(d\pi) = V$, and thus $d\pi$ can be interpreted as a section of $A^1(V)$ given locally by

$$d\pi = \sum \frac{\partial}{\partial y^i} \otimes dx^i.$$

From (3.4) and (3.5) we obtain

Proposition 3.1. *The Berwald connection D satisfies*

$$Dd\pi \equiv 0. \quad (3.6)$$

Any Finsler metric L defines a Riemannian structure G on the vertical subbundle V by

$$G \left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j} \right) = G_{ij}. \quad (3.7)$$

The homogeneity assumption for L implies $L^2 = G(\mathcal{E}, \mathcal{E})$ for the tautological section $\mathcal{E}$. Then, by the definition of the Berwald nonlinear connection θ , we have $\mathcal{X}(L^2) = 0$ for all $\mathcal{X} \in \Gamma(H)$. Thus L is constant along the horizontal subbundle H , i.e.,

$$\mathcal{X}(L) \equiv 0 \quad (3.8)$$

for all $\mathcal{X} \in \Gamma(H)$.

Since the vertical lift X^V is related to the horizontal lift X^H by

$$X^V = d\pi(X^H),$$

$[X^H, Y^H] - [X, Y]^H \in \Gamma(V)$ implies $d\pi([X^H, Y^H]) = d\pi([X, Y]^H) = [X, Y]^V$, and thus (3.6) implies

$$D_{X^H} Y^V - D_{Y^H} X^V = [X, Y]^V \quad (3.9)$$

for all $X, Y \in \Gamma(TM)$.

4. Landsberg spaces and Berwald spaces

The specific goal of this section is to recall some facts on Landsberg spaces and Berwald spaces which we need later on (see, e.g., [Ai3], [Ai-Ko], [Ic1], [Sz] for details).

Let (M, L) be a Finsler manifold, and let (φ_t) be the local flow generated by a vector field X in M , and (φ_t^H) the flow of the horizontal lift X^H of X with respect to the Berwald nonlinear connection θ . From (3.8), we have $\frac{d}{dt}\big|_{t=0}(\varphi_t^H)^*L = 0$. Therefore φ_t^H preserves the indicatrix $I_x := \{y \in T_x M \mid L(x, y) = 1\}$ for all $x \in M$:

$$I_{\varphi_t(x)} = \varphi_t^H(I_x). \quad (4.1)$$

Since each fibre $V_{(x,y)}$ over $(x, y) \in TM$ is the tangent space $T_y(T_x M)$ of $T_x M$ at $y \in T_x M$, each fibre $T_x M$ is a Riemannian space endowed the metric $G_x := G \upharpoonright T_x M$. A Finsler manifold (M, L) is called a *Landsberg space* if the parallel transport $P_{c(t)}$ along any curve c in M is an isometry from the initial Riemannian space $(T_{c(a)} M, G_{c(a)})$ to $(T_{c(t)} M, G_{c(t)})$ for all $t \in [a, b]$. Thus (M, L) is a Landsberg space if and only if

$$\mathcal{L}_{X^H} G = \frac{d}{dt}\bigg|_{t=0} (\varphi_t^H)^* G = 0 \quad (4.2)$$

for any $X \in \Gamma(TM)$. By the definition of D , this condition is equivalent to

$$D_{X^H} G = 0 \quad (4.3)$$

for all $X \in \Gamma(TM)$ (see [Ai-Ko]).

On the other hand, a Finsler manifold (M, L) is called a *Berwald space* if the parallel translation $P_{c(t)}$ is an isometry between the normed tangent spaces, i.e.,

$$\|v - w\| = \|P_{c(t)}(v) - P_{c(t)}(w)\| \quad (4.4)$$

is satisfied for all $v, w \in T_{c(a)} M$ and $t \in [a, b]$ (see [Ic1]). Then, by a well-known theorem due to SZABÓ [Sz], the Berwald connection D of a Berwald space (M, L) is induced from the Levi-Civita connection ∇^g of a Riemannian metric g on M , i.e.,

$$D_{X^H} Y^V = (\nabla_X^g Y)^V \quad (4.5)$$

for all $X, Y \in \Gamma(TM)$.

