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On the Banach algebra $(w_\infty(\Lambda), w_\infty(\Lambda))$ and applications to the solvability of matrix equations in $w_\infty(\Lambda)$

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Abstract. We apply the characterisation of the class $(w_\infty(\Lambda), w_\infty(\Sigma))$ and the fact that this is a Banach algebra to study the solvability in $w_\infty(\Lambda)$ of matrix equations of the form $\Delta_\rho^+ X = B$ and $\Delta_\rho X = B$, where Δ_ρ^+ and Δ_ρ are upper and lower triangular matrices. Finally, we obtain some results on infinite tridiagonal matrices considered as operators from $w_\infty(\Lambda)$ into itself, and study the solvability in $w_\infty(\Lambda)$ of matrix equations for tridiagonal matrices.

1. Introduction and known results

Let ω denote the set of all sequences $x = (x_k)_{k=0}^\infty$, and ℓ_∞, c_0 and ϕ be the sets of all bounded, null and finite sequences, respectively. We write e and $e^{(n)}$ $(n = 0, 1, \dots)$ for the sequences with $e_k = 1$ for all k, and $e_n^{(n)} = 1$ and $e_k^{(n)} = 0$ for $k \neq n$.

A *BK space* X is a Banach sequence space with continuous coordinates $P_k : X \to \mathbb{C}$ where $P_n(x) = x_k$ $(k = 0, 1, \dots)$ for all $x \in X$. A BK space $X \supset \phi$ is said to have AK if $x = \sum_{k=0}^\infty x_k e^{(k)}$ for every sequence $x = (x_k)_{k=0}^\infty \in X$.

Let X be a subset of ω. Then the set $X^\beta = \{a \in \omega : \sum_{k=0}^\infty a_k x_k$ converges for all $x \in X\}$ is called the *β-dual of X*.

Let $A=(a_{nk})_{k=0}^\infty$ be an infinite matrix of complex numbers and $x=(x_k)_{k=0}^\infty \in \omega$. Then we write $A_n = (a_{nk})_{k=0}^\infty$ $(n = 0, 1, \dots)$ and $A^k = (a_{nk})_{n=0}^\infty$ $(k = 0, 1, \dots)$ for

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the sequences in the n-th row and the k-th column of A, and $A_n(x) = \sum_{k=0}^{\infty} a_{nk}x_k$ provided the series converges. Given any subsets X and Y of ω, we write (X, Y) for the class of all infinite matrices A that map X into Y, that is $A_n \in X^\beta$ for all n, and $A(x) = (A_n(x))_{n=0}^{\infty} \in Y$.

Let X and Y be Banach spaces, and $B_X = \{x \in X : \|x\| \leq 1\}$ denote the unit ball in X. Then $\mathcal{B}(X, Y)$ denotes the Banach space of all bounded linear operators $L : X \to Y$ with the operator norm $\|L\| = \sup_{x \in B_X} \|L(x)\|$; $X^* = \mathcal{B}(X, \mathbb{C})$ is the *continuous dual* of X with the norm $\|f\| = \sup_{x \in B_X} |f(x)|$ for al $f \in X^*$. It is well known ([12, Theorem 4.2.8]) that if X and Y are BK spaces then $(X, Y) \subset \mathcal{B}(X, Y)$, that is, if $A \in (X, Y)$, then $L_A \in \mathcal{B}(X, Y)$ where $L_A(x) = A(x)$ for all $x \in X$.

Let $(\mu_n)_{n=0}^{\infty}$ be a non-decreasing sequence of positive reals tending to infinity. The set

$$\tilde{w}_\infty(\mu) = \left\{ x \in \omega : \sup_n \frac{1}{\mu_n} \sum_{k=0}^{n} |x_k| < \infty \right\}$$

was defined and studied in [9], where the concept of *exponentially bounded sequences* was introduced. When $\mu_n = n + 1$ $(n = 0, 1, \dots)$ then this set reduces w_∞, the set of all sequences that are strongly bounded by the Cesàro method of order 1 ([1]). A non-decreasing sequence $\Lambda = (\lambda_n)_{n=0}^{\infty}$ of positive is called *exponentially bounded* if there is an integer $m \geq 2$ such that for all non-negative integers ν there is at least one term λ_n in the interval $I_m^{(\nu)} = [m^\nu, m^{\nu+1} - 1]$. It was shown ([9, Lemma 1]) that a non-decreasing sequence $\Lambda = (\lambda_n)_{n=0}^{\infty}$ is exponentially bounded, if and only if there are reals $s \leq t$ such that for some subsequence $(\lambda_{n(\nu)})_{\nu=0}^{\infty}$

$$0 < s \leq \frac{\lambda_{n(\nu)}}{\lambda_{n(\nu+1)}} \leq t < 1 \quad \text{for all } \nu = 0, 1, \dots;$$

such a subsequence is called an *associated subsequence*.

If $\Lambda = (\lambda_n)_{n=0}^{\infty}$ is an exponentially bounded sequence, and $(\lambda_{n(\nu)})_{\nu=0}^{\infty}$ is an associated subsequence with $\lambda_{n(0)} = \lambda_0$, then we write K_ν $(\nu = 0, 1, \dots)$ for the set of all integers k with $n(\nu) \leq k \leq n(\nu + 1) - 1$, and define the sets

$$w_0(\Lambda) = \left\{ x \in \omega : \lim_{\nu \to \infty} \frac{1}{\lambda_{n(\nu+1)}} \sum_{k \in K_\nu} |x_k| = 0 \right\}$$

and

$$w_\infty(\Lambda) = \left\{ x \in \omega : \sup_\nu \frac{1}{\lambda_{n(\nu+1)}} \sum_{k \in K_\nu} |x_k| < \infty \right\}.$$

The next result is well known.

