

CR-submanifolds of a nearly cosymplectic manifold

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0. Introduction

Among all submanifolds of Kählerian manifolds, CR-submanifolds have been intensively studied from different points of view. Since A. BEJANCU [1] introduced the concept, several important results have been obtained, some of them being brought together in [2].

It is the purpose of the present paper to consider and study the concept of CR-submanifold in case of a nearly cosymplectic manifold. In the first section we present some general formulas and basic results from the theory of submanifolds in order to use them in the next sections. In the second section we obtain necessary and sufficient conditions for the integrability of distributions defined on a CR-submanifold of a nearly cosymplectic manifold. In the third section we are dealing with totally contact umbilical CR-submanifolds of a nearly cosymplectic manifold $\tilde{M}$. More precisely, we prove that a totally contact umbilical CR-submanifold of $\tilde{M}$ is totally contact geodesic provided the dimension of the anti-invariant distribution is greater than 1 or the invariant distribution is autoparallel. Finally, in the fourth section we find theorems of decomposition for totally contact geodesic CR-submanifolds of $\tilde{M}$.

1. Preliminaries

Let $\tilde{M}$ be a real $2n + 1$ -dimensional differentiable manifold and f , ξ , η be a tensor field of type (1,1), a vector field and a 1-form on $\tilde{M}$, respectively, satisfying

$$(1.1) \quad f^2 = -I + \eta \otimes \xi; \quad \eta(\xi) = 1; \quad f(\xi) = 0; \quad \eta \odot f = 0,$$

where I is the identity on the tangent bundle $T\tilde{M}$ of $\tilde{M}$.

Then, following BLAIR [4] we say that $\tilde{M}$ is an *almost contact manifold* and (f, ξ, η) is the *almost contact structure* on $\tilde{M}$.

Throughout the paper, all manifolds and maps are differentiable of class C^∞ . We denote by $\mathfrak{A}(\tilde{M})$ the algebra of the differentiable functions on $\tilde{M}$ and by $\Gamma(E)$ the $\mathfrak{A}(\tilde{M})$ -module of the sections of the vector bundle E over $\tilde{M}$.

Now, we suppose that there exists a Riemannian metric g on $\tilde{M}$ which satisfies

$$g(fX, fY) = g(X, Y) - \eta(X)\eta(Y), \quad \forall X, Y \in \Gamma(T\tilde{M}).$$

This is equivalent with

$$(1.2) \quad g(fX, Y) + g(X, fY) = 0, \quad \forall X, Y \in \Gamma(T\tilde{M}).$$

In this case, we say that (f, ξ, η, g) is an *almost contact metric structure* and $\tilde{M}$ is an *almost contact metric manifold*.

D.E. BLAIR has introduced in [4] the notion of a nearly cosymplectic structure as follows: an almost contact metric structure (f, ξ, η, g) is a *nearly cosymplectic* structure if and only if

$$(1.3) \quad (\tilde{\nabla}_X f)Y + (\tilde{\nabla}_Y f)X = 0, \quad \forall X, Y \in \Gamma(T\tilde{M}),$$

where $\tilde{\nabla}$ is the Levi-Civita connection on $\tilde{M}$.

It is easy to see (cf. [4], p. 84) that on a nearly cosymplectic manifold $\tilde{M}$, ξ is a Killing vector field, that is

$$(1.4) \quad g(\tilde{\nabla}_X \xi, Y) + g(\tilde{\nabla}_Y \xi, X) = 0, \quad \forall X, Y \in \Gamma(T\tilde{M}).$$

Let $\tilde{M}$ be an almost contact metric manifold and M be an m -dimensional submanifold of $\tilde{M}$ such that ξ is tangent to M . We say that M is a CR-submanifold of $\tilde{M}$ if there exist two maximal distributions D and $D^\perp$ such that

$$TM = D \oplus D^\perp \oplus \{\xi\},$$

and

$$(1.5) \quad fX \in \Gamma(D); fY \in \Gamma(TM^\perp), \quad \forall X \in \Gamma(D); Y \in \Gamma(D^\perp),$$

where $TM^\perp$ is the normal bundle of M , and $\{\xi\}$ is the 1-dimensional distribution spanned by ξ . Thus the distributions D and $D^\perp$ are invariant and anti-invariant respectively on M . It follows that the distributions $\{\xi\}$, D and $D^\perp$ are mutually orthogonal on each other.