Since we have $G(\mathcal{E}, \mathcal{E}) = L^2$, the tautological section $\mathcal{E}$ is a unit vector at every point $y \in I_x$. Further, the gradient vector field of the level hypersurface

$I_x \subset T_x M$ is given by

$$\begin{aligned} \sum G^{im} \frac{\partial L}{\partial y^m} \left(\frac{\partial}{\partial y^i} \right) &= \frac{1}{2L} \sum G^{im} \frac{\partial L^2}{\partial y^m} \left(\frac{\partial}{\partial y^i} \right) \\ &= \frac{1}{L} \sum G^{im} G_{lm} y^l \left(\frac{\partial}{\partial y^i} \right) = \frac{1}{L} \sum y^i \frac{\partial}{\partial y^i} \end{aligned}$$

at each point of I_x . Thus $\mathcal{E}$ may be considered as the outward-pointing unit normal vector field of the indicatrix I_x . Hence, for the volume form $d\mu = \sqrt{\det G} \, dy^1 \wedge \cdots \wedge dy^n$ on each tangential Riemannian space $(T_x M, G_x)$, the $(n-1)$ -form

$$d\mu_I = \iota(\mathcal{E})d\mu = \sum (-1)^{j-1} y^j \sqrt{\det G} \, dy^1 \wedge \cdots \wedge \check{dy}^j \wedge \cdots \wedge dy^n \quad (4.6)$$

defines a volume form of each indicatrix I_x , and the volume $\text{vol}(I_x)$ of I_x is given by $\text{vol}(I_x) = \int_{I_x} d\mu_I$.

The *averaged Riemannian metric* of G is a Riemannian metric g on M defined by

$$g(X, Y) = \frac{1}{\text{vol}(I_x)} \int_{I_x} G(X^V, Y^V) \, d\mu_I, \quad (4.7)$$

and the *averaged connection* of D is a linear connection ∇ on TM defined by

$$g(\nabla_X Y, Z) = \frac{1}{\text{vol}(I_x)} \int_{I_x} G(D_{X^H} Y^V, Z^V) \, d\mu_I \quad (4.8)$$

for all $X, Y, Z \in \Gamma(TM)$, respectively (see [Ma-Ra-Tr-Ze], [To-Et]). Since $\mathcal{E}$ satisfies (2.3), we have

Lemma 4.1 (cf. [Ai-Ko]). *If (M, L) is a Landsberg space, then*

$$\mathcal{L}_{X^H} d\mu_I = 0 \quad (4.9)$$

for every $X \in \Gamma(TM)$, and therefore the volume of the indicatrix I_x is constant.

From Lemma 4.1 we conclude that

$$X \left(\int_{I_x} f d\mu_I \right) = \int_{I_x} \mathcal{L}_{X^H}(f d\mu_I) = \int_{I_x} X^H(f) \, d\mu_I$$

for all $X \in \Gamma(TM)$ and $f \in C^\infty(TM)$ if (M, L) is a Landsberg space. This identity implies that ∇ is compatible with the averaged metric g . Further, (3.9) implies that ∇ is torsion-free.

Theorem 4.1 ([Ai3]). *If (M, L) is a Landsberg space, then the averaged connection ∇ of D is the Levi–Civita connection of the averaged Riemannian metric g of G .*

In particular, if (M, L) is a Berwald space, we have the the following well-known result.

Theorem 4.2. ([Sz], [Vi]) *If (M, L) is a Berwald space, then the Berwald connection D is induced by the Levi–Civita connection ∇^g of the averaged Riemannian metric g of G , i.e., D is given by (4.5) for all $X, Y \in \Gamma(TM)$.*

5. Rizza–Kähler manifolds

In the sequel of this paper we assume that (M, L) is a $2n$ -dimensional Finsler manifold which admits an almost complex structure J , i.e., an endomorphism J of TM such that $J \circ J = -I$, where I is the identity morphism of TM .