Theorem 1.1 ([9, Theorem 1 (a), (b)]). *Let $(\mu_n)_{n=0}^\infty$ be a non-decreasing sequence of positive reals tending to infinity, $\Lambda = (\lambda_n)_{n=0}^\infty$ be an exponentially bounded sequence and $(\lambda_{n(\nu)})_{n=0}^\infty$ be an associated subsequence. Then $\tilde{w}_\infty(\mu)$ is a BK space with the norm $\|\cdot\|_\mu^-$ defined by*

$$\|x\|_\mu^- = \sup_n \frac{1}{\mu_n} \sum_{k=0}^n |x_k|. \tag{1.1}$$

Moreover, we have $\tilde{w}_\infty(\Lambda) = w_\infty(\Lambda)$, and the norms $\|\cdot\|_\Lambda^-$ and $\|\cdot\|_\Lambda$are equivalent on $w_\infty(\Lambda)$, where

$$\|x\|_\Lambda = \sup_\nu \frac{1}{\lambda_{n(\nu+1)}} \sum_{k \in K_\nu} |x_k|. \tag{1.2}$$

Thus, in view of the previous result, we always assume, unless explicitly stated otherwise, that Λ is an exponentially bounded sequence and $(\lambda_{n(\nu)})_{n=0}^\infty$ is an associated subsequence, and we consider the spaces $w_0(\Lambda)$ and $w_\infty(\Lambda)$ with the norm $\|\cdot\| = \|\cdot\|_\Lambda$.

First we give the characterisaton of the class $(w_\infty(\Lambda), w_\infty(\Sigma))$. Let $\Lambda = (\lambda_k)_{k=0}^\infty$ and $\Sigma = (\sigma_m)_{m=0}^\infty$ be exponentially bounded sequences and $(\lambda_{k(\nu)})_{\nu=0}^\infty$ and $(\sigma_{m(\mu)})_{\mu=0}^\infty$ be associated subsequences. Furthermore, let K_ν $(\nu = 0, 1, \dots)$ and M_μ $(\mu = 0, 1, \dots)$ be the sets of all integers k and m with $k(\nu) \le k \le k(\nu+1) - 1$ and $m(\mu) \le m \le m(\mu+1) - 1$. We obtain the characterisation of the class $(w_\infty(\Lambda), w_\infty(\Sigma))$ as the special case $p = 1$ of [10, Theorem 3.1].

Theorem 1.2. *We have $A \in (w_\infty(\Lambda), w_\infty(\Sigma))$ if and only if*

$$\|A\|_{(w_\infty(\Lambda), w_\infty(\Sigma))} = \sup_\mu \left(\frac{1}{\sigma_{m(\mu+1)}} \max_{M(\mu) \subset M_\mu} \sum_{\nu=0}^\infty \lambda_{k(\nu+1)} \max_{k \in K_\nu} \left| \sum_{m \in M(\mu)} a_{mk} \right| \right) < \infty. \tag{1.3}$$

If $A \in (w_\infty(\Lambda), w_\infty(\Sigma))$, then

$$\|A\|_{(w_\infty(\Lambda), w_\infty(\Sigma))} \le \|L_A\| \le 4 \cdot \|A\|_{(w_\infty(\Lambda), w_\infty(\Sigma))}. \tag{1.4}$$

Finally we need the following known result.

Theorem 1.3 ([10, Theorem 4.2]). *The class $(w_\infty(\Lambda), w_\infty(\Lambda))$ is a Banach algebra with respect to the norm*

$$\|A\|_{(\Lambda,\Lambda)} = \sup\{\|A\|_\Lambda : \|x\|_\Lambda \le 1\} = \|L_A\| \quad \textit{for all } A \in (w_\infty(\Lambda), w_\infty(\Lambda)).$$

In this paper, we use the characterisation of the class $(w_\infty(\Lambda), w_\infty(\Lambda))$ in Theorem 1.2 and the fact that this class is a Banach algebra to study the solvability in $w_\infty(\Lambda)$ of matrix equations of the form $\Delta_\rho^+ X = B$ and $\Delta_\rho X = B$, where Δ_ρ^+ and Δ_ρ are upper and lower triangular matrices. Finally, we obtain some results on infinite tridiagonal matrices considered as operators from $w_\infty(\Lambda)$ into itself, and study the solvability in $w_\infty(\Lambda)$ of matrix equations for tridiagonal matrices; these results are new. Our results on the Banach algebra and the deduced solvability in $w_\infty(\Lambda)$ of matrix equations are also useful for statistical convergence, and for A-statistical convergence, where A is an operator represented by an infinite matrix. They can also be applied in spectral theory and the study of operators generators of analytic semi-group, and be used in the intuitionistic fuzzy normed space (IFNS).

2. The sets s_τ^0, $s_\tau^{(c)}$ and s_τ

We are going to study the solvability of certain matrix equations $AX = B$ in $w_\infty(\Lambda)$, where B belongs to the set $w_\infty(\Lambda)$ and A is an infinite matrix.

First, we recall some results on the sets s_τ^0, $s_\tau^{(c)}$ and s_τ which are needed in the sequel. These sets are closely related to the sets c_0, c and ℓ_∞. We also define and study the sets $\widehat{C}$, $\widehat{\Gamma}$, Γ and $\widehat{C}_1$.

We slightly change our notations to match those normally used in the theory of infinite systems of linear equations, in particular, we consider sequences x as one-column matrices X. It is also more convenient for the indices to run from 1 to ∞ instead of from 0 to ∞.

Thus, for a given infinite matrix $A = (a_{nk})_{n,k\geq 1}$, we consider the operators A_n for any integer $n \geq 1$, defined by

$$A_n(X) = \sum_{k=1}^{\infty} a_{nk} x_k \tag{2.1}$$

where $X = (x_n)_{n\geq 1}$ is a one-column matrix of complex numbers, and the series are assumed to be convergent. So we are led to the study of the infinite linear system

$$A_n(X) = b_n \quad \text{for } n = 1, 2, \ldots \tag{2.2}$$

where $B = (b_n)_{n\geq 1}$ is a one-column matrix and X the unknown ([2], [4], [5], [6], [7], [8]). The equations in (2.2) can be written in the form $AX = B$, where $AX = (A_n(X))_{n\geq 1}$. We also consider A as an operator from a sequence space into another sequence space.

By cs, we denote the set of all convergent series.