Remark 1.1. The above definition has been given by A. BEJANCU and N. PAPAGHIUC [3] for semi-invariant submanifolds of a Sasakian manifold. Several results for this case can be found in [1]-[3], [7].

We denote by P and Q the projection morphisms of TM to D and $D^\perp$ and obtain

$$(1.6) \quad X = PX + QX + \eta(X)\xi, \quad \forall X \in \Gamma(TM).$$

Now, we define the tensor field t of type (1.1) and the $\Gamma(TM^\perp)$ -valued 1-form ω as follows:

$$(1.7) \quad tX = fPX \text{ and } \omega Y = fQY, \quad \forall X \in \Gamma(D), Y \in \Gamma(D^\perp).$$

Finally, for any $N \in \Gamma(TM^\perp)$, we decompose the vector field fN :

$$(1.8) \quad fN = BN + CN,$$

where BN and CN are the tangent part and the normal part of fN , respectively. It is easy to check that t defines an f -structure on TM and C defines an f -structure on $TM^\perp$ in the sense of K. YANO [8].

Next we define the torsion tensor of f or the *Nijenhuis tensor* of f by

$$(1.9) \quad N_f(X, Y) = [fX, fY] + f^2[X, Y] - f[X, fY] - f[fX, Y], \quad \forall X, Y \in \Gamma(\tilde{M}).$$

We recall the Gauss and Weingarten formulas, respectively:

$$(1.10) \quad \begin{aligned} \tilde{\nabla}_X Y &= \nabla_X Y + h(X, Y); \\ \tilde{\nabla}_X N &= -A_N X + \nabla_N^\perp X, \quad \forall X, Y \in \Gamma(TM), N \in \Gamma(TM^\perp), \end{aligned}$$

where ∇ and $\nabla^\perp$ are the induced connections on TM and $TM^\perp$, respectively, A_N is the *shape operator* with respect to the section N and h is the *second fundamental form* of M . Then we have

$$(1.11) \quad g(h(X, Y), N) = g(A_N X, Y), \quad \forall X, Y \in \Gamma(TM), N \in \Gamma(TM^\perp).$$

Also we define the *fundamental 2-form* Φ of $\tilde{M}$ by

$$(1.12) \quad \Phi(X, Y) = g(X, fY), \quad \forall X, Y \in \Gamma(\tilde{M}).$$

For any $X, Y, Z, W \in \Gamma(\tilde{M})$ we recall the curvature tensor $\tilde{K}$ of the Levi-Civita connection $\tilde{\nabla}$ as

$$(1.13) \quad \tilde{K}(X, Y)Z = \tilde{\nabla}_X \tilde{\nabla}_Y Z - \tilde{\nabla}_Y \tilde{\nabla}_X Z - \tilde{\nabla}_{[X, Y]} Z.$$

We also recall the Gauss equation

$$(1.14) \quad \begin{aligned} &\tilde{K}(X, Y, Z, W) \\ &= K(X, Y, Z, W) + g(h(X, Z), h(Y, W)) - g(h(X, W), h(Y, Z)), \end{aligned}$$

for any $X, Y, Z, W \in \Gamma(TM)$, where $\tilde{K}(X, Y, Z, W) = g(\tilde{K}(X, Y)Z, W)$, and K is the curvature tensor on M . The Codazzi equation is given by

$$(\tilde{K}(X, Y)Z)^\perp = (\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z), \quad \forall X, Y \in \Gamma(\tilde{M}),$$

where $(\tilde{K}(X, Y)Z)^\perp$ is the normal component of $\tilde{K}(X, Y)Z$.

Now for each plane π spanned by orthonormal vectors X and Y in the tangent space $T_x M$, $x \in M$, we define the sectional curvature $K(\pi)$ by

$$(1.15) \quad K(\pi) = K_M(X \wedge Y) = g(K(X, Y)Y, X).$$

Next we have

Proposition 1.1. *Let $\tilde{M}$ be a nearly cosymplectic manifold. The Nijenhuis tensor of f is given by*

$$(1.16) \quad N_f(X, Y) = 4f((\tilde{\nabla}_Y f)X) + 2d\eta(X, Y)\xi + \eta(Y)\tilde{\nabla}_X \xi - \eta(X)\tilde{\nabla}_Y \xi, \quad \forall X, Y \in \Gamma(T\tilde{M}).$$

