

Proportionally modular numerical semigroups with embedding dimension three

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Abstract. In this paper we study numerical semigroups, generated by three positive integers, that are the set of solutions of a Diophantine inequality of the form $ax \bmod b \leq cx$. As a consequence, we show that, if these numerical semigroups are irreducible (this is, symmetric or pseudo-symmetric), then they are the set of solutions of a Diophantine inequality of the form $\alpha x \bmod \beta \leq x$.

1. Introduction

Let $\mathbb{N}$ be the set of nonnegative integers. A *numerical semigroup* is a subset S of $\mathbb{N}$ such that it is closed under addition, $0 \in S$ and $\mathbb{N} \setminus S$ is finite. If $A \subseteq \mathbb{N}$, we denote by $\langle A \rangle$ the submonoid of $(\mathbb{N}, +)$ generated by A , this is,

$$\langle A \rangle = \{\lambda_1 a_1 + \cdots + \lambda_n a_n \mid n \in \mathbb{N} \setminus \{0\}, a_1, \dots, a_n \in A, \lambda_1, \dots, \lambda_n \in \mathbb{N}\}.$$

It is well known (see for instance [11]) that $\langle A \rangle$ is a numerical semigroup if and only if $\gcd\{A\} = 1$, where $\gcd$ means greatest common divisor.

Let S be a numerical semigroup and let X be a subset of S . We say that X is a *system of generators* of S if $S = \langle X \rangle$. In addition, if no proper subset of X generates S , then we say that X is a *minimal system of generators* of S . Every

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numerical semigroup admits a unique minimal system of generators. Moreover, such system has finitely many elements (see [1], [11]).

If $n_1 < n_2 < \dots < n_e$ are the elements of the minimal system of generators of a numerical semigroup S , then n_1 , n_2 and e are known as the *multiplicity*, the *ratio*, and the *embedding dimension* of S , and they are denoted by $m(S)$, $r(S)$ and $e(S)$, respectively.

Let m, n be integers such that $n \neq 0$. We denote by $m \bmod n$ the remainder of the division of m by n . A *proportionally modular Diophantine inequality* (see [12]) is an expression of the form

$$ax \bmod b \leq cx \quad (1)$$

where a, b, c are positive integers. We call a, b and c the *factor*, the *modulus*, and the *proportion* of the inequality, respectively. Let $S(a, b, c)$ be the set of integer solutions of (1). Then $S(a, b, c)$ is a numerical semigroup. We say that a numerical semigroup is a *proportionally modular numerical semigroup* (PM-semigroup) if it is the set of integer solutions of a proportionally modular Diophantine inequality.

A *modular Diophantine inequality* (see [13]) is an expression of the form

$$ax \bmod b \leq x, \quad (2)$$

this is, it is a proportionally modular Diophantine inequality with proportion equal to one. A numerical semigroup is a *modular numerical semigroup* (M-semigroup) if it is the set of integer solutions of a modular Diophantine inequality. Therefore, every M-semigroup is a PM-semigroup, but the converse is false. In effect, from Example 26 in [12], we have that the numerical semigroup $\langle 3, 8, 10 \rangle$ is a PM-semigroup, but is not an M-semigroup.

A numerical semigroup is an *irreducible numerical semigroup* if it can not be expressed as an intersection of two numerical semigroups containing it properly. In [9] it is shown that the class of irreducible numerical semigroups is the union of two widely studied classes of numerical semigroups: the *symmetric numerical semigroups* and the *pseudo-symmetric numerical semigroups* (see [1], [2], [5]).

From Theorem 16 in [12] and Theorem 6 in [3] we deduce that the numerical semigroup $\langle n, n+1, \dots, 2n-2 \rangle$ is a symmetric PM-semigroup if n is an integer such that $n \geq 3$. Therefore, there exist irreducible PM-semigroups with arbitrary embedding dimension. However, this result is false for M-semigroups. In fact, as a consequence of the results in [7], [8], we have that every irreducible M-semigroup has embedding dimension less than or equal to three. More precisely, in this paper we will show that the irreducible M-semigroups are just the irreducible PM-semigroups with embedding dimension less than or equal to three.

We summarize the content of this article in the following way. From Remark 24 in [14], we know that, if S is a PM-semigroup, then $\gcd\{r(S), m(S)\} = 1$. For this reason, we will consider m, r integers such that $3 \leq m < r$ and $\gcd\{m, r\} = 1$. In Section 2 we will give in explicit form the elements of the set

$$\text{PM}(m, r) = \{S \mid S \text{ is a PM-semigroup, } m(S) = m, r(S) = r, \text{ and } e(S) = 3\}.$$

Moreover, we will see that $\text{PM}(m, r)$ has cardinality equal to $m + r - \lceil \frac{2r}{m} \rceil - 3$. We will compute positive integers a, b, c such that $S = S(a, b, c)$ for each $S \in \text{PM}(m, r)$.