Definition 5.1 ([Ri1], [Ic3]). A Finsler metric L is called a *complex Finsler metric* or *Rizza metric* if it satisfies

$$L \circ (aI + bJ)X = \sqrt{a^2 + b^2} L \circ X \quad (5.1)$$

for all $X \in \Gamma(TM)$ and $a, b \in \mathbb{R}$. Then the triplet (M, J, L) is called a *Rizza manifold*.

Example 5.1. Let h be a Hermitian metric on an almost complex manifold (M, J) . For any $X \in \Gamma(TM)$, we put $L \circ X = \sqrt{h(X, X)}$. Then, since $h(JX, X) + h(X, JX) = 0$ is satisfied, it is easily checked that L satisfies (5.1). Thus Rizza manifolds are natural generalizations of Hermitian manifolds.

Let ϕ_θ be the endomorphism of TM defined by $\phi_\theta = \cos \theta \cdot I + \sin \theta \cdot J$ for each $\theta \in \mathbb{R}$. Then we can write assumption (5.1) as $L \circ \phi_\theta X = L \circ X$ for any $\theta \in \mathbb{R}$. By direct calculation, we have $\phi_{\theta_1} \circ \phi_{\theta_2} = \phi_{\theta_1 + \theta_2}$ for all $\theta_1, \theta_2 \in \mathbb{R}$. Using this fact, we obtain

Theorem 5.1 ([Ic3], [Ri1]). *For any Finsler metric $\bar{L}$ on an almost complex manifold (M, J) , the function $L \circ X = \left(\frac{1}{2\pi} \int_0^{2\pi} \bar{L}(\phi_\theta X)^2 d\theta \right)^{1/2}$ defines a Rizza metric on (M, J) .*

The almost complex structure J of TM is lifted to that of V :

$$J^V X^V := (JX)^V \quad (5.2)$$

for any $X \in \Gamma(TM)$. Suppose that J^V is parallel with respect to the Berwald connection D :

$$DJ^V = 0. \quad (5.3)$$

From the definitions of D and J^V , it is obvious that $D_Z J^V = 0$ for all $Z \in \Gamma(V)$. Thus this assumption is equivalent to

$$D_{\mathcal{X}} J^V = 0 \quad (5.4)$$

for all $\mathcal{X} \in \Gamma(H)$. Since the Kähler condition in [Ic4] implies (5.3), the class of Rizza manifolds satisfying (5.3) includes the class of *Kaehlerian Finsler manifolds* in [Ic4]. To distinguish our Kählerity from that of [Ic1] or [Ab-Pa], we use a new terminology:

Definition 5.2. A Rizza manifold (M, J, L) is said to be *Rizza-Kähler* if (5.3) is satisfied.

Remark 5.1. Since the Kähler condition in [Le-Wo] implies the corresponding condition in [Ic4], our Rizza-Kähler manifolds are defined in wider sense than that of [Le-Wo]. A complex manifold M with a normal (a, b, f) -metric L discussed in [Ic-Ha] is an example of Rizza-Kähler manifolds.

The integrability tensor for J is the Nijenhuis tensor field N_J given by $N_J(X, Y) := [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY]$ for all $X, Y \in \Gamma(TM)$. From (3.9), the assumption (5.3) implies

$$\begin{aligned} (N_J(X, Y))^V &= ([X, Y] + J[JX, Y] + J[X, JY] - [JX, JY])^V \\ &= [X, Y]^V + J^V[JX, Y]^V + J^V[X, JY]^V - [JX, JY]^V \\ &= D_{X^H} Y^V - D_{Y^H} X^V + J^V(D_{(JX)^H} Y^V - D_{Y^H}(JX)^V) \\ &\quad + J^V(D_{X^H}(JY)^V - D_{(JY)^H} X^V) - D_{(JX)^H}(JY)^V + D_{(JY)^H}(JX)^V \\ &= D_{X^H} Y^V - D_{Y^H} X^V + J^V D_{(JX)^H} Y^V + D_{Y^H} X^V - D_{X^H} Y^V \\ &\quad - J^V D_{(JY)^H} X^V - J^V D_{(JX)^H} Y^V + J^V D_{(JY)^H} X^V = 0 \end{aligned}$$

for all $X, Y \in \Gamma(TM)$. Thus $N_J \equiv 0$, therefore J is integrable.