PROOF. Taking account of the fact that $\tilde{\nabla}$ is a torsion-free connection on $\tilde{M}$ and by using (1.1) and (1.9) we deduce:

$$(1.17) \quad N_f(X, Y) = (\tilde{\nabla}_X f)Y - (\tilde{\nabla}_Y f)X + f((\tilde{\nabla}_Y f)X - (\tilde{\nabla}_X f)Y), \quad \forall X, Y \in \Gamma(T\tilde{M}).$$

On the other hand, from (1.3) we infer

$$(1.18) \quad (\tilde{\nabla}_X f)Y = -(\tilde{\nabla}_Y f)X = f((\tilde{\nabla}_Y \eta)X) - ((\tilde{\nabla}_Y \eta)X)\xi - \eta(X)\tilde{\nabla}_Y \xi.$$

From (1.17) and (1.18) follows our assertion. $\square$

By using (1.4) and Proposition 1.1 we obtain

Corollary 1.1. *Let $\tilde{M}$ be a nearly cosymplectic manifold. Then we have*

$$(1.19) \quad N_f(\xi, Y) = 3\tilde{\nabla}_Y \xi, \quad \forall Y \in \Gamma(T\tilde{M}).$$

By straightforward calculation, and by using (1.19) we infer

$$(1.20) \quad \eta([X, Y]) = \frac{2}{3}g(X, N_f(\xi, Y)), \quad \forall X, Y \in \Gamma(D \oplus D^\perp).$$

2. Integrability of distributions on a CR-submanifold of a nearly cosymplectic manifold

The purpose of this paragraph is to find necessary and sufficient conditions for the integrability of distributions on a CR-submanifold of a nearly cosymplectic manifold. We recall that some related results have been obtained by S. IANUS [6]. First, by using (1.3), (1.10), (1.17) and (1.18), we infer

Proposition 2.1. *Let M be a CR-submanifold of a nearly cosymplectic manifold $\tilde{M}$. Then we have*

$$(2.1) \quad \begin{aligned} h(X, fY) - h(fX, Y) &= \frac{1}{2} \{fN_f(X, Y) + \eta(X)f\tilde{\nabla}_Y\xi - \\ &\eta(Y)f\tilde{\nabla}_X\xi\} + f([X, Y]) + 2\eta((\tilde{\nabla}_X f)Y)\xi + \nabla_Y fX - \nabla_X fY, \\ &\forall X, Y \in \Gamma(D \oplus (\xi)). \end{aligned}$$

Theorem 2.1. *Let M be a CR-submanifold of a nearly cosymplectic manifold $\tilde{M}$. The distribution D is integrable if and only if the following conditions are satisfied:*

$$(2.2) \quad h(X, fY) = h(fX, Y),$$

$$(2.3) \quad N_f(X, Y)^\top \in \Gamma(D),$$

and

$$(2.4) \quad N_f(\xi, Y)^\top \in \Gamma(D^\perp),$$

for any $X, Y \in \Gamma(D)$, where $N_f(\xi, Y)^\top$ is the tangent part of $N_f(\xi, Y)$.

PROOF. Suppose that D is integrable. Then from (1.9) it follows that for any $X, Y \in \Gamma(D)$, we have (2.3). Next, from (2.1) and (2.3) follows (2.2). Finally by using (1.19) and (1.20) we deduce (2.4). Conversely, suppose that (2.2), (2.3) and (2.4) are true. Then, by using (2.1), (2.2) and (2.3) we obtain

$$(2.5) \quad \begin{aligned} g([X, Y], fV) &= -g(f[X, Y], V) = 0, \\ &\forall X, Y \in \Gamma(D), V \in \Gamma(fD^\perp). \end{aligned}$$

On the other hand, from (1.20), by using (2.4) there follows $\eta([X, Y]) = 0$. Hence D is integrable. $\square$

Remark 2.1. From (2.1) it follows that (2.2) is equivalent with

$$(2.6) \quad g(h(X, fY) - h(fX, Y), fZ) = 0, \quad \forall X, Y \in \Gamma(D), Z \in \Gamma(D^\perp).$$

Corollary 2.1. *Let M be a CR-submanifold of a nearly cosymplectic manifold $\tilde{M}$. Then the distribution D is integrable if and only if (2.2) is satisfied, and*

$$(2.7) \quad (\tilde{\nabla}_Z f)Y \in \Gamma(D^\perp \oplus \{\xi\}), \quad \forall Y \in \Gamma(D), Z \in \Gamma(D^\perp \oplus \{\xi\}).$$