In Section 3 we will study the set

$$\text{Sy}(\text{PM}(m, r)) = \{S \in \text{PM}(m, r) \mid S \text{ is symmetric}\}.$$

We will compute the cardinality of this set from the divisors of m and r . We will give positive integers a, b such that $S = S(a, b, 1)$ for each $S \in \text{Sy}(\text{PM}(m, r))$.

Finally, Section 4 will be devoted to the study of the set

$$\text{PSy}(\text{PM}(n_1, n_2)) = \{S \in \text{PM}(n_1, n_2) \mid S \text{ is pseudo-symmetric}\}.$$

We will show that, if $m = 3$, then this set has cardinality equal to 1 and, if $m \geq 4$, then the cardinality is equal to the cardinality of $\{x \in \{m, r\} \mid x \text{ is odd}\}$. At the end, we will compute positive integers a, b such that $S = S(a, b, 1)$ for each $S \in \text{PSy}(\text{PM}(m, r))$.

2. PM-semigroups

Let $x_1, x_2, \dots, x_q$ be a sequence of integer numbers. We say that it is arranged in a convex form if one of the following conditions is satisfied,

- (1) $x_1 \leq x_2 \leq \dots \leq x_q$;
- (2) $x_1 \geq x_2 \geq \dots \geq x_q$;
- (3) There exists $h \in \{2, \dots, q-1\}$ such that $x_1 \geq \dots \geq x_h \leq \dots \leq x_q$.

As a consequence of Theorem 31 in [14] (see its proof and Corollary 18 of [14]) we have the next lemma.

Lemma 2.1. *A numerical semigroup S is a PM-semigroup if and only if there exists a convex arrangement $n_1, n_2, \dots, n_e$ of its set of minimal generators that satisfies the following conditions,*

- (1) $\gcd\{n_i, n_{i+1}\} = 1$ for all $i \in \{1, \dots, e-1\}$;
- (2) $(n_{i-1} + n_{i+1}) \equiv 0 \pmod{n_i}$ for all $i \in \{2, \dots, e-1\}$.

In what follows we will suppose that n_1, n_2 are integer numbers such that $3 \leq n_1 < n_2$ and $\gcd\{n_1, n_2\} = 1$. Moreover, to simplify the notation we will use the following sets,

- $A(n_1) = \{2, \dots, n_1 - 1\}$;
- $A(n_1, n_2) = \{\lceil \frac{2n_2}{n_1} \rceil, \dots, n_2 - 1\}$;
- $B(n_1) = \{k \in A(n_1) \text{ such that } k \mid n_1\}$;
- $B(n_1, n_2) = \{t \in A(n_1, n_2) \text{ such that } t \mid n_2\}$.

For the definition of $A(n_1, n_2)$, remember that, if $q \in \mathbb{Q}$ (where $\mathbb{Q}$ is the set of rational numbers), then we denote $\lceil q \rceil = \min\{z \in \mathbb{Z} \mid q \leq z\}$. Moreover, for the definitions of $B(n_1)$ and $B(n_1, n_2)$, if a, b are positive integers, then we denote by $a \mid b$ that a divides b .

Lemma 2.2. *If $k \in A(n_1)$, then $S = \langle n_1, n_2, kn_2 - n_1 \rangle$ is a PM-semigroup with $e(S) = 3$. Moreover, $n_1 < n_2 < kn_2 - n_1$.*

PROOF. It is clear that $n_1 < n_2 < kn_2 - n_1$ because $k \geq 2$ (by hypothesis) and $n_1 < n_2$ (by assumption). In order to give the proof, it is enough to see that $kn_2 - n_1 \notin \langle n_1, n_2 \rangle$. Otherwise, there will exist $\lambda, \mu \in \mathbb{N}$ such that $kn_2 - n_1 = \lambda n_1 + \mu n_2$. Then $(k - \mu)n_2 = (\lambda + 1)n_1$. Because $\gcd\{n_1, n_2\} = 1$, we deduce that $k - \mu \geq n_1$ which is a contradiction with the condition $k \in A(n_1)$. Finally, if we consider the arrangement $n_1, n_2, kn_2 - n_1$ of the set of minimal generators for S , then it is a PM-semigroup as a consequence of Lemma 2.1. $\square$

Lemma 2.3. *If $t \in A(n_1, n_2)$, then $S = \langle n_1, n_2, tn_1 - n_2 \rangle$ is a PM-semigroup with $e(S) = 3$. Moreover, $n_1 < n_2 < tn_1 - n_2$.*

PROOF. Let us have $t \in \mathbb{Z}$. Let us observe that $n_2 < tn_1 - n_2$ if and only if $t > \frac{2n_2}{n_1}$, which is equivalent to $t \geq \lceil \frac{2n_2}{n_1} \rceil$ because $n_1 \geq 3$. Using a similar reasoning as in Lemma 2.2, we prove that $e(S) = 3$. Finally, S is a PM-semigroup by Lemma 2.1 with the arrangement $tn_1 - n_2, n_1, n_2$ of the set of its minimal generators. $\square$

Let us denote the set

$$\text{PM}(n_1, n_2) = \{S \mid S \text{ is a PM-semigroup, } m(S) = n_1, r(S) = n_2, e(S) = 3\}.$$