2.1. The sets $\mathbf{D_\tau E}$, where $\mathbf{E}$ is any of the sets $\mathbf{c_0}$, $\mathbf{c}$, or $\boldsymbol{\ell_\infty}$. Throughout, we use the set

$$U^+ = \{(u_n)_{n\geq 1} \in \omega : u_n > 0 \text{ for all } n\}.$$

For a given sequence $\tau = (\tau_n)_{n\geq 1} \in U^+$, we define the infinite diagonal matrix $D_\tau = (\tau_n \delta_{nk})_{n,k\geq 1}$; D_τ can be considered as the operator $X = (x_n)_{n\geq 1} \mapsto D_\tau X = (\tau_n x_n)_{n\geq 1}$ from ω to itself. For any subset E of ω, we write

$$D_\tau E = \left\{(x_n)_{n\geq 1} \in \omega : \left(\frac{x_n}{\tau_n}\right)_n \in E\right\},$$

in particular,

$$D_\tau E = \begin{cases} s_\tau^0 & \text{if } E = c_0, \\ s_\tau^{(c)} & \text{if } E = c, \\ s_\tau & \text{if } E = \ell_\infty \end{cases}$$

([3]). Each of the spaces $D_\tau E$, where $E \in \{c_0, c, \ell_\infty\}$, is a *BK space normed* by

$$\|X\|_{s_\tau} = \sup_{n\geq 1}\left(\frac{|x_n|}{\tau_n}\right), \tag{2.3}$$

and s_τ^0 *has AK*.

Now let $\tau = (\tau_n)_{n\geq 1}, \nu = (\nu_n)_{n\geq 1} \in U^+$. We write $S_{\tau,\nu}$ for the set of infinite matrices $A = (a_{nm})_{n,m\geq 1}$ such that $\sup_{n\geq 1}\left(\sum_{k=1}^\infty |a_{nk}|\tau_k/\nu_n\right) < \infty$. The set $S_{\tau,\nu}$ is a Banach space with the norm

$$\|A\|_{S_{\tau,\nu}} = \sup_{n\geq 1}\left(\frac{1}{\nu_n}\sum_{k=1}^\infty |a_{nk}|\tau_k\right).$$

It was proved in [8] that $A \in (s_\tau, s_\nu)$ if and only if $A \in S_{\tau,\nu}$, that is, $(s_\tau, s_\nu) = S_{\tau,\nu}$.

When $s_\tau = s_\nu$ then $S_\tau = S_{\tau,\nu}$ is a *Banach algebra with identity*, normed by $\|A\|_{S_\tau} = \|A\|_{S_{\tau,\tau}}$ ([3, 5]).

If $\tau = (r^n)_{n\geq 1}$, we write S_r, s_r, $\overset{0}{s}_r$ and $s_r^{(c)}$ for S_τ, s_τ, s_τ^0 and $s_\tau^{(c)}$, respectively. When $r = 1$, we obtain $s_1 = \ell_\infty$, $\overset{0}{s}_1 = c_0$ and $s_1^{(c)} = c$. Thus we have $S_1 = S_e$. It is well known ([12, Example 8.4.5A]) that

$$(s_1, s_1) = (c_0, s_1) = (c, s_1) = S_1.$$

For any subset E of ω, we put

$$AE = \{Y \in \omega : Y = AX \text{ for some } X \in E\}. \tag{2.4}$$

If F is a subset of ω, we write

$$F(A) = F_A = \{X \in \omega : Y = AX \in F\}. \tag{2.5}$$

2.2. Some properties of the sequence $\mathbf{C(X)X}$. Here we deal with the operators represented by $C(\Lambda)$ and $\Delta(\Lambda)$. Let U be the set of all sequences $(u_n)_{n\geq 1}$ with $u_n \neq 0$ for all n. We define $C(\Lambda) = (c_{nk})_{n,k\geq 1}$ for $\Lambda = (\lambda_n)_{n\geq 1} \in U$, by

$$c_{nk} = \begin{cases} \dfrac{1}{\lambda_n} & \text{if } k \leq n \\ 0 & \text{otherwise.} \end{cases}$$

We write $C(\Lambda)^T = C^+(\Lambda)$, $C(e) = \Sigma$ and $\Sigma^+ = \Sigma^T$. The matrix $\Delta(\Lambda) = (c'_{nk})_{n,k\geq 1}$ with

$$c'_{nk} = \begin{cases} \lambda_n & \text{if } k = n \\ -\lambda_{n-1} & \text{if } k = n-1 \text{ and } n \geq 2 \\ 0 & \text{otherwise} \end{cases}$$

is the inverse of $C(\Lambda)$ ([3], [4]). We need to recall some results given in [3], [7]. For this, we consider the following sets

$$\widehat{C_1} = \left\{ X = (x_n)_{n\geq 1} \in U^+ : \sup_n \left[\frac{1}{x_n} \left(\sum_{k=1}^{n} x_k \right) \right] < \infty \right\},$$

$$\widehat{C} = \left\{ X = (x_n)_{n\geq 1} \in U^+ : \left(\frac{1}{x_n} \left(\sum_{k=1}^{n} x_k \right) \right)_{n\geq 1} \in c \right\},$$

$$\widehat{C_1^+} = \left\{ X \in U^+ \bigcap cs : \sup_n \left[\frac{1}{x_n} \left(\sum_{k=n}^{\infty} x_k \right) \right] < \infty \right\},$$

$$\Gamma = \left\{ X \in U^+ : \overline{\lim}_{n\to\infty} \left(\frac{x_{n-1}}{x_n} \right) < 1 \right\},$$

$$\widehat{\Gamma} = \left\{ X \in U^+ : \lim_{n\to\infty} \left(\frac{x_{n-1}}{x_n} \right) < 1 \right\},$$

$$\Gamma^+ = \left\{ X \in U^+ : \overline{\lim}_{n\to\infty} \left(\frac{x_{n+1}}{x_n} \right) < 1 \right\}.$$

Note that $X \in \Gamma^+$ if and only if $1/X \in \Gamma$. We will see in Lemma 2.1 that if $X \in \widehat{C_1}$, then $x_n \to \infty$ $(n \to \infty)$. Furthermore, $X \in \Gamma$ if and only if there is an integer $q \geq 1$ such that

$$\gamma_q(X) = \sup_{n\geq q+1} \left(\frac{x_{n-1}}{x_n} \right) < 1.$$

Writing

$$[C(X)X]_n = \frac{1}{x_n} \left(\sum_{k=1}^{n} x_k \right),$$

we obtain the following the following result from [5, Proposition 2.1].