PROOF. Let $X, Y \in \Gamma(D)$, $W \in \Gamma(D^\perp)$. Then by using (1.3) (1.16) and (1.18) we deduce

$$(2.8) \quad g(N_f(fX, Y), W) = 4g(f((\tilde{\nabla}_Y f)fX), W) = 4g((\tilde{\nabla}_W f)Y, X).$$

By straightforward calculation we infer

$$(2.9) \quad g(N_f(\xi, Y), fX) = 3g(f((\tilde{\nabla}_Y f)\xi), fX) = -3g((\tilde{\nabla}_\xi f)Y, X).$$

Finally, by using Theorem 2.1, (2.8) and (2.9) we obtain the assertion $\square$

From Corollary 2.1 we obtain

Corollary 2.2. *Let M be a CR-submanifold of a nearly cosymplectic manifold $\tilde{M}$. Then the distribution D is integrable if and only if (2.2) is satisfied, and*

$$(2.10) \quad N_f(Y, Z) \in \Gamma(fD^\perp \oplus \{\xi\}), \quad \forall Y \in \Gamma(D), \quad Z \in \Gamma(D^\perp).$$

Now we recall

Proposition 2.2 (S. IANUS [6]). *Let M be a CR-submanifold of a nearly cosymplectic manifold $\tilde{M}$. The distribution $D \oplus \{\xi\}$ is integrable if and only if*

$$(2.11) \quad h(X, fY) = h(fX, Y), \quad \forall X, Y \in \Gamma(D \oplus \{\xi\}),$$

and

$$(2.12) \quad N_f(X, Y) \in \Gamma(D \oplus \{\xi\}), \quad \forall X, Y \in \Gamma(D \oplus \{\xi\}).$$

From Proposition 2.2 and by using (1.16) we infer

Corollary 2.3. *Let M be a CR-submanifold of a nearly cosymplectic manifold $\tilde{M}$. Then the distribution $D \oplus \{\xi\}$ is integrable if and only if (2.11) is satisfied, and*

$$(\tilde{\nabla}_Z f)Y \in \Gamma(D \oplus \{\xi\}), \quad \forall Y \in \Gamma(D \oplus \{\xi\}), \quad Z \in \Gamma(D^\perp).$$

Now from Corollary 2.3 and by using (1.3) we infer

Corollary 2.4. *Let M be a CR-submanifold of a nearly cosymplectic manifold $\tilde{M}$. Then the distribution $D \oplus \{\xi\}$ is integrable if and only if (2.10) is satisfied, and*

$$N_f(Y, Z)^\top \in \Gamma(D^\perp \oplus \{\xi\}), \quad \forall Y \in \Gamma(D \oplus \{\xi\}), \quad Z \in \Gamma(D^\perp).$$

Next, we are concerned with the integrability of distributions $D^\perp$ and $D^\perp \oplus \{\xi\}$, on a CR-submanifold in a nearly cosymplectic manifold. First we note that, for an almost contact metric manifold $\tilde{M}$ (See [4]) we have

$$(2.13) \quad 3d\Phi(Y, Z, X) = g((\tilde{\nabla}_Y f)X, Z) + g((\tilde{\nabla}_Z f)Y, X) + g((\tilde{\nabla}_X f)Z, Y),$$

for any $X, Y, Z \in \Gamma(T\tilde{M})$. Then, by using (1.3) and (2.13) we infer

$$(2.14) \quad d\Phi(Y, Z, X) = g((\tilde{\nabla}_Z f)Y, X), \quad \forall X, Y, Z \in \Gamma(T\tilde{M}).$$

On the other hand, by straightforward calculation we deduce

$$(2.15) \quad 3d\Phi(Y, Z, X) = g([Y, Z], fX), \quad \forall X \in \Gamma(D), Y, Z \in \Gamma(D^\perp \oplus \{\xi\}).$$

From (1.20), (2.14) and (2.15) we deduce

Proposition 2.3. *Let M be a CR-submanifold of a nearly cosymplectic manifold $\tilde{M}$. The distribution $D^\perp$ is integrable if and only if*

$$(2.16) \quad g((\tilde{\nabla}_Z f)Y, X) = 0, \quad \forall X \in \Gamma(D), Y, Z \in \Gamma(D^\perp),$$

and

$$(2.17) \quad g((\tilde{\nabla}_\xi f)Y, fZ) = 0, \quad \forall Y, Z \in \Gamma(D^\perp).$$

