Publ. Math. Debrecen
85/1-2 (2014), 93–100
DOI: 10.5486/PMD.2014.5815

On the Diophantine equation $(x-1)^k + x^k + (x+1)^k = y^n$

By ZHONGFENG ZHANG (Zhaoqing)

Abstract. In this paper, we study the Diophantine equation $(x-1)^k + x^k + (x+1)^k = y^n$, $n > 1$, and completely solve it for $k = 2, 3, 4$.

1. Introduction

The Diophantine equation

$$1^k + 2^k + \cdots + x^k = y^n$$

was studied by LUCAS [9] for $(k, n) = (2, 2)$ and SCHÄFFER [12] for the general situation. There are many results on this equation, for example, see [1], [8], [11]. A generalization is to consider the equation

$$(x+1)^k + (x+2)^k + \cdots + (x+m)^k = y^n.$$

When $m = 3$, redefining variables, we will consider the equation $(x-1)^k + x^k + (x+1)^k = y^n$. CASSELS [6] proved that $x = 0, 1, 2, 24$ are the only integer solutions to this equation for $k = 3$, $n = 2$.

Our result in this paper is the following.

Theorem 1.1. *Let $k = 2, 3, 4$, then the equation*

$$(x-1)^k + x^k + (x+1)^k = y^n \tag{1}$$

has no integer solutions (x, y) with $n > 1$, unless $(x, y, k, n) = (1, \pm 3, 3, 2)$, $(2, \pm 6, 3, 2)$, $(24, \pm 204, 3, 2)$, $(\pm 4, \pm 6, 3, 3)$ or $(x, y, k) = (0, 0, 3)$.

Mathematics Subject Classification: 11D41, 11D61.
Key words and phrases: Diophantine equations, modular form, Thue equations.
This research was supported by the Guangdong Provincial Natural Science Foundation (No. S2012040007653) and NSF of China (No. 11271142).

2. Some preliminary results

In this section, we present some lemmas which will help us to prove Theorem 1.1. The first lemma is due to Nagell [10] and the second one is just Theorem 0.1 of [6].

Lemma 2.1. *If $2 \nmid D$ and $D \geq 3$, then the equation*

$$2 + Dx^2 = y^n, n > 2$$

has no integer solutions (x, y, n) with $n \nmid h(-2D)$, where $h(-2D)$ is the class number of $\mathbb{Q}(\sqrt{-2D}\,)$.

Lemma 2.2. *The equation*

$$3x(x^2 + 2) = y^2$$

has only the integer solutions $(x, y) = (0, 0)$, $(1, \pm 3)$, $(2, \pm 6)$, $(24, \pm 204)$.

A special case of Theorem 1.5 in [4] which we need in this paper is the following result.

Lemma 2.3. *Let $p \geq 5$ be a prime, $\alpha \geq 2$ be an integer, then the equation*

$$x^p + 3^\alpha y^p = 2z^3$$

has no solutions in coprime integers with $|xy| > 1$.

In order to discuss the small exponents for $k = 4$ we also need the following result.

Lemma 2.4. *Let $p \geq 3$ be a prime and (x, y) be an integer solution to equation*

$$3x^2 - 10 = y^p, \tag{2}$$

then

$$(\sqrt{3}x + \sqrt{10}\,)^2 = (11 + 2\sqrt{30}\,)^i (a + b\sqrt{30}\,)^p$$

for some integers a, b, i with $-\frac{p-1}{2} \leq i \leq \frac{p-1}{2}$.