Theorem 2.4. *$\text{PM}(n_1, n_2)$ is equal to the union of*

$$\text{PM}_1(n_1, n_2) = \{\langle n_1, n_2, kn_2 - n_1 \rangle \mid k \in A(n_1)\}$$

and

$$\text{PM}_2(n_1, n_2) = \{\langle n_1, n_2, tn_1 - n_2 \rangle \mid t \in A(n_1, n_2)\}.$$

PROOF. By applying Lemmas 2.2 and 2.3, $\text{PM}_1(n_1, n_2) \cup \text{PM}_2(n_1, n_2) \subseteq \text{PM}(n_1, n_2)$. In order to prove the other inclusion, let us have $S \in \text{PM}(n_1, n_2)$. Then S is minimally generated by $\{n_1, n_2, n_3\}$ with $n_1 < n_2 < n_3$. From Lemma 2.1, we deduce that $(n_1 + n_3) \equiv 0 \pmod{n_2}$ or $(n_2 + n_3) \equiv 0 \pmod{n_1}$. If $(n_1 + n_3) \equiv 0 \pmod{n_2}$, then $n_3 = kn_2 - n_1$ for an integer $k \geq 2$. Moreover, $k \leq n_1 - 1$ because $n_3 \in \langle n_1, n_2 \rangle$ in other case, and this is a contradiction with the fact that $\{n_1, n_2, n_3\}$ is the minimal system of generators of S . On the other hand, if $(n_2 + n_3) \equiv 0 \pmod{n_1}$, then $n_3 = tn_1 - n_2$ for an integer $t \geq \lceil \frac{2n_2}{n_1} \rceil$. Moreover, $t \leq n_2 - 1$ because $n_3 \notin \langle n_1, n_2 \rangle$. $\square$

Now we compute the cardinality of $\text{PM}(n_1, n_2)$. First, we need a lemma that is the key.

Lemma 2.5. *Let k, t be positive integers such that $k \leq n_1 - 1$ and $t \leq n_2 - 1$. Then $kn_2 - n_1 = tn_1 - n_2$ if and only if $k = n_1 - 1$ and $t = n_2 - 1$.*

PROOF. It is obvious that $kn_2 - n_1 = tn_1 - n_2$ if and only if $(k + 1)n_2 = (t + 1)n_1$. Because $\gcd\{n_1, n_2\} = 1$, $k \leq n_1 - 1$, and $t \leq n_2 - 1$, we deduce that $(k + 1)n_2 = (t + 1)n_1$ if and only if $k = n_1 - 1$ and $t = n_2 - 1$. $\square$

Let us have a set A . We denote by $\#A$ the cardinality of A .

Corollary 2.6. $\#\text{PM}(n_1, n_2) = n_1 + n_2 - \lceil \frac{2n_2}{n_1} \rceil - 3$.

PROOF. From Lemma 2.5, we have that $\text{PM}_1(n_1, n_2) \cap \text{PM}_2(n_1, n_2)$ has cardinality equal to one. By applying Theorem 2.4, we have the conclusion. $\square$

To clarify the previous results, we give an example.

Example 2.1. We want to compute $\text{PM}(5, 7)$. First, from Corollary 2.6, we know that $\#\text{PM}(5, 7) = 5 + 7 - \lceil \frac{14}{5} \rceil - 3 = 6$. Second, by applying Theorem 2.4, we have that the elements of $\text{PM}(5, 7)$ are,

- $S_1 = \langle 5, 7, 2 \cdot 7 - 5 \rangle = \langle 5, 7, 9 \rangle$;
- $S_2 = \langle 5, 7, 3 \cdot 7 - 5 \rangle = \langle 5, 7, 16 \rangle$;
- $S_3 = \langle 5, 7, 4 \cdot 7 - 5 \rangle = \langle 5, 7, 23 \rangle = \langle 5, 7, 6 \cdot 5 - 7 \rangle$;
- $S_4 = \langle 5, 7, 3 \cdot 5 - 7 \rangle = \langle 5, 7, 8 \rangle$;
- $S_5 = \langle 5, 7, 4 \cdot 5 - 7 \rangle = \langle 5, 7, 13 \rangle$;
- $S_6 = \langle 5, 7, 5 \cdot 5 - 7 \rangle = \langle 5, 7, 18 \rangle$.

Let S be a PM-semigroup. By definition, there exist positive integers a, b and c such that $S = S(a, b, c) = \{x \in \mathbb{N} \mid ax \bmod b \leq cx\}$. We finish this section giving, for each $S \in \text{PM}(n_1, n_2)$, a triplet (a, b, c) such that $S = S(a, b, c)$. For this purpose, we introduce some concepts and results.