Now let $Y, Z \in \Gamma(D^\perp)$, $X \in \Gamma(D)$. Then from (1.3) and (1.8) we infer

$$(2.18) \quad g(h(X, Y), fZ) - g(h(X, Z), fY) = g(Z, (\tilde{\nabla}_X f)Y) = g(X, (\tilde{\nabla}_Y f)Z).$$

By using Proposition 2.3 and relation (2.18) we infer

Corollary 2.5. *Let M be a CR-submanifold of a nearly cosymplectic manifold $\tilde{M}$. The distribution $D^\perp$ is integrable if and only if*

$$(2.19) \quad g(h(X, Y), fZ) = g(h(X, Z), fY), \quad \forall X \in \Gamma(D), Y, Z \in \Gamma(D^\perp),$$

and

$$(2.20) \quad g(\tilde{\nabla}_Y \xi, Z) = 0, \quad \forall Y, Z \in \Gamma(D^\perp).$$

By using Proposition 2.3, (1.3), (1.6) and (1.16), we deduce

Corollary 2.6. *Let M be a CR-submanifold of a nearly cosymplectic manifold $\tilde{M}$. The distribution $D^\perp$ is integrable if and only if*

$$(2.21) \quad g(Y, N_f(X, Z)) = 0, \quad \forall X \in \Gamma(D \oplus \{\xi\}), Y, Z \in \Gamma(D^\perp).$$

Proposition 2.4. *Let M be a CR-submanifold of a nearly cosymplectic manifold $\tilde{M}$. The distribution $D^\perp \oplus \{\xi\}$ is integrable if and only if*

$$(2.22) \quad g((\tilde{\nabla}_Z f)Y, X) = 0, \quad \forall X \in \Gamma(D), Y, Z \in \Gamma(D^\perp \oplus \{\xi\}).$$

The PROOF of this assertion is immediate from (2.14) and (2.15) $\square$

Now from Proposition 2.4 we infer

Corollary 2.7 (S. IANUS [6]). *The distribution $D^\perp \oplus \{\xi\}$ of a CR-submanifold of a nearly cosymplectic manifold $\tilde{M}$ is integrable if and only if*

$$(2.23) \quad g(X, A_{fY}Z - A_{fZ}Y) = 0, \quad \forall X \in \Gamma(D), Y, Z \in \Gamma(D^\perp \oplus \{\xi\}).$$

Corollary 2.8. *Let M be a CR-submanifold of a nearly cosymplectic manifold $\tilde{M}$. Then the distribution $D^\perp \oplus \{\xi\}$ is integrable if and only if*

$$N_f(X, Z)^\top \in \Gamma(D), \quad \forall X \in \Gamma(D), \quad Z \in \Gamma(D^\perp).$$

PROOF. Let $X \in \Gamma(D)$, $Y \in \Gamma(D^\perp)$, $Z \in \Gamma(D^\perp \oplus \{\xi\})$. Then by using (1.16), we infer

$$(2.24) \quad g((\tilde{\nabla}_Z f)Y, fX) = -g(Y, (\tilde{\nabla}_Z f)fX) = \frac{1}{4}g(Y, N_f(X, Z)).$$

Now, our assertion follows from Proposition 2.4 and (2.24) $\square$

3. Totally contact umbilical CR-submanifold of a nearly cosymplectic manifold

The purpose of this paragraph is to establish some properties of totally umbilical CR-submanifolds of a nearly cosymplectic manifold $\tilde{M}$.

Definition 3.1. We say that a CR-submanifold M of the nearly cosymplectic manifold $\tilde{M}$ is *totally contact umbilical* if there exists a normal vector field H so that

$$(3.1) \quad h(X, Y) = g(fX, fY)H + \eta(X)h(Y, \xi) + \eta(Y)h(X, \xi), \\ \forall X, Y \in \Gamma(TM).$$

We say that M is *totally contact geodesic* if $H = 0$, that is,

$$(3.2) \quad h(X, Y) = \eta(X)h(Y, \xi) + \eta(Y)h(X, \xi), \quad \forall X, Y \in \Gamma(TM).$$