Proof. From equation (2) one has $(3x^2 - 10)^2 = y^{2p}$, that is

$$(3x^2 + 10)^2 - 30(2x)^2 = (y^2)^p.$$

It is easy to see $2 \nmid x, 5 \nmid x$, then $\gcd(3x^2+10, 2x) = 1$, together with the fact that the class number of $\mathbb{Q}(\sqrt{30}\,)$ is 2 and $11+2\sqrt{30}$ be the fundamental unit of this field, we have

$$3x^2 + 10 + 2x\sqrt{30} = (11+2\sqrt{30}\,)^i (a+b\sqrt{30}\,)^p$$

with $-\frac{p-1}{2} \le i \le \frac{p-1}{2}$, that is

$$\begin{aligned}(11+2\sqrt{30}\,)^i (a+b\sqrt{30}\,)^p &= 3x^2 + 10 + 2x\sqrt{30} \\ &= (\sqrt{3}x + \sqrt{10}\,)^2, -\frac{p-1}{2} \le i \le \frac{p-1}{2}. \qquad \square\end{aligned}$$

3. The modular approach

Let E be an elliptic curve over $\mathbb{Q}$ of conductor N. For a prime of good reduction l we write $\#E(\mathbb{F}_l)$ for the number of points on E over the finite field $\mathbb{F}_l$, and let $a_l(E) = l+1-\#E(\mathbb{F}_l)$. By a *newform* f, we will always mean a cuspidal newform of weight 2 with respect to $\Gamma_0(N_0)$ for some positive integer N_0, and N_0 will be called the *level* of f. Write $f = q + \sum_{i\ge 2} c_i q^i$ the q-expansion of f, then c_n will be called the *Fourier coefficients* of f. Let $\mathbb{K} = \mathbb{Q}(c_2, c_3, \dots)$ be the field obtained by adjoining to $\mathbb{Q}$ the Fourier coefficients of f, then $\mathbb{K}$ is a finite and totally real extension of $\mathbb{Q}$ (see e.g. [7], Chapter 15).

We shall say that the curve E arises modulo p from the newform f (and write $E \sim_p f$) if there is a prime ideal $\mathfrak{p}$ of $\mathbb{K}$ above p such that for all but finitely many primes l we have $a_l(E) \equiv c_l \pmod{\mathfrak{p}}$ (see [7], Definition 15.2.1).

We have the following result, which is just Lemma 2.1 of [5].

Proposition 3.1. *Assume that $E \sim_p f$. There exists a prime ideal $\mathfrak{p}$ of $\mathbb{K}$ above p such that for all primes l,*

(i) *if $l \nmid pNN_0$ then $a_l(E) \equiv c_l \pmod{\mathfrak{p}}$,*

(ii) *if $l \| N$ but $l \nmid pN_0$ then $\pm(l+1) \equiv c_l \pmod{\mathfrak{p}}$.*

Moreover, if f is rational, then the above can be relaxed slightly as follows: for all primes l,

(i) *if $l \nmid NN_0$ then $a_l(E) \equiv c_l \pmod{p}$,*

(ii) *if $l \,||\, N$ but $l \nmid N_0$ then $\pm(l+1) \equiv c_l \pmod{p}$.*

4. Proof of Theorem 1.1

Proof of Theorem 1.1. Without loss of generality, we assume $n = p$ and p is a prime. Expanding the left hand side of equation (1), one has

(i) $3x^2 + 2 = y^p$ when $k = 2$;

(ii) $3x(x^2 + 2) = y^p$ when $k = 3$;

(iii) $3x^4 + 12x^2 + 2 = y^p$ when $k = 4$.

We will discuss them separately.

(1) $k = 2$

Applying Lemma 2.1, we conclude that there are no integer solutions for $p \geq 3$ since $h(-6) = 2$. When $p = 2$, the equation $3x^2 + 2 = y^2$ modulo 3 yields a contradiction.

(2) $k = 3$

From the result of Cassels, that is Lemma 2.2, one has $(x, y) = (0, 0)$, $(1, \pm 3)$, $(2, \pm 6)$, $(24, \pm 204)$ for $p = 2$.