Let $\mathbb{Q}_0^+$ be the set of nonnegative rational numbers. If $A \subseteq \mathbb{Q}_0^+$, we denote by $\langle A \rangle$ the submonoid of $(\mathbb{Q}_0^+, +)$ generated by A , this is,

$$\langle A \rangle = \{\lambda_1 a_1 + \cdots + \lambda_n a_n \mid n \in \mathbb{N} \setminus \{0\}, a_1, \dots, a_n \in A, \lambda_1, \dots, \lambda_n \in \mathbb{N}\}.$$

It is clear that $S(A) = \langle A \rangle \cap \mathbb{N}$ is a submonoid of $(\mathbb{N}, +)$. If α, β are two rational numbers such that $\alpha < \beta$, then we denote $[\alpha, \beta] = \{x \in \mathbb{Q} \mid \alpha \leq x \leq \beta\}$. The next result is a consequence of Lemmas 12 and 21 in [12].

Lemma 2.7. *Let S be a numerical semigroup. Then S is a PM-semigroup if and only if there exist α, β positive rational numbers such that $S = S([\alpha, \beta])$.*

And this one is consequence of Lemma 6 and Corollary 9 in [12].

Lemma 2.8. *Let a, b, c be positive integers such that $c < a < b$. Then $S(a, b, c) = S([\frac{b}{a}, \frac{b}{a-c}])$.*

Following the ideas in [14], we say that $\frac{a_1}{b_1} < \frac{a_2}{b_2} < \cdots < \frac{a_r}{b_r}$ is a Bézout sequence if $a_1, a_2, \dots, a_r, b_1, b_2, \dots, b_r$ are positive integers such that $a_{i+1}b_i - a_i b_{i+1} = 1$ for all $i \in \{1, 2, \dots, r-1\}$. The following result is Theorem 12 in [14].

Lemma 2.9. *If $\frac{a_1}{b_1} < \frac{a_2}{b_2} < \cdots < \frac{a_r}{b_r}$ is a Bézout sequence, then*

$$\langle a_1, a_2, \dots, a_r \rangle = S\left(\left[\frac{a_1}{b_1}, \frac{a_r}{b_r}\right]\right).$$

In the rest of the paper, we denote by u, v the unique positive integers such that $\frac{n_1}{u} < \frac{n_2}{v}$ is a Bézout sequence and $1 < \frac{n_1}{u}$.

Proposition 2.10. (1) *If $k \in A(n_1)$, then*

$$\langle n_1, n_2, kn_2 - n_1 \rangle = \{x \in \mathbb{N} \mid u(kn_2 - n_1)x \bmod n_1(kn_2 - n_1) \leq kx\}.$$

(2) *If $t \in A(n_1, n_2)$, then*

$$\langle n_1, n_2, tn_1 - n_2 \rangle = \{x \in \mathbb{N} \mid n_2(tu - v)x \bmod n_2(tn_1 - n_2) \leq tx\}.$$

PROOF. (1) It is clear that $\frac{n_1}{u} < \frac{n_2}{v} < \frac{kn_2 - n_1}{kv - u}$ is a Bézout sequence. From Lemma 2.9, we have

$$\begin{aligned} \langle n_1, n_2, kn_2 - n_1 \rangle &= S \left(\left[\frac{n_1}{u}, \frac{kn_2 - n_1}{kv - u} \right] \right) \\ &= S \left(\left[\frac{n_1(kn_2 - n_1)}{u(kv - u)}, \frac{n_1(kn_2 - n_1)}{n_1(kv - u)} \right] \right). \end{aligned}$$

By using Lemma 2.8, we have the conclusion.

(2) The proof is similar to the previous one with the Bézout sequence given by $\frac{tn_1 - n_2}{tu - v} < \frac{n_1}{u} < \frac{n_2}{v}$. $\square$

Example 2.2. Let S_1 be the numerical semigroup that appeared in Example 2.1, this is, $S_1 = \langle 5, 7, 2 \cdot 7 - 5 \rangle$. Let us observe that $\frac{5}{3} < \frac{7}{4} < \frac{2 \cdot 7 - 5}{2 \cdot 4 - 3} = \frac{9}{5}$ is a Bézout sequence. Therefore,

$$S_1 = \langle 5, 7, 9 \rangle = S \left(\left[\frac{5}{3}, \frac{9}{5} \right] \right) = S \left(\left[\frac{45}{27}, \frac{45}{25} \right] \right) = \{x \in \mathbb{N} \mid 27x \bmod 45 \leq 2x\}.$$

3. Symmetric PM-semigroups

Let S be a numerical semigroup. The greatest integer that does not belong to S is called the Frobenius number of S (see [6]) and denoted by $F(S)$. In [9] it is shown that a numerical semigroup S is irreducible if and only if it is maximal (with respect to the inclusion order) in the set of all numerical semigroups with fixed Frobenius number. Therefore, on applying [2], we have that a numerical semigroup S is symmetric (respectively, pseudo-symmetric) if and only if S is irreducible and $F(S)$ is odd (respectively, even).

The next result is Proposition 39 in [14].