Lemma 3.1. *Let M be a CR-submanifold of a nearly cosymplectic manifold $\tilde{M}$. Then we have*

$$(3.3) \quad QA_{fZ}W = QA_{fW}Z, \quad \forall Z, W \in \Gamma(D^\perp).$$

PROOF. Let $U, Z, W \in \Gamma(D^\perp)$. Then by using (1.3), (1.6) and (1.10) we infer

$$2Bh(Z, W) = -QA_{fZ}W - QA_{fW}Z,$$

which is equivalent with

$$(3.4) \quad 2g(A_{fU}Z, W) = g(A_{fZ}W, U) + g(A_{fW}Z, U).$$

From (3.4) we deduce

$$(3.5) \quad 2g(A_{fZ}W, U) = g(A_{fW}U, Z) + g(A_{fU}W, Z).$$

Now our assertion follows from (3.4) and (3.5). $\square$

Lemma 3.2. *If M is a totally contact umbilical proper CR-submanifold of a nearly cosymplectic manifold $\tilde{M}$, then either $\dim.D^\perp = 1$, or the normal vector field H is orthogonal to $fD^\perp$.*

PROOF. Let $X \in \Gamma(TM)$, $Y \in \Gamma(D^\perp)$. Then by using (3.1) we deduce

$$(3.6) \quad g(h(X, X), fY) = g(fX, fX)g(H, fY).$$

Suppose $\dim.D^\perp > 1$. Then there exists a unit vector field $Z \in \Gamma(D^\perp)$ orthogonal to Y . Further, from (1.11) and (3.1) we deduce

$$\begin{aligned} g(H, fY) &= g(A_{fY}Z, Z) = g(QA_{fY}Z, Z) \\ &= g(QA_{fZ}Y, Z) = g(h(Y, Z), fZ) = 0, \end{aligned}$$

which proves that H is perpendicular to $fD^\perp$. $\square$

Proposition 3.1. *Let M be a totally contact umbilical proper CR-submanifold of a nearly cosymplectic manifold M with $\dim.D^\perp > 1$. Then M is totally contact geodesic.*

PROOF. Since $\dim.D^\perp > 1$, from Lemma 3.2 and relation (1.8) there follows

$$(3.7) \quad fH = CH.$$

By using (1.3) we infer

$$(3.8) \quad (\tilde{\nabla}_X f)fX = 0, \quad \forall X \in \Gamma(TM).$$

Now from (3.7) and (3.8) and (1.3) obtain

$$\begin{aligned} 0 &= g((\tilde{\nabla}_X f)fX, H) = -g(fX, (\tilde{\nabla}_X f)H) \\ &= g(fX, PA_{fH}X) - g(fX, fPA_HX) \\ &= g(h(X, fX), fH) - g(h(X, X), H) \\ &= -g(X, X)g(H, H), \quad \forall X \in \Gamma(D), \end{aligned}$$

which proves that $H = 0$. Therefore M is totally contact geodesic. $\square$

If $\dim.D^\perp = 1$, we deduce a sufficient condition for totally contact umbilical proper CR-submanifolds to be totally geodesic.

Definition 3.1. The distribution D (resp. $D^\perp \oplus \{\xi\}$) is *autoparallel* if and only if, we have $\nabla_X Y \in \Gamma(D), \forall X, Y \in \Gamma(D)$ ($\nabla_X Y \in \Gamma(D^\perp \oplus \{\xi\}), \forall X, Y \in \Gamma(D^\perp \oplus \{\xi\})$ resp.).

Theorem 3.1. *Let M be a totally contact umbilical proper CR-submanifold of a nearly cosymplectic manifold $\tilde{M}$. If the distribution D is autoparallel then M is totally contact geodesic.*

PROOF. Let $X \in \Gamma(D)$ so that $g(X, X) = 1$. Then by using (1.3, (1.10) and (3.1) we deduce

$$\begin{aligned} 0 &= g((\tilde{\nabla}_X f)X, BH) = -g(X, (\tilde{\nabla}_X f)BH) \\ &= g(X, A_{fBH}X) + g(fP\nabla_X BH, X) \\ &= g(H, fBH) + g(BH, \nabla_X fX) = -g(BH, BH), \end{aligned}$$

which implies that $BH = 0$. From this point the proof of Proposition 3.1 applies. $\square$

Proposition 3.2. *Let $\tilde{M}$ be a nearly cosymplectic manifold. Then we have*

$$\begin{aligned} (3.9) \quad &\tilde{K}(X, Y, fY, fX) - \tilde{K}(X, Y, Y, X) \\ &= \|(\tilde{\nabla}_X f)Y\|^2 + \|(\tilde{\nabla}_X \eta)Y\|^2, \quad \forall X, Y \in \Gamma(T\tilde{M}) \end{aligned}$$

The proof follows from (1.2), (1.3) and (1.13) by straightforward calculation.