When $p \geq 3$ we obtain $(x, y) = (0, 0)$ if $xy = 0$. Now we assume $xy \neq 0$. Since $\gcd(x, x^2 + 2) = \gcd(x, 2) \in \{1, 2\}$, then equation

$$3x(x^2 + 2) = y^p \tag{3}$$

implies one of following cases:

(a) $x = 3^{p-1}u^p, x^2 + 2 = v^p, 2 \nmid v$; (b) $x = u^p, x^2 + 2 = 3^{p-1}v^p, 2 \nmid v$;

(c) $x = 2^{p-1} \times 3^{p-1}u^p, x^2 + 2 = 2v^p$; (d) $x = 2^{p-1}u^p, x^2 + 2 = 2 \times 3^{p-1}v^p$.

Firstly, in case (a), we can write equation (3) as $3^{p-2}(-3u^2)^p + v^p = 2$ and find it has no integer solutions when $p \geq 5$ by Lemma 2.3. When $p = 3$, one has $(3^2u^3)^2 = v^3 - 2$, modulo 9 yields a contradiction.

In case (b), equation (3) turns into $(-u^2)^p + 3^{p-1}v^p = 2$ and applying Lemma 2.3 we know it has no integer solutions when $p \geq 5$. The left equation for $p = 3$ can be written as $u^6 + 2 = 9v^3$, and no integer solutions exists since $u^6 + 2 \equiv 2, 3 \pmod 9$.

In case (c), equation (3) becomes $v^p - 2^{2p-3} \times 3^{2p-2}u^{2p} = 1$, applying Theorem 1.1 of [2], we find that the equation has no nonzero integer solutions (u, v) for $p \geq 3$.

Finally, in case (d), one has $3^{p-1}v^p - 2^{2p-3}u^{2p} = 1$, also from Theorem 1.1 of [2], we know $(u, v, p) = (\pm 4, 1, 3)$, which yields $(x, y) = (\pm 4, \pm 6)$.

(3) $k = 4$

From the equation $3(x^2+2)^2-10=y^2$ we know $2 \nmid x$, then $3(x^2+2)^2-10 \equiv \pm 3 \not\equiv y^2 \pmod{10}$, that is there are no integer solutions for $p=2$. When $p=3$, one has $3(x^2+2)^2-10=y^3$, that is $(9x^2+18)^2-270=(3y)^3$. Applying Magma to calculate the integer points on the elliptic curve $y^2=x^3-270$, we conclude that it has no integer solutions in this case.

We proceed to prove the equation

$$3x^4+12x^2+2=y^p \tag{4}$$

has no integer solutions for prime $p \geq 11$. The remaining cases $p=5,7$ will be treated at the end of the paper.

Let $u=x^2+2, v=y$, and write equation (4) as

$$3u^2-10=v^p.$$

It is easy to see $\gcd(u,v)=1$ and $uv \neq 0$. Suppose $p \geq 7$. To a possible solution (u,v), we associate the Frey curve (see [3])

$$E_u : Y^2 = X^3 + 6uX^2 + 30X,$$

with conductor $N = 2^6 \times 3^2 \operatorname{rad}(10v) = 2^7 \times 3^2 \times 5 \operatorname{rad}_{\{2,5\}}(v)$ where

$$\operatorname{rad}_{\{2,5\}}(v) = \prod_{p|v,\, p\neq 2,5} p.$$

Then, by the result of Bennett and Skinner [3], there is a newform of level $N(E_u)_p = 2^7 \times 3^2 \times 5 = 5760$ such that $E_u \sim_p f$.

Let l be a prime and $u \equiv r \pmod{l}$. Since $u=x^2+2$, one has the following table:

l	r
7	2, 3, 4, 6
11	0, 2, 3, 5, 6, 7
13	1, 2, 3, 5, 6, 11, 12
17	0, 1, 2, 3, 4, 6, 10, 11, 15
19	0, 2, 3, 6, 7, 8, 9, 11, 13, 18

Recall the definition of a_l and c_l in Section 3, that is $a_l = a_l(E) = l+1-\#E(\mathbb{F}_l)$, and $c_l = c_l(f)$ the Fourier coefficient of f. Therefore, calculating by Pari we obtain

(i) $7|N$ or $a_7(E_u) \in \{0,-4\}$; (ii) $a_{11}(E_u) \in \{0, \pm 2, -4, \pm 6\}$; (iii) $13|N$ or $a_{13}(E_u) \in \{\pm 2, -6\}$; (iv) $17|N$ or $a_{17}(E_u) \in \{2, \pm 6\}$; (v) $a_{19}(E_u) \in \{\pm 6\}$.