Lemma 3.1. *Let S be a numerical semigroup with minimal system of generators $\{m_1, m_2, m_3\}$. Let us suppose that $\gcd\{m_1, m_2\} = \gcd\{m_2, m_3\} = 1$ and $dm_2 = m_1 + m_3$. Then S is symmetric if and only if $d = \gcd\{m_1, m_3\}$.*

In the following result it is shown what elements of $\text{PM}(n_1, n_2)$ are symmetric. We denote $\text{Sy}(\text{PM}(n_1, n_2)) = \{S \in \text{PM}(n_1, n_2) \mid S \text{ is symmetric}\}$.

Proposition 3.2. *$\text{Sy}(\text{PM}(n_1, n_2))$ is equal to the union of*

$$\text{Sy}(\text{PM}_1(n_1, n_2)) = \{\langle n_1, n_2, kn_2 - n_1 \rangle \mid k \in B(n_1)\}$$

and

$$\text{Sy}(\text{PM}_2(n_1, n_2)) = \{\langle n_1, n_2, tn_1 - n_2 \rangle \mid t \in B(n_1, n_2)\}.$$

PROOF. Let us take $k \in A(n_1)$. From Lemmas 2.2 and 3.1, the numerical semigroup $\langle n_1, n_2, kn_2 - n_1 \rangle$ is symmetric if and only if $k = \gcd\{n_1, kn_2 - n_1\}$, but this equality is equivalent to the condition $k|n_1$.

If $t \in A(n_1, n_2)$, we use Lemmas 2.3 and 3.1, and a similar reasoning to the previous one.

By using Theorem 2.4 we finish the proof. $\square$

As a consequence of Proposition 3.2 and Lemma 2.5, we have the following result.

Corollary 3.3. $\# \text{Sy}(\text{PM}(n_1, n_2)) = \#B(n_1) + \#B(n_1, n_2)$.

Again, we give an example to clarify the previous results.

Example 3.1. Let us show that there do not exist symmetric PM-semigroups such that the embedding dimension is 3, the multiplicity is 5, and the ratio is 49. In fact, if $k \in \{2, 3, 4\}$ and $t \in \{\lceil \frac{98}{5} \rceil, \dots, 48\} = \{20, \dots, 48\}$, then $k \nmid 5$ and $t \nmid 49$. By applying Corollary 3.3, we have the statement.

Now, let us compute $\text{Sy}(\text{PM}(10, 21))$. Since $B(10) = \{2, 5\}$ and $B(10, 21) = \{7\}$, from Proposition 3.2, we have that

$$\text{Sy}(\text{PM}(10, 21)) = \{\langle 10, 21, 32 \rangle, \langle 10, 21, 95 \rangle, \langle 10, 21, 49 \rangle\}.$$

In the introduction, we noted that every irreducible PM-semigroup with embedding dimension equal to three is an M-semigroup. In the next result we prove this fact for symmetric PM-semigroups.

Proposition 3.4. (1) If $k \in B(n_1)$, then

$$\langle n_1, n_2, kn_2 - n_1 \rangle = \left\{ x \in \mathbb{N} \mid u \left(n_2 - \frac{n_1}{k} \right) x \mod n_1 \left(n_2 - \frac{n_1}{k} \right) \leq x \right\}.$$

(2) If $t \in B(n_1, n_2)$, then

$$\langle n_1, n_2, tn_1 - n_2 \rangle = \left\{ x \in \mathbb{N} \mid \frac{n_2}{t}(tu - v)x \mod \frac{n_2}{t}(tn_1 - n_2) \leq x \right\}.$$

PROOF. Let us observe that, if d is a common divisor of a , b , c , then the inequalities $ax \mod b \leq cx$ and $\frac{a}{d}x \mod \frac{b}{d} \leq \frac{c}{d}x$ have the same set of solutions. From this fact and Proposition 2.10, we have the proof. $\square$

In [8] it is shown that, if S is a symmetric M-semigroup, then $e(S) \leq 3$. On the other hand, from Lemmas 2.8 and 2.9, $\langle n_1, n_2 \rangle = S\left(\left[\frac{n_1}{u}, \frac{n_2}{v}\right]\right) = S\left(\left[\frac{n_1 n_2}{un_2}, \frac{n_1 n_2}{n_1 v}\right]\right) = \{x \in \mathbb{N} \mid un_2 x \mod n_1 n_2 \leq x\}$. Therefore, $\langle n_1, n_2 \rangle$ is an M-semigroup. Moreover, it is well known (see for instance [2]) that every numerical semigroup with embedding dimension equal to two is symmetric. From all this, we can enunciate the next result.

Proposition 3.5. *S is a symmetric M-semigroup with $m(S) = n_1$ and $r(S) = n_2$ if and only if S is in one of the following three cases,*

- (1) $S = \langle n_1, n_2 \rangle$;
- (2) $S = \langle n_1, n_2, kn_2 - n_1 \rangle$ with $k \in B(n_1)$;
- (3) $S = \langle n_1, n_2, tn_1 - n_2 \rangle$ with $t \in B(n_1, n_2)$.

4. Pseudo-symmetric PM-semigroups

Our purpose is to study the pseudo-symmetric PM-semigroups with embedding dimension equal to three. For this aim, we begin with a series of lemmas.