Definition 3.2. Let M be a CR-submanifold of a nearly cosymplectic manifold $\tilde{M}$. If $X \in \Gamma(D \oplus \{\xi\})$ and $Y \in \Gamma(D^\perp)$, (or $X \in \Gamma(D)$ and $Y \in \Gamma(D^\perp \oplus \{\xi\})$) then we say that the sectional curvature $K(\pi)$ generated by X and Y is a CR-sectional curvature.

Theorem 3.2. *Let M be a totally contact umbilical proper CR-submanifold of a nearly cosymplectic manifold $\tilde{M}$. Then any CR-sectional curvature of $\tilde{M}$ is non negative.*

PROOF. By using (1.4) we deduce

$$K(X \wedge \xi) = g(\tilde{\nabla}_X \xi, \tilde{\nabla}_X \xi) \geq 0, \quad X \in \Gamma(D \oplus D^\perp).$$

Next, by using (3.1) and the equation of Codazzi we obtain

$$\begin{aligned} K(X, Y, fX, fY) &= g(Y, fX)g(\nabla_X^\perp H, fY) - g(Y, fX)g(\nabla_Y^\perp H, fY) = 0, \\ &\quad \forall X \in \Gamma(D), Y \in \Gamma(D^\perp). \end{aligned}$$

Finally by using (3.9) we infer

$$K(X \wedge Y) = \|(\tilde{\nabla}_X f)Y\|^2 + \|(\tilde{\nabla}_X \eta)Y\|^2,$$

which proves our assertion. $\square$

From Theorem 3.2 we obtain

Corollary 3.1. *There exist no proper totally contact umbilical CR-submanifolds of negatively curved nearly cosymplectic manifolds.*

4. Totally contact geodesic CR-submanifolds of a nearly cosymplectic manifold

First we prove

Lemma 4.1. *Let M be a totally contact geodesic CR-submanifold of a nearly cosymplectic manifold $\tilde{M}$. If (2.7) holds, then D is autoparallel.*

PROOF. Let $X, Y \in \Gamma(D)$. Then, for $Z \in \Gamma(D^\perp \oplus \{\xi\})$, and by using (1.3), (1.10) and (3.2) we deduce:

$$(4.1) \quad 0 = g((\tilde{\nabla}_Z f)Y, X) = g((\nabla_Z f)Y, X) = g(Z, \nabla_Y fX),$$

which proves our assertion. $\square$

From Corollary 2.1 and Lemma 4.1 we infer

Corollary 4.1. *Let M be a totally contact geodesic CR-submanifold of a nearly cosymplectic manifold $\tilde{M}$. If (2.7) holds, then D is integrable and its leaves are totally geodesically immersed in both M and $\tilde{M}$.*

PROOF. Let M be totally contact geodesic and $X, Y \in \Gamma(D)$, $Z \in \Gamma(D^\perp \oplus \{\xi\})$, and M_1 a leaf of the distribution D . Denote by ∇_1 the Riemannian connection on M_1 and by h_1 the second fundamental form of the immersion $M_1 \hookrightarrow M$. Then from Gauss' formula we deduce

$$(4.2) \quad \nabla_X Y = \nabla_{1X} Y + h_1(X, Y).$$

By using Lemmas 4.1 and (4.2) it follows that M_1 is totally geodesic in M . From (3.2) follows $h_1(X, Y) = 0$. Finally, by using the formulas of Gauss for the immersion of M_1 in to both M and $\tilde{M}$ and for the immersion of M into $\tilde{M}$ we obtain that M_1 is totally geodesic in $\tilde{M}$. $\square$

Lemma 4.2. *Let M be a totally contact geodesic CR-submanifold of a nearly cosymplectic manifold $\tilde{M}$. Then the distribution $D^\perp \oplus \{\xi\}$ is autoparallel.*

PROOF. By using (1.7) and (1.8) we obtain

$$(4.3) \quad \begin{aligned} g((\tilde{\nabla}_X f)Y, Z) &= g(\tilde{\nabla}_X fY - f\tilde{\nabla}_X Y, Z) \\ &= g(-A_{\omega Y}X + \nabla_X^\perp \omega Y - t\nabla_X Y \\ &\quad - \omega \nabla_X Y - Bh(X, Y) - Ch(X, Y), Z) \\ &= -g(h(X, Z), \omega Y) - g(t\nabla_X Y + Bh(X, Y), Z) = 0, \end{aligned}$$

for any $X \in \Gamma(D)$, $Y, Z \in \Gamma(D^\perp \oplus \{\xi\})$.