Lemma 2.1. *Let $\tau \in U^+$. Then*

i) $(\tau_{n-1}/\tau_n)_{n\geq 1} \to 0$ $(n \to \infty)$ *if and only if* $[C(\tau)\tau]_n \to 1$ $(n \to \infty)$.

ii) a) $\tau \in \widehat{C}$ *if and only if* $(\tau_{n-1}/\tau_n)_{n=0}^{\infty} \in c$,

b) $[C(\tau)\tau]_n \to l$ $(n \to \infty)$ *if and only if* $\tau_{n-1}/\tau_n \to 1 - 1/l$ $(n \to \infty)$.

iii) *If $\tau \in \widehat{C_1}$ then there are $K > 0$ and $\gamma > 1$ such that*

$$\tau_n \geq K\gamma^n \quad \text{for all } n. \tag{2.6}$$

iv) *The condition $\tau \in \Gamma$ implies $\tau \in \widehat{C_1}$ and there is a real $b > 0$ such that*

$$[C(\tau)\tau]_n \leq \frac{1}{1-\gamma_q(\tau)} + b\gamma_q(\tau)^n \quad \text{for } n \geq q+1. \tag{2.7}$$

v) *The condition $\tau \in \Gamma^+$ implies $\tau \in \widehat{C_1^+}$.*

It was also shown in [7] that

$$\widehat{C} = \widehat{\Gamma} \subset \Gamma \subset \widehat{C_1}.$$

3. The Characterizations of some operators mapping $w_\infty(\Lambda)$ into itself

In this section, we apply Theorem 1.3 to infinite matrices such as Δ_ρ^+, Δ_ρ considered as operators from $w_\infty(\Lambda)$ into $D_a w_\infty(\Lambda)$.

Throughout, we assume that $\Lambda = (\lambda_n)_{n\geq 1}$ is an exponentially bounded sequence, and $(\lambda_{n_i})_{i\geq 1}$ is an associated subsequence.

As an immediate consequence of Theorem 1.3, we see that $(w_\infty(\Lambda), w_\infty(\Lambda))$ is a *Banach algebra* with the norm

$$\|A\|^-_{(\Lambda,\Lambda)} = \sup_{X\neq 0}\left(\frac{\|AX\|^-_\Lambda}{\|X\|^-_\Lambda}\right). \tag{3.1}$$

where $\|\cdot\|^-$ is the norm defined in (1.1).

Since the sequence Λ with $\lambda_n = n$ $(n = 1, 2\dots)$ is exponentially bounded, we obtain in particular that (w_∞, w_∞) is a Banach algebra with the norm in (3.1) where

$$\|X\|^- = \sup_n\left(\frac{1}{n}\sum_{k=1}^n |x_k|\right).$$

3.1. Some properties of the matrix map $\boldsymbol{\Delta}_\rho^+$.

3.1.1. *The map Δ_ρ^+ considered as an operator from $w_\infty(\Lambda)$ into itself.* Let $\rho = (\rho_n)_{n\geq 1}$ and consider the infinite matrix $\Delta_\rho^+ = \{(\Delta_\rho^+)_{nk}\}$ defined by

$$(\Delta_\rho^+)_{nk} = \begin{cases} 1 & (k=n) \\ -\rho_n & (k=n+1) \\ 0 & (\text{otherwise}) \end{cases} \quad \text{for all } n,\ k.$$

First we see that if ρ and $(\lambda_{n+1}/\lambda_n)_{n\geq 1} \in \ell_\infty$ then $\Delta_\rho^+ \in (w_\infty(\Lambda), w_\infty(\Lambda))$. Indeed, we have

$$\Delta_\rho^+ x_n = x_n - \rho_n x_{n+1} \quad \text{for all } n$$

and

$$\begin{aligned}\frac{1}{\lambda_n}\sum_{k=1}^{n}|x_k - \rho_k x_{k+1}| &\leq \frac{1}{\lambda_n}\sum_{k=1}^{n}|x_k| + \frac{\lambda_{n+1}}{\lambda_n}\sup_n|\rho_n|\frac{1}{\lambda_{n+1}}\sum_{k=2}^{n+1}|x_k| \\ &\leq \left(1 + \sup_n\left(\frac{\lambda_{n+1}}{\lambda_n}\right)\sup_n|\rho_n|\right)\|X\|^-_{w_\infty(\Lambda)}.\end{aligned}$$

for all $X \in w_\infty(\Lambda)$ and for all n. This shows that $\Delta_\rho^+ X \in w_\infty(\Lambda)$ for all $X \in w_\infty(\Lambda)$.

More precisely we have the following result.

Theorem 3.1. *Let $\Lambda \in U^+$ be an exponentially bounded sequence and assume*

$$\overline{\lim}_{n\to\infty}\left(\frac{\lambda_{n+1}}{\lambda_n}\right) < \infty \quad \textit{and} \quad \overline{\lim}_{n\to\infty}|\rho_n| < \frac{1}{\overline{\lim}_{n\to\infty}\left(\frac{\lambda_{n+1}}{\lambda_n}\right)}. \tag{3.2}$$

For given $B \in w_\infty(\Lambda)$ the equation $\Delta_\rho^+ X = B$ has a unique solution in $w_\infty(\Lambda)$ given by

$$x_n = b_n + \sum_{i=n+1}^{\infty}\left(\prod_{j=n}^{i-1}\rho_j\right)b_i \quad \textit{for all } n. \tag{3.3}$$

Proof. Let

$$\Sigma_\rho^{+(N)} = \begin{pmatrix} \left[\Delta_\rho^{+(N)}\right]^{-1} & & 0 \\ & 1 & \\ 0 & & . \end{pmatrix},$$

where $\Delta_\rho^{+(N)}$ is the finite matrix whose elements are those of the N first rows and columns of Δ_ρ^+. The finite matrix $\Delta_\rho^{+(N)}$ is invertible, since it is an upper triangle. We get $\Delta_\rho^+\Sigma_\rho^{+(N)} = (a_{nk})_{n,k\geq 1}^\infty$, with $a_{nn} = 1$ for all n; $a_{n,n+1} = -\rho_n$ for all $n \geq N$; and $a_{nk} = 0$ otherwise. For any given $X \in w_\infty(\Lambda)$, we have $(I - \Delta_\rho^+\Sigma_\rho^{+(N)})X = (\xi_n(X))_{n\geq 1}$ with $\xi_n(X) = 0$ for all $n \leq N-1$ and $\xi_n(X) = \rho_n x_{n+1}$ for all $n \geq N$. Now we put $K_N = \sup_{n\geq N}(\lambda_{n+1}/\lambda_n)\sup_{k\geq N}|\rho_k|$ and show that (3.2) implies $K_N < 1$ for N large enough. For this let $\varepsilon > 0$ be given. From (3.2), we have $\overline{\lim}_{n\to\infty}|\rho_n| = l < \infty$, since $\overline{\lim}_{n\to\infty}\lambda_{n+1}/\lambda_n \geq 1$. Then there is an integer N_1 such that

$$\sup_{n\geq N_1}|\rho_n| < l + \varepsilon.$$

Now since $\overline{\lim}_{n\to\infty}\lambda_{n+1}/\lambda_n < \infty$, (3.2) implies $\overline{\lim}_{n\to\infty}\lambda_{n+1}/\lambda_n = L < 1/l$, and as above there is an integer N_2 such that

$$\sup_{n\geq N_2}\left(\frac{\lambda_{n+1}}{\lambda_n}\right) < L + \varepsilon.$$