Proposition 5.1 ([Ic4]). *If the Berwald connection D satisfies (5.3), then J is integrable, i.e., (M, J) is a complex manifold.*

Remark 5.2. This proposition was first proved by ICHIJYŌ [Ic4] in terms of the *Cartan connection* of (M, L) . The essential fact we need in the proof above is the condition (3.9) of D .

We suppose that (5.3) is satisfied. Since (M, J) is a complex manifold, we identify TM with the holomorphic tangent bundle over (M, J) . Then each fibre $T_x M$ over $x \in M$ is a complex manifold with complex structure J_x . Since $\mathcal{L}_{X^H}(JY)^V, J^V(\mathcal{L}_{X^H}Y^V) \in \Gamma(V)$ for all $X, Y \in \Gamma(TM)$, we obtain $(D_{X^H}J^V)(Y^V) = D_{X^H}(J^VY^V) - J^V(D_{X^H}Y^V) = (\mathcal{L}_{X^H}J^V)Y^V$.

Proposition 5.2. *If (M, J, L) is a Rizza–Kähler manifold, then*

$$\mathcal{L}_{X^H}J^V = 0 \quad (5.5)$$

for all $X \in \Gamma(TM)$.

A real vector field X on a complex manifold (M, J) is real holomorphic if $X^{1,0} := (X - \sqrt{-1}JX)/2$ is a holomorphic vector field on (M, J) . A vector field X is real holomorphic if and only if the flow (φ_t) generated by X is a holomorphic map of (M, J) , i.e., $d\varphi_t \circ J_x = J_{\varphi_t(x)} \circ d\varphi_t$ is satisfied for every $x \in M$. Thus X is real holomorphic if and only if $\mathcal{L}_X J = 0$, i.e., X is an infinitesimal automorphism of J .

Let X be a real holomorphic vector field on a Rizza–Kähler manifold (M, J, L) . Then, since (5.5) implies $d\varphi_t^H \circ J_x^V = J_{\varphi_t(x)}^V \circ d\varphi_t^H$, the flow $\varphi_t^H : T_x M \setminus \{0\} \rightarrow T_{\varphi_t(x)} M \setminus \{0\}$ generated by the horizontal lift X^H of X is a holomorphic map for every $x \in M$.

Theorem 5.2. *If (M, J, L) is a Rizza–Kähler manifold, then (M, L) is a Berwald space.*

PROOF. Let $\varphi_t^H : T_x M \setminus \{0\} \rightarrow T_{\varphi_t(x)} M \setminus \{0\}$ be the flow generated by the horizontal lift X^H of any real holomorphic vector field X . By Proposition 5.2, each φ_t^H is a holomorphic map. Since we are always concerned with M of $\dim_{\mathbb{C}} M \geq 2$, the isolated singularity $\{0\}$ of φ_t^H is removable by Hartogs' theorem (see, e.g., [Hu]), and thus each φ_t^H may be extended to a holomorphic map on the whole of $T_x M$ for every $x \in M$.