Next, from (4.3) we infer

$$\begin{aligned} 0 &= g((\tilde{\nabla}_X f)Y, Z) = g(X, (\nabla_Y f)Z) = -g(X, t\nabla_Y Z) \forall X \in \Gamma(D); \\ &\quad Y \in \Gamma(D^\perp), Z \in \Gamma(D^\perp \oplus \{\xi\}), \end{aligned}$$

which proves that $\nabla_Y Z \in \Gamma(D^\perp \oplus \{\xi\})$. $\square$

From Lemma 4.2 we deduce

Corollary 4.2. *Let M be a totally contact geodesic CR-submanifold of a nearly cosymplectic manifold $\tilde{M}$. Then the distribution $D^\perp \oplus \{\xi\}$ is integrable.*

Lemma 4.3. *Let M be a totally contact geodesic CR-submanifold of a nearly cosymplectic manifold $\tilde{M}$. Then any leaf of the distribution $D^\perp \oplus \{\xi\}$ is totally geodesic in M and $\tilde{M}$.*

PROOF. By using (3.1) and Lemma 4.2 our assertion follows. $\square$

Lemma 4.4. *Let M be a totally contact geodesic CR-submanifold of a nearly cosymplectic manifold $\tilde{M}$. If*

$$(4.4) \quad (\nabla_Z f)Y \in \Gamma(D \oplus \{\xi\}), \quad \forall Y \in \Gamma(D \oplus \{\xi\}), \quad Z \in \Gamma(D^\perp),$$

then the distribution $D \oplus \{\xi\}$ is autoparallel.

PROOF. Let $X, Y \in \Gamma(D \oplus \{\xi\})$. Then $Z \in \Gamma(D^\perp)$ and by using (1.3), (1.10) and (3.2) we infer

$$(4.5) \quad 0 = g((\tilde{\nabla}_Z f)Y, X) = -g((\nabla_X f)Y, Z) = g(\nabla_Y fX, Z),$$

which proves our assertion. $\square$

From Corollary 2.4 and Lemma 4.4 we obtain

Corollary 4.3. *Let M be a totally contact geodesic CR-submanifold of a nearly cosymplectic manifold $\tilde{M}$ such that (4.4) holds. Then the distribution $D \oplus \{\xi\}$ is integrable and its leaves are totally geodesically immersed in both M and $\tilde{M}$.*

Lemma 4.5. *Let M be a totally contact geodesic CR-submanifold of a nearly cosymplectic manifold $\tilde{M}$ such that (2.16) and (2.17) hold. Then the distribution $D^\perp$ is autoparallel.*

PROOF. Let $X \in \Gamma(D)$, $Y, Z \in \Gamma(D^\perp)$. By using (1.3), (1.10) and (3.2) we infer

$$(4.6) \quad 0 = g((\tilde{\nabla}_Z f)Y, X) = -g(Y, (\tilde{\nabla}_Z f)X) = g(\nabla_Z Y, fX),$$

and

$$0 = g((\tilde{\nabla}_\xi f)Y, fZ) = g(\xi, (\tilde{\nabla}_Y f)fZ) = -g(\xi, \nabla_Y Z).$$

From (4.6) and (4.7) there follows our assertion. $\square$

Lemma 4.5. *Let M be a totally contact geodesic CR-submanifold of a nearly cosymplectic manifold $\tilde{M}$ such that (2.16) and (2.17) hold. Then the distribution $D^\perp$ is integrable and its leaves are totally geodesically immersed in M and $\tilde{M}$.*

By using the lemmas 4.1, 4.2 and 4.3 we deduce

Theorem 4.1. *Let M be a totally contact geodesic CR-submanifold of a nearly cosymplectic manifold $\tilde{M}$. If the distribution D is integrable, then M is locally the Riemannian product $M_1 \times M_2$ where M_1 and M_2 are the leaves of the distributions D and $D^\perp \oplus \{\xi\}$, respectively.*

Theorem 4.2. *Let M be a totally contact geodesic CR-submanifold of a nearly cosymplectic manifold $\tilde{M}$. If the distribution $D \oplus \{\xi\}$ is integrable and $\tilde{\nabla}_Y \xi \in \Gamma(D \oplus \{\xi\})$ for any $Y \in \Gamma(D^\perp)$, then M is locally the Riemannian product $M_1 \times M_2$ where M_1 and M_2 are the leaves of the distributions $D \oplus \{\xi\}$ and $D^\perp$, respectively.*

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