

On the m -ary expansion of a multiple of $(m^k - 1)/(m - 1)$

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Abstract. We prove an extremal result on the m -ary expansion of $n(m^k - 1)/(m - 1)$.

1. Introduction

Recently, Z.-W. SUN [3] proved that for any $n, k \geq 1$,

$$\frac{1}{(2^k - 2)n + 1} \binom{(2^k - 1)n}{n} \binom{2(2^k - 1)n}{(2^k - 1)n}$$

is divisible by

$$2^{k-1} \binom{2n}{n}.$$

One key of Sun's proof is the following curious lemma:

For positive integers n and k , the number of 1's in the binary expansion of $n(2^k - 1)$ is at least k .

In fact, he got a stronger result [3, Lemma 3.2]:

Theorem 1. *For a prime p and positive integers n and k , the sum of all digits in the expansion of $n(p^k - 1)$ in base p is at least $k(p - 1)$.*

Motivated by Theorem 1, Sun made the following conjecture.

Conjecture 1. *Suppose that n, m, k are positive integers and $m \geq 2$.*

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- (i) *There are at least k non-zero digits in the m -ary expansion of $n(m^k - 1)/(m - 1)$.*
- (ii) *The sum of all digits in the m -ary expansion of $n(m^k - 1)$ is at least $k(m - 1)$.*

In fact, Conjecture 1 is not new. A solution of Part (i) of Conjecture 1 for the base $m = 10$ has been given in [1]. Furthermore, the second part of Conjecture 1 for $m = 10$ is also proved in [2]. Of course, those two proofs can be extended to any base $m \geq 2$ without any difficulty.

However, in this short note, we shall prove the following stronger result.

Theorem 2. *Suppose that $m \geq 2$, $k \geq 1$ and $a_1, a_2, \dots, a_k \in \mathbb{N} = \{0, 1, 2, \dots\}$ are not all zero.*

- (i) *If $a_k m^{k-1} + a_{k-1} m^{k-2} + \dots + a_2 m + a_1$ is a multiple of $(m^k - 1)/(m - 1)$, then*

$$\sum_{j=1}^k \left\lceil \frac{a_j}{m} \right\rceil \geq k,$$

where $\lceil x \rceil = \min\{z \in \mathbb{Z} : z \geq x\}$.

- (ii) *Suppose that*

$$a_k m^{k-1} + a_{k-1} m^{k-2} + \dots + a_2 m + a_1 \equiv 0 \pmod{m^k - 1}.$$

Then

$$\sum_{j=1}^k \left\lfloor \frac{a_j}{m-1} \right\rfloor \geq k,$$

where $\lfloor x \rfloor = \max\{z \in \mathbb{Z} : z \leq x\}$.

Let us give an explanation why Theorem 2 implies Conjecture 1. Write

$$n(m^k - 1)/(m - 1) = b_h m^{h-1} + b_{h-1} m^{h-2} + \dots + b_2 m + b_1,$$

where $0 \leq b_i < m$. Let

$$a_j = \sum_{\substack{1 \leq i \leq h \\ i \equiv j \pmod{k}}} b_i.$$

Since all b_i are less than m , we have

$$\left\lceil \frac{a_j}{m-1} \right\rceil \leq |\{1 \leq i \leq h : i \equiv j \pmod{k}, b_i > 0\}|.$$

On the other hand,

$$\sum_{i=1}^h b_i m^{i-1} \equiv \sum_{j=1}^k m^{j-1} \sum_{\substack{1 \leq i \leq h \\ i \equiv j \pmod{k}}} b_i = \sum_{j=1}^k a_j m^{j-1} \equiv 0 \pmod{(m^k - 1)/(m - 1)}.$$

It follows from Part (i) of Theorem 2 that

$$|\{1 \leq i \leq h : b_i > 0\}| \geq \sum_{j=1}^k \left\lceil \frac{a_j}{m-1} \right\rceil \geq \sum_{j=1}^k \left\lceil \frac{a_j}{m} \right\rceil \geq k.$$