Let S be a numerical semigroup and let $\{m_1, m_2, m_3\}$ be its minimal system of generators. We define the numbers

$$c_i = \min \{x \in \mathbb{N} \setminus \{0\} \mid xm_i \in \langle \{m_1, m_2, m_3\} \setminus \{m_i\} \rangle\}, \quad i \in \{1, 2, 3\}.$$

The next result is Theorem 10 in [10].

Lemma 4.1. *Let S be a numerical semigroup with embedding dimension 3. Then S is pseudo-symmetric if and only if, for some rearrangement of its generators $\{m_1, m_2, m_3\}$, we have that $c_1m_1 = (c_2 - 1)m_2 + m_3$, $c_2m_2 = (c_3 - 1)m_3 + m_1$, and $c_3m_3 = (c_1 - 1)m_1 + m_2$.*

Lemma 4.2. *Let S be a numerical semigroup with minimal system of generators $\{m_1, m_2, m_3\}$. If $k \in \mathbb{N}$ and $km_2 = m_1 + m_3$, then $k = c_2$.*

PROOF. If $c_2 < k$, then we have that c_2m_2 is either multiple of m_1 or multiple of m_3 . Let us assume that $c_2m_2 = \lambda m_1$, with $\lambda \in \mathbb{N}$. Then $m_1 + m_3 = km_2 = (k - c_2)m_2 + \lambda m_1$. Therefore, $m_3 = (k - c_2)m_2 + (\lambda - 1)m_1$. We conclude that $m_3 \in \langle m_1, m_2 \rangle$, in contradiction with the fact that $\{m_1, m_2, m_3\}$ is a minimal system of generators. $\square$

The following result is Theorem 14 in [7].

Lemma 4.3. *Let t, n be two positive integers such that $t < n$ and t divides n . Then $S = \langle \frac{n}{t}, t + 2, \frac{n+t+2}{2} \rangle$ is a pseudo-symmetric modular numerical semigroup with Frobenius number $n - t - 2$ if and only if $\frac{n}{t}$ is odd and $\gcd\{t + 2, \frac{n}{t}\} = 1$.*

For our purpose, the next result is fundamental.

Proposition 4.4. *The following conditions are equivalent.*

- (1) S is a pseudo-symmetric PM-semigroup with $e(S) = 3$.

- (2) *There exists an arrangement m_1, m_2, m_3 of the minimal generators of S such that m_1 is odd, $\gcd\{m_1, m_2\} = 1$, and $m_3 = \frac{m_1+1}{2}(m_2 - 2) + 1$.*
- (3) *S is a pseudo-symmetric M-semigroup with $e(S) = 3$.*

PROOF. (1) $\Rightarrow$ (2). By applying Lemmas 2.1 and 4.2, we deduce that there exists an arrangement m_1, m_2, m_3 of the minimal generators of S such that $\gcd\{m_1, m_2\} = \gcd\{m_2, m_3\} = 1$ and $c_2 m_2 = m_1 + m_3$. Besides, as S has embedding dimension 3, it is clear that m_1, m_2 and m_3 are greater than or equal to 3. As $c_2 m_2 = m_1 + m_3$, from Lemma 4.1 we deduce that either $c_1 = 2$ or $c_3 = 2$. Let us suppose, without loss of generality, that $c_3 = 2$. By using again Lemma 4.1, we have $c_2 m_2 = m_1 + m_3$ and $2m_3 = (c_1 - 1)m_1 + m_2$. Therefore, $2c_2 m_2 = 2m_1 + 2m_3 = 2m_1 + (c_1 - 1)m_1 + m_2$ and, in consequence, $(2c_2 - 1)m_2 = (c_1 + 1)m_1$. Since $\gcd\{m_1, m_2\} = 1$ and $1 \leq c_1 \leq m_2$, we have $c_1 + 1 = m_2$ and $m_1 = 2c_2 - 1$. Thus m_1 is odd. Moreover, $m_3 = c_2 m_2 - m_1 = c_2(c_1 + 1) - 2c_2 + 1 = c_2(c_1 - 1) + 1 = \frac{m_1+1}{2}(m_2 - 2) + 1$.

(2) $\Rightarrow$ (3). Let $t = m_2 - 2$ and $n = m_1(m_2 - 2)$. Then $\frac{n}{t} = m_1$ is odd and $\gcd\{t + 2, \frac{n}{t}\} = \gcd\{m_2, m_1\} = 1$. Thus, in view of Lemma 4.3, we have that $S = \langle m_1, m_2, m_3 \rangle = \langle \frac{n}{t}, t + 2, \frac{n+t+2}{2} \rangle$ is a pseudo-symmetric modular numerical semigroup. To conclude the proof it suffices now to show that S has embedding dimension 3. However, we can deduce it by the assert that all numerical semigroups with embedding dimension 2 are symmetric (see [4]). Therefore, these ones are not pseudo-symmetric.

(3) $\Rightarrow$ (1). It is obvious. $\square$

The next result gives us information about the arrangement of m_1, m_2, m_3 in the previous proposition.