Since $lL < 1$, for ε small enough, taking $N = \max\{N_1, N_2\}$, we then have

$$K_N \leq (l+\varepsilon)(L+\varepsilon) = lL + l\varepsilon + L\varepsilon + \varepsilon^2 < 1.$$

Now we obtain

$$\begin{aligned}\left\|(I - \Delta_\rho^+\Sigma_\rho^{+(N)})X\right\|_{w_\infty(\Lambda)}^- &= \sup_{n\geq N}\left(\frac{1}{\lambda_n}\sum_{k=N}^{n}|\rho_k x_{k+1}|\right)\\ &\leq \sup_{n\geq N}\left[\left(\sup_{k\geq N}|\rho_k|\right)\frac{1}{\lambda_n}\sum_{k=N+1}^{n+1}|x_k|\right]\\ &\leq \sup_{n\geq N}\left(\frac{\lambda_{n+1}}{\lambda_n}\right)\sup_{k\geq N}|\rho_k|\sup_{n\geq N}\left(\frac{1}{\lambda_{n+1}}\sum_{k=N+1}^{n+1}|x_k|\right)\\ &\leq \left[\sup_{n\geq N}\left(\frac{\lambda_{n+1}}{\lambda_n}\right)\sup_{k\geq N}|\rho_k|\right]\|X\|_{w_\infty(\Lambda)}^-.\end{aligned}$$

Then $\|(I - \Delta_\rho^+\Sigma_\rho^{+(N)})X\|_{w_\infty(\Lambda)}^- \leq K_N\|X\|_{w_\infty(\Lambda)}^-$ and we conclude

$$\left\|I - \Delta_\rho^+\Sigma_\rho^{+(N)}\right\|_{(\Lambda,\Lambda)}^- = \sup_{X\neq 0}\left(\frac{\left\|(I - \Delta_\rho^+\Sigma_\rho^{(N)})X\right\|_{w_\infty(\Lambda)}^-}{\|X\|_{w_\infty(\Lambda)}^-}\right) \leq K_N < 1.$$

Then (3.2) implies $\|I - \Delta_\rho^+\Sigma_\rho^{+(N)}\|_{(\Lambda,\Lambda)}^- \leq K_N < 1$ and $\Delta_\rho^+\Sigma_\rho^{+(N)}$ has a unique inverse in the Banach algebra $(w_\infty(\Lambda), w_\infty(\Lambda))$. Since obviously $\Sigma_\rho^{+(N)}$ is bijective from $w_\infty(\Lambda)$ into itself, the operators defined by $\Delta_\rho^+\Sigma_\rho^{+(N)}$ and $\Delta_\rho^+ = (\Delta_\rho^+\Sigma_\rho^{+(N)})(\Sigma_\rho^{+(N)})^{-1}$ are bijective from $w_\infty(\Lambda)$ into itself. So, for any given $B \in w_\infty(\Lambda)$, the equation $\Delta_\rho^+ X = B$ has a unique solution in $w_\infty(\Lambda)$. Finally $(\Delta_\rho^+)^{-1} = \sum_{i=0}^\infty (I - \Delta_\rho^+)^i = \left[[(\Delta_\rho^+)^T]^{-1}\right]^T$ and an elementary calculation shows that

$$(\Delta_\rho^+)^{-1} = \left[\left[(\Delta_\rho^+)^T\right]^{-1}\right]^T = \begin{pmatrix} 1 & \rho_1 & \rho_1\rho_2 & . & . \\ & 1 & \rho_2 & \rho_2\rho_3 & . \\ & & . & . & \\ 0 & & & 1 & \rho_n \\ & & & & . \end{pmatrix}.$$

We then obtain (3.3). This completes the proof. □

Proposition 3.2. *For given $B \in w_\infty(\Lambda)$, the equation $\Delta_\rho^+ X = B$ has a unique solution in $w_\infty(\Lambda)$ given by* (3.3) *in the following cases*

i) $\Lambda \in \widehat{C_1}$ *and* $\overline{\lim}_{n\to\infty}(\lambda_{n+1}/\lambda_n)|\rho_n| < 1$*;*

ii) $\Lambda \notin \widehat{C_1}$ *and condition* (3.2) *holds.*

Proof. i) The condition $\Lambda \in \widehat{C_1}$ implies $w_\infty(\Lambda) = s_\Lambda$. Furthermore, the condition $\overline{\lim}_{n\to\infty}(\lambda_{n+1}/\lambda_n)|\rho_n| < 1$ implies that there is an integer N such that

$$\sup_{n\geq N}\left(\frac{\lambda_{n+1}}{\lambda_n}|\rho_n|\right) < 1.$$

Now since $(w_\infty(\Lambda), w_\infty(\Lambda)) = (s_\Lambda, s_\Lambda) = S_\Lambda$, we deduce

$$\left\|I - \Delta_\rho^+\Sigma_\rho^{+(N)}\right\|^-_{(\Lambda,\Lambda)} = \left\|I - \Delta_\rho^+\Sigma_\rho^{+(N)}\right\|_{S_\lambda} = \sup_{n\geq N}\left(\frac{\lambda_{n+1}}{\lambda_n}|\rho_n|\right) < 1.$$

So we have shown i).

ii) This is a direct consequence of Theorem 3.1. □

Remark 3.3. Note that since $\overline{\lim}_{n\to\infty}(\lambda_{n+1}/\lambda_n)|\rho_n| \leq \overline{\lim}_{n\to\infty}(\lambda_{n+1}/\lambda_n)\overline{\lim}_{n\to\infty}|\rho_n|$, condition (3.2) is weaker than the condition $\overline{\lim}_{n\to\infty}(\lambda_{n+1}/\lambda_n)|\rho_n| < 1$ in Proposition 3.2 ii).

Putting $\Lambda^- = (1, \lambda_1, \dots, \lambda_{n-1}, \dots)$, we easily deduce the following result.