Similarly, if $n(m^k - 1) = b_h m^{h-1} + \dots + b_1$ with $0 \leq b_i < m$ and let

$$a_j = \sum_{\substack{1 \leq i \leq h \\ i \equiv j \pmod{k}}} b_i,$$

by Part (ii) of Theorem 2, we have

$$\sum_{i=1}^h b_i = \sum_{j=1}^k a_j \geq (m-1) \sum_{j=1}^k \left\lfloor \frac{a_j}{m-1} \right\rfloor \geq (m-1)k.$$

We shall prove Theorem 2 in the next section.

2. Proof of Theorem 2

Proposition 1. Suppose that $m \geq 2$, $k \geq 1$ and $a_1, a_2, \dots, a_k \in \mathbb{N}$ are not all zero. Let q be a divisor of $m^k - 1$. Suppose that $\tau(x_1, \dots, x_k)$ is a nonnegative integer-valued symmetric function satisfying that

$$\tau(x_1, x_2, \dots, x_{k-1}, md + b) \geq \tau(x_1 + d, x_2, \dots, x_{k-1}, b)$$

for every $0 \leq b < m$ and $d \geq 1$. If

$$a_k m^{k-1} + a_{k-1} m^{k-2} + \dots + a_2 m + a_1 \equiv 0 \pmod{q},$$

then

$$\tau(a_1, a_2, \dots, a_k) \geq \min_{1 \leq t \leq (m^k - 1)/q} \{\tau^*(tq)\},$$

where

$$\tau^*(h) = \tau(c_1, c_2, \dots, c_k)$$

provided $0 \leq h < m^k$ has the m -ary expansion $h = c_k m^{k-1} + c_{k-1} m^{k-2} + \dots + c_1$.

PROOF. Let

$$S = \left\{ (a_1, \dots, a_k) : a_1, \dots, a_k \in \mathbb{N} \text{ are not all zero, } \sum_{j=1}^k a_j m^{j-1} \equiv 0 \pmod{q} \right\}.$$

For $\mathbf{x} = (a_1, \dots, a_k)$, define

$$\sigma(\mathbf{x}) = a_1 + a_2 + \dots + a_k.$$

Let

$$t_0 = \min_{(a_1, \dots, a_k) \in S} \tau(a_1, \dots, a_k)$$

and

$$T = \{(a_1, \dots, a_k) \in S : \tau(a_1, \dots, a_k) = t_0\}.$$

Clearly T is non-empty. Choose an $\mathbf{x}^\circ = (a_1^\circ, \dots, a_k^\circ) \in T$ such that

$$\sigma(\mathbf{x}^\circ) = \min_{(a_1, \dots, a_k) \in T} \sigma(a_1, \dots, a_k).$$

Since

$$m \cdot \sum_{j=1}^k a_j m^{j-1} = \sum_{j=1}^k a_j m^j \equiv a_k + \sum_{j=1}^{k-1} a_j m^j \pmod{q},$$

$(a_1, a_2, \dots, a_k) \in S$ implies $(a_k, a_1, a_2, \dots, a_{k-1}) \in S$. Furthermore, note that both τ and σ are symmetric. So by the definition of T and the choice of $\mathbf{x}^\circ$, without loss of generality, we may assume that $a_k^\circ \geq \max\{a_1^\circ, a_2^\circ, \dots, a_{k-1}^\circ\}$. We shall prove that $a_k^\circ < m$. Assume on the contrary that $a_k^\circ \geq m$. Write $a_k^\circ = md + b$ with $0 \leq b < m$ and $d \geq 1$. Then

$$\sum_{j=1}^k a_j^\circ m^{j-1} = (md+b)m^{k-1} + \sum_{j=1}^{k-1} a_j^\circ m^{j-1} \equiv bm^{k-1} + (d+a_1^\circ)m^{k-1} + \sum_{j=2}^{k-1} a_j^\circ m^{j-1} \pmod{q}.$$