For rational newforms at level 5760 numbered in Stein's Table [13] , we get a bound for p by Proposition 3.1, that is from $p|a_l(E_u) - c_l(f)$ when $l \nmid N$ or $p| \pm (l+1) - c_l(f)$ when $l|N$. We list these bounds in the following table.

l	f	p
7	$f_{2+i}, f_{16+j}, f_{26+k}, f_{40+m}, 1 \le i,k \le 6, 1 \le j,m \le 8$	≤ 5
11	$f_{15}, f_{16}, f_{39}, f_{40},$	≤ 5
13	$f_1, f_2, f_9, f_{10}, f_{12}, f_{14}, f_{26}, f_{33}, f_{34}, f_{35}, f_{37}$	≤ 7
17	f_{25}	≤ 3
19	$f_{11}, f_{13}, f_{36}, f_{38}$	≤ 7

For the nonrational newforms $f_{49}, f_{50}, \ldots, f_{64}$, we using $p = l$ or $p | N_{\mathbb{K}/\mathbb{Q}}(a_l(E_u) - c_l(f))$ or $p | N_{\mathbb{K}/\mathbb{Q}}(\pm(l+1) - c_l(f))$ to bound p.

For $f = f_{49}$, one has $c_{13}^2(f) - 20 = 0$, $c_{17}^2(f) - 20 = 0$. Take $l = 13$, then $N_{\mathbb{K}/\mathbb{Q}}(a_l(E_u) - c_l(f)) = \pm 16$, $N_{\mathbb{K}/\mathbb{Q}}(\pm(l+1) - c_l(f)) = 2^4 \times 11$, which implies $p \le 5$ or $p = 11, 13$. Take $l = 17$, then $N_{\mathbb{K}/\mathbb{Q}}(a_l(E_u) - c_l(f)) = \pm 16$, $N_{\mathbb{K}/\mathbb{Q}}(\pm(l+1) - c_l(f)) = 2^4 \times 19$, which implies $p \le 5$ or $p = 17, 19$. Combining these two bounds yields $p \le 5$.

For $f = f_{56}, f_{60}, f_{61}$, take $l = 13, 17$, and for the left 12 nonrational newforms take $l = 7, 13$, then the same argument as f_{49}, we get $p \le 5$.

From the discussion above, we know there is no newform of level 5760 corresponding to E_u when $p \ge 11$. It remains to deal with the prime $p = 5, 7$. We prove that there are no integer solutions to equation

$$3x^2 - 10 = y^p$$

for $p = 5, 7$.

We discuss the case $p = 5$ in detail. By Lemma 2.4 we get

$$(\sqrt{3}x + \sqrt{10})^2 = (11 + 2\sqrt{30})^i (a + b\sqrt{30})^5 \tag{5}$$

for some integers a, b, i with $-2 \le i \le 2$. Replacing x by $-x$, we only need to consider the cases $0 \le i \le 2$.

If $i = 0$, expanding both sides of equation (5) we obtain

$$2x = 5a^4 b + 300a^2 b^3 + 900b^5,$$

so that $5 | x$, an impossibility.

If $i = 1$, equation (5) can be written as

$$(\sqrt{3}x + \sqrt{10})^2 = (11 + 2\sqrt{30})(a + b\sqrt{30})(a + b\sqrt{30})^4,$$

thus

$$(11 + 2\sqrt{30})(a + b\sqrt{30}) = (\sqrt{3}u + \sqrt{10}v)^2$$

for some integers u, v. Expanding this equality we get

$$\begin{cases} 11a + 60b = 3u^2 + 10v^2 \\ 2a + 11b = 2uv, \end{cases}$$

that is

$$\begin{cases} a = 33u^2 + 110v^2 - 120uv \\ b = -6u^2 - 20v^2 + 22uv. \end{cases}$$

Substitution into

$$\sqrt{3}x + \sqrt{10} = (\sqrt{3}u + \sqrt{10}v)(a + b\sqrt{30})^2$$

yields the Thue equation

$$-1188u^5 + 10845u^4v - 39600u^3v^2 + 72300u^2v^3 - 66000uv^4 + 24100v^5 = 1.$$

According to Magma one obtains no integer solutions.