Lemma 4.5. *Let m_1, m_2, m_3 be integers such that they are greater than or equal to 4 and $m_3 = \frac{m_1+1}{2}(m_2 - 2) + 1$. Then $m_3 = \max\{m_1, m_2, m_3\}$.*

PROOF. It is easy to see that $m_3 = \frac{m_1+1}{2}(m_2 - 2) + 1 \geq \frac{m_1+1}{2}2 + 1$. Therefore, $m_3 \geq m_1$. In the same way, $m_3 = \frac{m_1+1}{2}(m_2 - 2) + 1 \geq 2(m_2 - 2) + 1$, and $m_3 \geq m_2$. $\square$

From Theorem 7 in [9], we deduce the next result.

Lemma 4.6. *S is a pseudo-symmetric numerical semigroup with $m(S) = e(S) = 3$ if and only if S is minimally generated by $\{3, x + 3, 2x + 3\}$, where x is a positive integer such that $x \not\equiv 0 \pmod{3}$.*

We denote $\text{PSy}(\text{PM}(n_1, n_2)) = \{S \in \text{PM}(n_1, n_2) \mid S \text{ is pseudo-symmetric}\}$.

Proposition 4.7. $S \in \text{PSy}(\text{PM}(n_1, n_2))$ if and only if $(n_1$ is odd and $S = \langle n_1, n_2, \frac{n_1+1}{2}n_2 - n_1 \rangle)$ or $(n_2$ is odd, $n_1 \geq 4$, and $S = \langle n_1, n_2, \frac{n_2+1}{2}n_1 - n_2 \rangle)$.

PROOF. If $k \in A(n_1)$, from Lemma 2.2, we know that $\langle n_1, n_2, kn_2 - n_1 \rangle$ is a PM-semigroup with $e(S) = 3$ and $n_1 < n_2 < kn_2 - n_1$. If $n_1 = 3$, by applying Lemma 4.6, we deduce that $\langle n_1, n_2, kn_2 - n_1 \rangle$ is pseudo-symmetric if and only if $n_2 - 3 = kn_2 - 3 - n_2$, and this equality is equivalent to $k = 2 = \frac{n_1+1}{2}$.

If $n_1 \geq 4$, from Proposition 4.4 and Lemma 4.5, we deduce that $\langle n_1, n_2, kn_2 - n_1 \rangle$ is pseudo-symmetric if and only if n_1 is odd and $kn_2 - n_1 = \frac{n_1+1}{2}(n_2 - 2) + 1$ or n_2 is odd and $kn_2 - n_1 = \frac{n_2+1}{2}(n_1 - 2) + 1$. The first case is possible if and only if $k = \frac{n_1+1}{2}$. The second case is not possible because, if $kn_2 - n_1 = \frac{n_2+1}{2}(n_1 - 2) + 1$, then $3n_1 = (2k - n_1 + 2)n_2$. Because $\gcd\{n_1, n_2\} = 1$, we have that 3 is a multiple of n_2 , but $n_2 \geq 4$.

If $t \in A(n_1, n_2)$, from Lemma 2.3, we know that $\langle n_1, n_2, tn_1 - n_2 \rangle$ is a PM-semigroup with $e(S) = 3$ and $n_1 < n_2 < tn_1 - n_2$. If $n_1 = 3$, again from Lemma 4.6, we deduce that $\langle n_1, n_2, tn_1 - n_2 \rangle$ is pseudo-symmetric if and only if $t = n_2 - 1$. Therefore, $\langle n_1, n_2, (n_2 - 1)n_1 - n_2 \rangle = \langle 3, n_2, 2n_2 - 3 \rangle = \langle n_1, n_2, \frac{n_1+1}{2}n_2 - n_1 \rangle$.

If $n_1 \geq 4$, again from Proposition 4.4 and Lemma 4.5, $\langle n_1, n_2, tn_1 - n_2 \rangle$ is pseudo-symmetric if and only if n_1 is odd and $tn_1 - n_2 = \frac{n_1+1}{2}(n_2 - 2) + 1$ or n_2 is odd and $tn_1 - n_2 = \frac{n_2+1}{2}(n_1 - 2) + 1$. Now, the second case is possible if and only if $t = \frac{n_2+1}{2}$. The first case is not possible because, if $tn_1 - n_2 = \frac{n_1+1}{2}(n_2 - 2) + 1$, then $3n_2 = (2t - n_2 + 2)n_1$. Because $\gcd\{n_1, n_2\} = 1$, now we have that 3 is a multiple of n_1 , but $n_1 \geq 4$.

By applying Theorem 2.4, we finish the proof. $\square$

As an immediate consequence of the previous proposition we have the next result.

Corollary 4.8. (1) If $n_1 = 3$, then $\#\text{PSy}(\text{PM}(n_1, n_2)) = 1$.

(2) If $n_1 \geq 4$, then $\#\text{PSy}(\text{PM}(n_1, n_2)) = \#\{x \in \{n_1, n_2\} \mid x \text{ is odd}\}$.