Let $\eta^a = y^a + \sqrt{-1}y^{(a)}$ ($1 \leq a \leq m$, $(a) = m+a$) be the complex coordinates on each fiber of the holomorphic tangent bundle over (M, J) naturally induced from the given local complex coordinate system $(z^1, \dots, z^m)$ ($n = 2m$) on M . Denoting by $\mathcal{N}_b^a$ the coefficients of the Berwald nonlinear connection H in the complex coordinate system (z^a, η^a) in TM , the horizontal lifts $(\partial/\partial z^b)^H$ of the members of the local frame field $(\partial/\partial z^b)_{1 \leq b \leq m}$ are given by

$$\left(\frac{\partial}{\partial z^b} \right)^H = \frac{\partial}{\partial z^b} - \sum \mathcal{N}_b^a \frac{\partial}{\partial \eta^a},$$

where the coefficients $\mathcal{N}_b^a$ are holomorphic in $\eta = (\eta^1, \dots, \eta^m)$. Further, the relations between the coefficients N_j^i and $\mathcal{N}_b^a$ are given by $\mathcal{N}_b^a := N_b^a + \sqrt{-1}N_b^{(a)}$ and

$$(N_j^i) = \begin{pmatrix} N_b^a & N_b^{(a)} \\ -N_b^{(a)} & N_b^a \end{pmatrix}, \quad 1 \leq i, j \leq n = 2m.$$

The power series expansions of $\mathcal{N}_b^a(z, \eta)$ with respect to $(\eta^1, \dots, \eta^m)$ are of the form

$$\mathcal{N}_b^a(z, \eta) = \sum_{c_1, \dots, c_m \geq 0} \mathcal{N}_{bc_1 \dots c_m}^a(z) (\eta^1)^{c_1} \dots (\eta^m)^{c_m},$$

and thus

$$\begin{aligned} N_b^a(x, y) + \sqrt{-1}N_b^{(a)}(x, y) \\ = \sum \mathcal{N}_{bc_1 \dots c_m}^a(z) \left(y^1 + \sqrt{-1}y^{(1)}\right)^{c_1} \dots \left(y^m + \sqrt{-1}y^{(m)}\right)^{c_m}. \end{aligned}$$

Since the real coefficients N_j^i satisfy the homogeneity condition $N_j^i \circ m_\lambda = \lambda N_j^i$ for all $\lambda > 0$, the surviving terms in the RHS of the above relation are given by $c_1 + \dots + c_m = 1$:

$$N_b^a(x, y) + \sqrt{-1}N_b^{(a)}(x, y) = \sum \mathcal{N}_{bc}^a(z) \left(y^c + \sqrt{-1}y^{(c)}\right).$$

If we put $\mathcal{N}_{bc}^a(z) = \Gamma_{bc}^a(x) + \sqrt{-1}\Gamma_{bc}^{(a)}(x)$, then we obtain

$$\begin{aligned} \sum \mathcal{N}_{bc}^a(z) \left(y^c + \sqrt{-1}y^{(c)}\right) &= \sum \left(\Gamma_{bc}^a(x)y^c - \Gamma_{bc}^{(a)}(x)y^{(c)}\right) \\ &\quad + \sqrt{-1} \sum \left(\Gamma_{bc}^a(x)y^{(c)} + \Gamma_{bc}^{(a)}(x)y^c\right). \end{aligned}$$

Consequently we have

$$N_b^a = \sum \left(\Gamma_{bc}^a y^c - \Gamma_{bc}^{(a)} y^{(c)}\right), \quad N_b^{(a)} = \sum \left(\Gamma_{bc}^a y^{(c)} + \Gamma_{bc}^{(a)} y^c\right)$$

This shows that the real coefficients N_j^i are of the forms $N_j^i = \sum \Gamma_{jk}^i y^k$, and the coefficients N_j^i of H are polynomials of degree one in $(y^1, \dots, y^n)$. This shows that (M, L) is a Berwald space by SZABÓ's theorem[Sz]. $\square$