Corollary 3.4. *We assume $\Lambda \in \widehat{C_1}$. Then we have*

i) $w_\infty(\Lambda) = s_\Lambda$*;*

ii) *ii)* $w_\infty(\Lambda)(\Delta^+) = w_\infty(\Lambda^-) = s_{\Lambda^-}$ *and, for any given* $B \in w_\infty(\Lambda)$, *the equation*

$$\Delta^+ X = B \tag{3.4}$$

has infinitely many solutions in $w_\infty(\Lambda^-)$ *given by* $x_n = u - \sum_{k=1}^{n-1} b_k$, *where* u *is an arbitrary scalar.*

Proof. i) Part (i) comes from [4].

ii) It can easily be seen that $\Lambda \in \widehat{C_1}$ implies $\Lambda^- \in \widehat{C_1}$ and since $w_\infty(\Lambda) = s_\Lambda$, we only have to show that $s_\Lambda(\Delta^+) = s_{\Lambda^-}$. From the inequality

$$\frac{\lambda_{n-1}}{\lambda_n} \leq \sup_n \left(\frac{\sum_{k=1}^n \lambda_k}{\lambda_n} \right) < \infty \quad \text{for all } n,$$

we deduce that $\sup_n(\lambda_{n-1}/\lambda_n) < \infty$ and $\Delta^+ \in (s_{\Lambda^-}, s_\Lambda)$. Then, for any given $B \in s_\Lambda$, the solutions of the equation $\Delta^+ X = B$ are given by $x_1 = -u$, and

$$x_n = u + \sum_{k=1}^{n-1} b_k \text{ for } n \geq 2, \quad \text{where } u \text{ is an arbitrary scalar.} \tag{3.5}$$

So there exists a real $K > 0$, such that

$$\frac{|x_n|}{\lambda_{n-1}} = \frac{\left|u + \sum_{k=1}^{n-1} b_k\right|}{\lambda_{n-1}} \leq \sup_n \left(\frac{|u| + K\left(\sum_{k=1}^{n-1} \lambda_k\right)}{\lambda_{n-1}} \right) < \infty,$$

since $\sup_n(|u|/\lambda_{n-1}) < \infty$. So $X \in s_{\Lambda^-}$ and we conclude that Δ^+ is surjective from s_{Λ^-} into s_Λ. Then $\Lambda \in \widehat{C_1}$ implies

$$s_\Lambda(\Delta^+) = s_{\Lambda^-}.$$

The previous argument shows that the equation $\Delta^+ X = B$ has infinitely many solutions in $w_\infty(\Lambda^-) = s_{\Lambda^-}$ given by $x_n = u - \sum_{k=1}^{n-1} b_k$. This completes the proof. ☐

3.1.2. *Properties of* $\Delta^+(\mu)$ *considered as an operator from* $D_\tau w_\infty(\Lambda)$ *to* $D_{\tau\mu} w_\infty(\Lambda)$. Here we consider the set

$$D_\tau w_\infty(\Lambda) = \left\{ X = (x_n)_{n\geq 1} : \sup_n \left(\frac{1}{\lambda_n} \sum_{k=1}^n \left| \frac{x_k}{\tau_k} \right| \right) < \infty \right\}.$$

We need the following lemma.

Lemma 3.5. *Let* E *and* F *be linear subspaces of* ω *and assume* $A \in (E, F)$. *If* A *is a surjective map from* E *to* F *and* $\operatorname{Ker} A = \{0\}$, *then*

$$F(A) = E.$$

Proof. Let $X \in F(A)$. Then $Y = AX \in F$. Now we show that $AX \in F$ implies $X \in E$. For this, we assume that $AX \in F$ and $X \notin E$. Since A is surjective from E to F, there is $X_0 \in E$ such that $Y = AX = AX_0$ and $X - X_0 \in Ker A$. So $X = X_0 \in E$ which gives a contradiction. Thus we have shown $F(A) \subset E$. The inclusion $E \subset F(A)$ is immediate. This shows the lemma. $\square$

Note, for instance, that for $0 < r < 1$, we have $\|I - \Delta^+\|_{S_r} = r < 1$. This means that Δ^+ is bijective from s_r to itself. Since $e \in \operatorname{Ker} \Delta^+ \setminus \{\tau\}$, we have $\Delta^+ e = 0 \in s_\tau$ and $e \in s_\tau(\Delta^+) \setminus s_\tau$. This shows $s_r(\Delta^+) \neq s_r$.

We have as a direct consequence of Theorem 3.1

Corollary 3.6. *Let $\Lambda, \mu, \tau \in U^+$ and assume*

$$\overline{\lim}_{n\to\infty} \frac{\tau_{n+1}}{\tau_n} < \frac{1}{\overline{\lim}_{n\to\infty}\left(\frac{\lambda_{n+1}}{\lambda_n}\right)}. \tag{3.6}$$

i) *Then Δ^+ is bijective from $D_\tau w_\infty(\Lambda)$ into itself and $\Delta^+(\mu)$ is bijective from $D_\tau w_\infty(\Lambda)$ into $D_{\tau\mu} w_\infty(\Lambda)$.*

ii) *$(D_\tau w_\infty(\Lambda))(C^+(\mu)) = D_{\tau\mu} w_\infty(\Lambda)$.*

Proof. i) Here we have that $\Delta^+ X = B$ with $B, X \in D_\tau w_\infty(\Lambda)$ (X being the unknown) is equivalent to $(D_{1/\tau}\Delta^+ D_\tau)X' = D_{1/\tau}B$ where $D_{1/\tau}B \in w_\infty(\Lambda)$ and $X' = D_{1/\tau}X \in w_\infty(\Lambda)$. So $\rho_n = \tau_{n+1}/\tau_n$ for all n and, by Theorem 3.1, the operator Δ^+ is bijective from $D_\tau w_\infty(\Lambda)$ into itself. Now since $\Delta^+(\mu) = D_\mu \Delta^+$, we easily conclude that Δ^+ is bijective from $D_\tau w_\infty(\Lambda)$ into itself and D_μ is bijective from $D_\tau w_\infty(\Lambda)$ into $D_{\tau\mu} w_\infty(\Lambda)$ and $\Delta^+(\mu)$ is bijective from $D_\tau w_\infty(\Lambda)$ into $D_{\tau\mu} w_\infty(\Lambda)$.