If $i = 2$, we write equation (5) as

$$(\sqrt{3}x + \sqrt{10})^2 = (11 + 2\sqrt{30})^2(a + b\sqrt{30})(a + b\sqrt{30})^4,$$

and then

$$a + b\sqrt{30} = (\sqrt{3}u + \sqrt{10}v)^2$$

for some integers u, v, therefore

$$\sqrt{3}x + \sqrt{10} = (11 + 2\sqrt{30})(\sqrt{3}u + \sqrt{10}v)^5.$$

Expanding the right hand side of the equation yields the Thue equation

$$54u^5 + 495u^4v + 1800u^3v^2 + 3300u^2v^3 + 3000uv^4 + 1100v^5 = 1$$

and again we find no integer solutions after appealing to Magma.

For the case $p = 7$, the same argument as in case $p = 5$, solving the corresponding Thue equations, we know the equation $3x^2 - 10 = y^7$ has no integer solutions. From the discussion above, this completes the proof of Theorem 1.1. □

Acknowledgments. The author is grateful to the referee for his/her carefully reading the manuscript and making valuable suggestions.

References

[1] M. Bennett, K. Győry and Á. Pintér, On the Diophantine equation $1^k + 2^k + \ldots + x^k = y^n$, *Compositio Math.* **140** (2004), 1417–1431.

[2] M. Bennett, K. Győry, M. Mignotte and Á. Pintér, Binomial Thue equations and polynomial powers, *Compositio Math.* **142** (2006), 1103–1121.

[3] M. Bennett and C. Skinner, Ternary Diophantine equations via Galois representations and modular forms, *Canad. J. Math.* **56** (2004), 23–54.

[4] M. Bennett, N. Vatsal and S. Yazdani, Ternary Diophantine equations of signature $(p, p, 3)$, *Compositio Math.* **140** (2004), 1399–1416.

[5] Y. Bugeaud, M. Mignotte and S. Siksek, A multi-Frey approach to some multi-parameter families of Diophantine equations, *Canad. J. Math.* **60** (2008), 491–519.

[6] J. Cassels, A Diophantine equation, *Glasgow Math. J.* **27** (1985), 11–18.

[7] H. Cohen, Number Theory, Vol. II: Analytic and Modern Tools, GTM 240, *Springer-Verlag, New York*, 2007.

[8] M. Jacobson, Á. Pintér and P. G. Walsh, A computational approach for solving $y^2 = 1^k + 2^k + \cdots + x^k$, *Math. Comp.* **72** (2003), 2099–2110.

[9] É. Lucas, Problem 1180, *Nouvelle Ann. Math.* **14** (1875), 336.

[10] T. Nagell, Contributions to the theory of a category of diophantine equations of the second degree with two unknowns, *Nova Acta Soc. Sei. Upsal.* **16** (1955), 1–38.

[11] Á. Pintér, On the power values of power sums, *J. Number Theory* **125** (2007), 412–423.

[12] J. Schäffer, The equation $1^p + 2^p + \ldots n^p = m^q$, *Acta Math.* **95** (1956), 155–189.

[13] W. Stein, Arithmetic data about every weight 2 newform on $\Gamma_0(N)$, Current web address `http://modular.math.washington.edu/Tables`.

ZHONGFENG ZHANG
SCHOOL OF MATHEMATICS
AND STATISTICS
ZHAOQING UNIVERSITY
ZHAOQING 526061
P.R. CHINA

E-mail: zh12zh31f@aliyun.com

(Received March 19, 2013; revised August 11, 2013)