6. Kähler metrics associated with Rizza–Kähler metrics

Let (M, J, L) be a Rizza manifold. If the metric G on V defined by (3.7) satisfies the Hermitian condition

$$G(J^V \mathcal{Z}, J^V \mathcal{W}) = G(\mathcal{Z}, \mathcal{W}) \quad (6.1)$$

for all $\mathcal{Z}, \mathcal{W} \in \Gamma(V)$, then L is the norm function $L(x, y) = \sqrt{\sum g_{ij} y^i y^j}$ of certain Riemannian metric $g = \sum g_{ij} dx^i \otimes dx^j$ (see [He], [Ic4]). Thus, in [Ic4], the following Riemannian structure K on V has been introduced:

$$K(\mathcal{Z}, \mathcal{W}) := \frac{1}{2} [G(\mathcal{Z}, \mathcal{W}) + G(J^V \mathcal{Z}, J^V \mathcal{W})], \quad (6.2)$$

where $\mathcal{Z}, \mathcal{W} \in \Gamma(V)$. Obviously, K satisfies the Hermitian condition, but K is never obtained from a Finsler metric. Hence K is a *generalized Finsler structure*. If (M, J, L) is a Rizza–Kähler manifold, then (4.3) and (5.3) show that the Berwald connection D of (M, J, L) satisfies

$$D_{X^H} K = 0 \quad (6.3)$$

for all $X \in \Gamma(TM)$. The aim of this section is to show that D is induced from the Levi–Civita connection of a Kähler metric on (M, J) .

Let (M, J, L) be a Rizza–Kähler manifold. Then, from Theorem 5.2, (M, L) is a Berwald space, i.e., the Berwald connection D is induced from the Levi–Civita connection ∇^g of the averaged Riemannian metric g defined by (4.7). Further, from (4.5), we obtain

$$[(\nabla_X^g J)Y]^V = (D_{X^H} J^V)Y^V = 0$$

for all $X, Y \in \Gamma(TM)$, and thus ∇^g is a complex connection of (M, J) .

Let k be the averaged Riemannian metric of K :

$$k(X, Y) := \frac{1}{\text{vol}(I_x)} \int_{I_x} K(X^V, Y^V) d\mu_I. \quad (6.4)$$

Theorem 6.1. *Let (M, J, L) be a Rizza–Kähler manifold. Then (M, L) is a Berwald space, and its Berwald connection D is induced from the Levi–Civita connection of the averaged Kähler metric k on M .*

PROOF. First we show that the metric k is a Hermitian metric on (M, J) . Indeed, from the definitions of K and k , we have

$$\begin{aligned} k(JX, JY) &= \frac{1}{\text{vol}(I_x)} \int_{I_x} K(J^V X^V, J^V Y^V) d\mu_I \\ &= \frac{1}{\text{vol}(I_x)} \int_{I_x} K(X^V, Y^V) d\mu_I = k(X, Y). \end{aligned}$$

Furthermore (4.5) implies

$$\begin{aligned} k(\nabla_Z X, Y) &= \frac{1}{\text{vol}(I_x)} \int_{I_x} K((\nabla_Z^g X)^V, Y^V) d\mu_I \\ &= \frac{1}{\text{vol}(I_x)} \int_{I_x} K(D_{Z^H} X^V, Y^V) d\mu_I. \end{aligned}$$

Then from (6.3) we obtain

$$\begin{aligned} (\nabla_Z^g k)(X, Y) &= Z(k(X, Y)) - k(\nabla_Z^g X, Y) - k(X, \nabla_Z^g Y) \\ &= \frac{1}{\text{vol}(I_x)} \int_{I_x} [Z^H K(X^V, Y^V) - K(D_{Z^H} X^V, Y^V) - K(X, D_{Z^H} Y^V)] d\mu_I \\ &= \frac{1}{\text{vol}(I_x)} \int_{I_x} (D_{Z^H} K)(X^V, Y^V) d\mu_I = 0. \end{aligned}$$

Since ∇^g is torsion-free, k is a Kähler metric on M . $\square$

Therefore, if (M, J, L) is a Rizza–Kähler manifold, then (M, J) is a Kähler manifold. As is well-known, since a Hopf manifold never admits any Kähler metric, there exists no Rizza–Kähler metric on such a manifold. However, any Hopf manifold admits a locally conformal Kähler metric ([Va]). Hence, in the next section, we shall consider conformal changes of Rizza metrics.