Because $\gcd\{n_1, n_2\} = 1$, then $\#\{x \in \{n_1, n_2\} \mid x \text{ is odd}\} \geq 1$. Therefore, $\text{PSy}(\text{PM}(n_1, n_2))$ is always nonempty.

Example 4.1. From Proposition 4.7, it is easy to check that

- $\text{PSy}(\text{PM}(5, 7)) = \{\langle 5, 7, 16 \rangle, \langle 5, 7, 13 \rangle\}$;
- $\text{PSy}(\text{PM}(5, 6)) = \{\langle 5, 6, 13 \rangle\}$;
- $\text{PSy}(\text{PM}(6, 7)) = \{\langle 6, 7, 17 \rangle\}$.

From Proposition 4.4, we know that every pseudo-symmetric PM-semigroup with embedding dimension equal to three is an M-semigroup. Therefore, for each numerical semigroup $S \in \text{PSy}(\text{PM}(n_1, n_2))$ there exist a, b positive integers such that $S = S(a, b, 1) = \{x \in \mathbb{N} \mid ax \bmod b \leq x\}$. The next proposition give us a new proof of this fact. First we introduce a lemma which is Theorem 10 in [7].

Lemma 4.9. *Let a, b be integers such that $2 \leq a < b$ and $\gcd\{a-1, b\} = 1$. Let $d = \gcd\{a, b\}$. Then $S(a, b, 1)$ is a pseudo-symmetric numerical semigroup if and only if $0 < a(d+2) \bmod b < d+2$. Moreover, if this is the case, then $S(a, b, 1) = \langle \frac{b}{d}, d+2, \frac{b+d+2}{2} \rangle$.*

Now we are in conditions to show the announced result.

Proposition 4.10. (1) *If n_1 is odd, then*

$$\left\langle n_1, n_2, \frac{n_1+1}{2}n_2 - n_1 \right\rangle = \{x \in \mathbb{N} \mid u(n_2-2)x \bmod n_1(n_2-2) \leq x\}.$$

(2) *If $n_1 \geq 4$ and n_2 is odd, then*

$$\left\langle n_1, n_2, \frac{n_2+1}{2}n_1 - n_2 \right\rangle = \{x \in \mathbb{N} \mid (n_2-v)(n_1-2)x \bmod n_2(n_1-2) \leq x\}.$$

PROOF. (1) Let us have $a = (n_2-2)u$, $b = (n_2-2)n_1$. Because $\gcd\{u, n_1\} = 1$, then $\gcd\{a, b\} = n_2-2$. Let us see that $\gcd\{a-1, b\} = 1$. Let us have $p = \gcd\{a-1, b\}$. Then $p \mid ((n_2-2)u-1)$ and $p \mid ((n_2-2)n_1)$. Therefore, $p \mid n_1$. Because $(n_2-2)u-1 = un_2-2u-1 = vn_1+1-2u-1 = vn_1-2u$, then $p \mid n_1$ and $p \mid (vn_1-2u)$. Therefore, $p \mid n_1$ and $p \mid (2u)$. Applying that n_1 is odd and $\gcd\{n_1, u\} = 1$, we have $p = 1$.

Now, because $(n_2-2)un_2 \bmod ((n_2-2)n_1) = (n_2-2)(un_2 \bmod n_1) = n_2-2$ and $0 < n_2-2 < n_2$, applying Lemma 4.9, we have that

$$\left\langle n_1, n_2, \frac{(n_2-2)n_1 + n_2}{2} \right\rangle = \{x \in \mathbb{N} \mid u(n_2-2)x \bmod n_1(n_2-2) \leq x\}.$$

In order to conclude, we must note that $\frac{(n_2-2)n_1 + n_2}{2} = \frac{n_1+1}{2}n_2 - n_1$.

(2) If we consider $a = (n_1-2)(n_2-v)$ and $b = (n_1-2)n_2$, the proof of this case is analogous to the previous one. Therefore, we omit it. $\square$

If S is a pseudo-symmetric M-semigroup, in [7] it is proven that $e(S) = 3$. Therefore, we can give the next result.

Proposition 4.11. *S is a pseudo-symmetric M-semigroup with $m(S) = n_1$ and $r(S) = n_2$ if and only if one of following cases holds,*

- (1) n_1 is odd and $S = \langle n_1, n_2, \frac{n_1+1}{2}n_2 - n_1 \rangle$;
- (2) n_2 is odd, $n_1 \geq 4$ and $S = \langle n_1, n_2, \frac{n_2+1}{2}n_1 - n_2 \rangle$.

In the introduction we stated that the irreducible M-semigroups are just the irreducible PM-semigroups with embedding dimension less than or equal to three. As a consequence of Proposition 3.4, the commentary after it, and Proposition 4.4 (or Proposition 4.10), we show the above mentioned result.

Proposition 4.12. *Let S be an irreducible numerical semigroup such that $e(S) \leq 3$. Then S is a PM-semigroup if and only if it is an M-semigroup.*

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