Hence $\mathbf{x}^\Delta := (a_1^\circ + d, a_2^\circ, \dots, a_{k-1}^\circ, b) \in S$. Note that now

$$\tau(a_1^\circ, \dots, a_k^\circ) = \tau(a_1^\circ, a_2^\circ, \dots, a_{k-1}^\circ, md+b) \geq \tau(a_1^\circ + d, a_2^\circ, \dots, a_{k-1}^\circ, b).$$

It follows that $\mathbf{x}^\Delta$ also lies in T . But clearly

$$\sigma(\mathbf{x}^\circ) - \sigma(\mathbf{x}^\Delta) = a_1^\circ + a_k^\circ - (a_1^\circ + d + b) = (m-1)d \geq 1,$$

i.e., $\sigma(\mathbf{x}^\Delta) < \sigma(\mathbf{x}^\circ)$. This evidently leads to a contradiction with our choice of $\mathbf{x}^\circ$.

So we must have $a_k^\circ < m$, i.e.,

$$\max\{a_1^\circ, a_2^\circ, \dots, a_k^\circ\} \leq m-1.$$

Thus $a_k^\circ m^{k-1} + \dots + a_1^\circ = t_0 q$ for some $1 \leq t_0 \leq (m^k - 1)/q$. And

$$\tau(a_1^\circ, a_2^\circ, \dots, a_k^\circ) = \tau^*(t_0 q) \geq \min_{1 \leq t \leq (m^k - 1)/q} \{\tau^*(tq)\}.$$

□

Now let

$$\tau_1(x_1, \dots, x_k) = \sum_{i=1}^k \left\lceil \frac{x_i}{m} \right\rceil$$

and

$$\tau_2(x_1, \dots, x_k) = \sum_{i=1}^k \left\lfloor \frac{x_i}{m-1} \right\rfloor.$$

We shall verify τ_1 and τ_2 satisfy the requirements of Proposition 1. Evidently τ_1 and τ_2 are symmetric. Note that

$$\left\lceil \frac{md+b}{m} \right\rceil = d + \left\lceil \frac{b}{m} \right\rceil,$$

and

$$\left\lceil \frac{x_1+d}{m} \right\rceil \leq \left\lceil \frac{x_1}{m} \right\rceil + \left\lceil \frac{d}{m} \right\rceil \leq \left\lceil \frac{x_1}{m} \right\rceil + d.$$

Clearly it follows that

$$\tau_1(x_1, x_2, \dots, x_{k-1}, md+b) \geq \tau_1(x_1+d, x_2, \dots, x_{k-1}, b).$$

Moreover, if $m > 2$, then

$$\left\lfloor \frac{md+b}{m-1} \right\rfloor \geq d + \left\lfloor \frac{b}{m-1} \right\rfloor,$$

and

$$\left\lfloor \frac{x_1+d}{m-1} \right\rfloor \leq \left\lfloor \frac{x_1}{m-1} \right\rfloor + \left\lfloor \frac{d}{m-1} \right\rfloor + 1 \leq \left\lfloor \frac{x_1}{m-1} \right\rfloor + d.$$

That is,

$$\tau_2(x_1, x_2, \dots, x_{k-1}, md+b) \geq \tau_2(x_1+d, x_2, \dots, x_{k-1}, b).$$

Of course, apparently the above inequality also holds when $m = 2$. Finally, note that for each $1 \leq t \leq m-1$,

$$\tau_1^*\left(t \cdot \frac{m^k-1}{m-1}\right) = \tau_1^*\left(\sum_{j=1}^k t \cdot m^{k-j}\right) = \sum_{j=1}^k \left\lceil \frac{t}{m} \right\rceil = k,$$

and

$$\tau_2^*(m^k-1) = \tau_2^*\left(\sum_{j=1}^k (m-1) \cdot m^{k-j}\right) = \sum_{j=1}^k \left\lfloor \frac{m-1}{m-1} \right\rfloor = k.$$

Thus applying Proposition 1, we immediately get Theorem 2.

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