7. Locally conformal Rizza–Kähler manifolds

First define the operator $d_\nabla : L \rightarrow d_\nabla L$ for a Finsler metric L by

$$i(X)d_\nabla L = X^H(L), \quad X \in \Gamma(TM), \quad (7.1)$$

where X^H is the horizontal lift of X with respect to a linear connection ∇ . A Finsler manifold (M, L) is said to be *locally conformal Berwald* (*l.c. Berwald* in short) if there exists a torsion-free linear connection ∇ on TM such that

$$d_\nabla L = \beta \otimes L \quad (7.2)$$

for a closed 1-form β on M ([Ai1], [Ai3]). Such a space is a special type of the so-called *Wagner spaces* (see [Ha-Ic]).

Let (M, J, L) be a Rizza manifold. We consider a conformal change

$$L^\alpha := e^{\sigma_\alpha} L \quad (7.3)$$

of L by a local function σ_α defined on an open subset $U_\alpha \subset M$

Definition 7.1. A Rizza manifold (M, J, L) is called a *locally conformal Rizza–Kähler manifold* (l.c. *Rizza–Kähler manifold* in short) if there exists an open cover $(U_\alpha)_{\alpha \in A}$ of M and a family $(\sigma_\alpha)_{\alpha \in A}$ of functions $\sigma_\alpha : U_\alpha \rightarrow \mathbb{R}$ such that all L^α 's are Rizza–Kähler metrics on U_α .

From Theorem 5.2, the Finsler metric L^α is a Berwald metric on U_α . Since the Berwald connection D^α of L^α satisfies $D^\alpha J^V = 0$, Proposition 5.1 implies that J is integrable, and thus (M, J) is a complex manifold.

For the Hermitian metric K on V defined by (6.2), we denote by K^α the Hermitian metric on $V \upharpoonright \pi^{-1}(U_\alpha)$ obtained by the conformal change $K^\alpha = e^{2\sigma_\alpha} K$. Then the averaged Riemannian metric k^α of K^α is defined by

$$k^\alpha(X, Y) = \frac{1}{\text{vol}(I_x^\alpha)} \int_{I_x^\alpha} K^\alpha(X^V, Y^V) d\mu_{I^\alpha}$$

for all $X, Y \in \Gamma(TM)$, where $I_x^\alpha = e^{-\sigma_\alpha} I_x$ is the indicatrix at $x \in M$ with respect to L^α , and $d\mu_{I^\alpha}$ is the volume form on I_x^α :

$$\begin{aligned} d\mu_{I^\alpha} &= \sum (-1)^{i-1} \sqrt{\det G^\alpha} w^i dw^1 \wedge \cdots \wedge \check{dw}^i \wedge \cdots \wedge dw^n \\ &= e^{n\sigma_\alpha(x)} \sum (-1)^{i-1} \sqrt{\det G} w^i dw^1 \wedge \cdots \wedge \check{dw}^i \wedge \cdots \wedge dw^n \end{aligned}$$

at $w = (w^1, \dots, w^n) \in I_x^\alpha$, where G^α is the metric obtained by the conformal change $G^\alpha = e^{2\sigma_\alpha} G$. Since the isomorphism $\psi_\alpha : (T_x M, G_x) \ni y \rightarrow \psi_\alpha(y) = e^{-\sigma_\alpha} y \in (T_x M, G_x^\alpha)$ is an isometry, we have

$$\begin{aligned} \psi_\alpha^*(d\mu_{I^\alpha}) &= e^{n\sigma_\alpha(x)} \sum (-1)^{i-1} \sqrt{\det G \circ \psi_\alpha} e^{-n\sigma_\alpha(x)} y^i dy^1 \wedge \cdots \wedge \check{dy}^i \wedge \cdots \wedge dy^n \\ &= \sum (-1)^{i-1} \sqrt{\det G} y^i dy^1 \wedge \cdots \wedge \check{dy}^i \wedge \cdots \wedge dy^n = d\mu_I, \end{aligned}$$