ii) Since $\Delta^+(\mu)$ is bijective from $D_\tau w_\infty(\Lambda)$ into $D_{\tau\mu} w_\infty(\Lambda)$, $C^+(\mu) = \Sigma^+ D_{1/\mu} = (\Delta^+(\mu))^{-1}$ is bijective from $D_{\tau\mu} w_\infty(\Lambda)$ into $D_\tau w_\infty(\Lambda)$. Now the equation $C^+(\mu)X = 0$ is equivalent to $\Sigma^+ Y = 0$ with $Y = D_{1/\mu}X = (y_n)_{n\geq 1}$ and $Y \in cs$. Then we have

$$\sum_{k=n}^{\infty} y_k = 0 \quad \text{for } n = 1, 2, \dots .$$

We conclude $y_1 = y_2 = \cdots = y_n = 0$ for all n and $\operatorname{Ker} \Sigma^+ = \operatorname{Ker} C^+(\mu) = \{0\}$. By Lemma 3.5, we conclude $(D_\tau w_\infty(\Lambda))(C^+(\mu)) = D_{\tau\mu} w_\infty(\Lambda)$. $\square$

Corollary 3.7. *If $\tau \in \Gamma^+$, then the equation $\Delta^+ X = B$ has a unique solution in $D_\tau w_\infty$ for any given $B \in D_\tau w_\infty$.*

Proof. Indeed by Corollary 3.6, the condition $\tau \in \Gamma^+$ implies that (3.2) is satisfied for $\lambda_n = n$, that is,

$$\overline{\lim}_{n\to\infty} \frac{\tau_{n+1}}{\tau_n} < \frac{1}{\overline{\lim}_{n\to\infty}\left(\frac{n+1}{n}\right)} = 1. \qquad \square$$

These results lead to the next remarks.

Remark 3.8. Note that if $\Lambda \in \Gamma^+$ then (3.6) yields $1 < 1/\overline{\lim}_{n\to\infty}(\lambda_{n+1}/\lambda_n)$ and Δ^+ is bijective from $w_\infty(\Lambda)$ into itself.

Remark 3.9. Using the inverse of the infinite matrix $(\Delta_\rho^+)^T$ ([2]), we see that, under (3.2), we also have that $\sum_{k=1}^n |x_k| = O(\lambda_n)$ $(n \to \infty)$ implies

$$\sum_{k=1}^{n}\left|x_k + \sum_{i=k+1}^{\infty}\left(\prod_{j=k}^{i-1}\rho_j\right)x_i\right| = O(\lambda_n) \quad (n\to\infty) \quad \text{for all } X \in \omega.$$

Now we consider an example.

Example 3.10. We choose $\lambda_n = n$ and $\rho_n = a_n - 1/n^\alpha$ for all n, where $(a_n)_n \in c$ is a positive sequence with $a_n \to l$ for some $l < 1$ $(n \to \infty)$, and $\alpha > 0$. We see that if $\sum_{k=1}^n |x_k| = O(n)$ $(n \to \infty)$ then

$$\sum_{k=1}^{n}\left|x_k + \sum_{i=k+1}^{\infty}\left[\prod_{j=k}^{i-1}\left(a_j - \frac{1}{j^\alpha}\right)\right]x_i\right| = O(n) \quad (n\to\infty)$$

for all $X \in \omega$.

Remark 3.11. Note that, by Corollary 3.4 ii), we have $\Lambda \in \widehat{C_1}$ if and only if $w_\infty(\Lambda)(\Delta^+) = w_\infty(\Lambda^-)$. Indeed, since $w_\infty(\Lambda) = s_\Lambda$, it is enough to assume that $s_\Lambda(\Delta^+) = s_{\Lambda^-}$. If we take $B = \Lambda \in s_\Lambda$, then the solutions of the equation $\Delta^+ X = B$ belong to s_{Λ^-}, and are given by $x_n = x_1 - \sum_{k=1}^{n-1}\lambda_k$, where x_1 is an arbitrary scalar and

$$\frac{x_n}{\lambda_{n-1}} = \frac{x_1}{\lambda_{n-1}} - \frac{1}{\lambda_{n-1}}\left(\sum_{k=1}^{n-1}\lambda_k\right) = O(1) \quad (n\to\infty).$$

Putting $x_1 = 0$, we conclude that $\Lambda \in \widehat{C_1}$.

3.2. Some properties of the matrix map $\boldsymbol{\Delta_\rho}$.

3.2.1. *On the operator Δ_ρ mapping $w_\infty(\Lambda)$ into itself.* Now we study the triangle $\Delta_\rho = (\Delta_\rho^+)^T$, that is,

$$\Delta_\rho = \begin{pmatrix} 1 & & & & \\ -\rho_1 & 1 & & 0 & \\ & \cdot & \cdot & & \\ & & -\rho_{n-1} & 1 & \cdot \\ & & & \cdot & \cdot \end{pmatrix}.$$

considered as an operator that maps $w_\infty(\Lambda)$ to itself.

As in Subsection 3.1.1, we easily see that $\Delta_\rho \in (w_\infty(\Lambda), w_\infty(\Lambda))$, if $(\lambda_{n-1}/\lambda_n)_{n\geq 2} \in \ell_\infty$.

Theorem 3.12. *Let $\Lambda \in U^+$ be an exponentially bounded sequence. We assume*

$$\overline{\lim}_{n\to\infty}|\rho_n| < \frac{1}{\overline{\lim}_{n\to\infty}\left(\frac{\lambda_{n-1}}{\lambda_n}\right)}. \tag{3.7}$$

Then, for any given $B \in w_\infty(\Lambda)$, the equation $\Delta_\rho X = B$ has a unique solution in $w_\infty(\Lambda)$ given by $x_1 = b_1$ and 66

$$x_n = b_n + \sum_{k=1}^{n-1}\left(\prod_{j=k}^{n-1}\rho_j\right)b_k \quad \text{for all } n \geq 2. \tag{3.8}$$