which implies $\text{vol}(I_x^\alpha) = \int_{I_x^\alpha} d\mu_{I^\alpha} = \int_{I_x} \psi_\alpha^*(d\mu_{I^\alpha}) = \int_{I_x} d\mu_I = \text{vol}(I_x)$. Thus the averaged Riemannian metric k^α obtained from K^α is given by

$$k^\alpha(X, Y) = \frac{1}{\text{vol}(I_x^\alpha)} \int_{I_x^\alpha} K^\alpha(X^V, Y^V) d\mu_{I^\alpha}$$

$$\begin{aligned}
&= \frac{1}{\text{vol}(I_x)} \int_{I_x} (K^\alpha(X^V, Y^V) \circ \psi_\alpha) d\mu_I \\
&= \frac{e^{2\sigma_\alpha}}{\text{vol}(I_x)} \int_{I_x} K(X^V, Y^V) d\mu_I = e^{2\sigma_\alpha} k(X, Y)
\end{aligned}$$

for all $X, Y \in \Gamma(TM)$, i.e., k^α is given by the conformal change

$$k^\alpha = e^{2\sigma_\alpha} k \quad (7.4)$$

for the Hermitian metric k defined by (6.4). From Theorem 6.1, the metric k^α is a local Kähler metric for all $\alpha \in A$. Since the Kähler form Ω^α of k^α is given by $\Omega^\alpha = e^{2\sigma_\alpha} \Omega$ for the one Ω of k , we obtain $d\Omega = -2d\sigma_\alpha \wedge \Omega$ and, hence $(d\sigma_\alpha - d\sigma_\beta) \wedge \Omega = 0$. By the non-degeneracy of Ω , we have $d\sigma_\alpha = d\sigma_\beta$ on the intersection $U_\alpha \cap U_\beta \neq \emptyset$. Therefore the family $(d\sigma_\alpha)_{\alpha \in A}$ of exact local one-forms $d\sigma_\alpha$ glues up to a global 1-form β_L on M which implies $d\Omega = -2\beta_L \wedge \Omega$. Thus (M, J, k) is an l.c. Kähler manifold ([Va]).

Theorem 7.1. *Let (M, J, L) be an l.c. Rizza–Kähler manifold. Then the associated Hermitian manifold (M, J, k) is an l.c. Kähler manifold.*

Let ∇^α be the averaged connection determined by the local connection D^α with respect to the local metric G^α by the formula (4.8). Since each local connection ∇^α is compatible with the local Kähler metric k^α , we have

$$0 = \nabla^\alpha k^\alpha = e^{2\sigma_\alpha} (2d\sigma_\alpha \otimes k + \nabla^\alpha k),$$

and thus we obtain

$$\nabla^\alpha k = -2\beta_L \otimes k.$$

Therefore ∇^α is the Weyl connection of (M, k) with the Lee form β_L of the conformal class represented by k . The uniqueness of the Weyl connection implies that the local connections ∇^α glue up to a global torsion-free linear connection ∇ . Since each local connection D^α is induced from ∇^α , the connection ∇^α satisfies $d_{\nabla^\alpha} L^\alpha = 0$, i.e., $d_{\nabla} L = -d\sigma_\alpha \otimes L$. Hence we obtain

$$d_{\nabla} L = -\beta_L \otimes L \quad (7.5)$$

on M . Therefore we have

Theorem 7.2. *If (M, J, L) is an l.c. Rizza–Kähler manifold, then the underlying real Finsler manifold (M, L) is l.c. Berwald.*

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TADASHI AIKOU
DEPARTMENT OF MATHEMATICS
AND COMPUTER SCIENCE
GRADUATE SCHOOL OF SCIENCE
AND ENGINEERING
KAGOSHIMA UNIVERSITY
JAPAN

E-mail: aikou@sci.kagoshima-u.ac.jp

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