Proof. Let $\Sigma_\rho^{(N)}$ be defined by

$$\begin{pmatrix} \left[\Delta_\rho^{(N)}\right]^{-1} & & 0 \\ & 1 & \\ 0 & & . \end{pmatrix},$$

where $\Delta_\rho^{(N)}$ is the finite matrix whose elements are those of the N first rows and columns of Δ_ρ. The finite matrix $\Delta_\rho^{(N)}$ is invertible, since it is a triangle. We get $\Sigma_\rho^{(N)}\Delta_\rho = (a_{nk})_{n,k\geq 1}$, with $a_{nn} = 1$ for all n; $a_{n,n-1} = -\rho_{n-1}$ for all $n \geq N+1$; and $a_{nm} = 0$ otherwise. For any given $X \in w_\infty(\Lambda)$, we have $(I - \Sigma_\rho^{(N)}\Delta_\rho)X = (\xi_n(X))_{n\geq 1}$ with $\xi_n(X) = 0$ for all $n \leq N$ and $\xi_n(X) = \rho_{n-1}x_{n-1}$ for all $n \geq N+1$. Now put $K'_N = \sup_{n\geq N+1}(\lambda_{n-1}/\lambda_n)\sup_{k\geq N+1}|\rho_{k-1}|$. Then we have as in Theorem 3.1

$$\sup_{n\geq N+1}\left[\left(\sup_{N\leq k\leq n}|\rho_k|\right)\frac{\lambda_{n-1}}{\lambda_n}\right] \leq K'_N < 1 \quad \text{for all } n \geq N. \tag{3.9}$$

Furthermore, we obtain

$$\left\|\left(I - \Sigma_\rho^{(N)}\Delta_\rho\right)X\right\|^-_{w_\infty(\Lambda)} = \sup_{n\geq N+1}\left(\frac{1}{\lambda_n}\sum_{k=N+1}^{n}|\rho_{k-1}x_{k-1}|\right)$$

$$\leq \sup_{n\geq N+1}\left[\left(\sup_{N+1\leq k\leq n}|\rho_{k-1}|\right)\frac{\lambda_{n-1}}{\lambda_n}\right]\sup_{n\geq N}\left(\frac{1}{\lambda_{n-1}}\sum_{k=N}^{n-1}|x_k|\right) \leq K'_N\|X\|^-_{w_\infty(\Lambda)}.$$

Finally, using (3.9), we get

$$\left\|I - \Sigma_\rho^{(N)}\Delta_\rho\right\|^-_{(\Lambda,\Lambda)} \leq K'_N < 1,$$

and we conclude, reasoning as Theorem 3.1. □

Remark 3.13. Note that since Λ is a non-decreasing sequence, we do not need to assume $\overline{\lim}_{n\to\infty}(\lambda_{n-1}/\lambda_n) < \infty$ in Theorem 3.12.

We immediately get

Corollary 3.14. *Under* (3.7), *we have* $w_\infty(\Lambda)(\Delta_\rho) = w_\infty(\Lambda)$.

Corollary 3.15. i) *We assume that* (3.7) *holds. Then*

$$\sup_n \left(\frac{1}{\lambda_n} \sum_{k=1}^{n} |x_k| \right) < \infty$$

implies

$$\sup_n \left\{ \frac{1}{\lambda_n} \sum_{k=2}^{n} \left| x_k + \sum_{j=1}^{k-1} \left(\prod_{i=j}^{k-1} \rho_i \right) x_j \right| \right\} < \infty \quad \textit{for all } X.$$

ii) *If* $\Lambda \in \Gamma$ *and* $\overline{\lim}_{n\to\infty}|\rho_n| < 1$, *then*

$$\sup_n \left(\frac{|x_k|}{\lambda_n} \right) < \infty$$

implies

$$\sup_n \left\{ \frac{1}{\lambda_n} \left| x_n + \sum_{j=1}^{n-1} \left(\prod_{i=j}^{n-1} \rho_i \right) x_j \right| \right\} < \infty \quad \textit{for all } X.$$

Proof. i) Part i) is an immediate consequence of Theorem 3.12.

ii) Since $\Lambda \in \Gamma$, that is, $\overline{\lim}_{n\to\infty}(\lambda_{n-1}/\lambda_n) < 1$, we have

$$\overline{\lim}_{n\to\infty}|\rho_n| < 1 < \frac{1}{\overline{\lim}_{n\to\infty}(\lambda_{n-1}/\lambda_n)}$$

and condition (3.7) follows. Now, by Corollary 3.4, we have $w_\infty(\Lambda) = s_\Lambda$ and, by Theorem 3.12, Δ_ρ is bijective from s_Λ into itself. We conclude, using identity (3.8). $\square$

Remark 3.16. Putting $\Delta_1 = \Delta$, we see that if $\Lambda \in \Gamma$ then $w_\infty(\Lambda)(\Delta) = w_\infty(\Lambda)$. Indeed by Theorem 3.12, condition (3.7) implies $\overline{\lim}_{n\to\infty}(\lambda_{n-1}/\lambda_n) < 1$, that is, $\Lambda \in \Gamma$. Since $\Gamma \subset \widehat{C_1}$, applying Corollary 3.4, we conclude $w_\infty(\Lambda) = s_\Lambda$ and $s_\Lambda(\Delta) = s_\Lambda$.

These results lead to consider the sets $[C(\Lambda), C(\mu)]_\infty$ and $[C(\Lambda), \Delta_\rho(\mu)]_\infty$, ([5]). In this way we obtain the following results.

3.2.2. *On the sets* $[C(\Lambda), C(\mu)]_\infty$ *and* $[C(\Lambda), \Delta_\rho(\mu)]_\infty$. Now we can give some applications of the previous results. For this, we define for $\Lambda, \mu \in U^+$ the set

$$[C(\Lambda), C(\mu)]_\infty = \{X \in \omega : C(\Lambda)(|C(\mu)X|) \in \ell_\infty\}.$$

Then we have

$$[C(\Lambda), C(\mu)]_\infty = \left\{X \in \omega : \sup_n \left(\frac{1}{\lambda_n} \sum_{k=1}^{n} \frac{1}{\mu_k} \left|\sum_{i=1}^{k} x_i\right|\right) < \infty\right\}.$$

It was shown in [5, Theorem 3.1] that if $\Lambda, \Lambda\mu \in \widehat{C_1}$ then

$$[C(\Lambda), C(\mu)]_\infty = s_{\Lambda\mu}.$$

This result can be improved here as follows. We put $C_1 = C((n)_{n\geq 1})$. The following result holds.

Proposition 3.17. *Let* $\mu, \Lambda \in U^+$.

i) *If*

$$\overline{\lim}_{n\to\infty} \frac{\mu_{n-1}}{\mu_n} < \frac{1}{\overline{\lim}_{n\to\infty} \dfrac{\lambda_{n-1}}{\lambda_n}},$$

then $[C(\Lambda), C(\mu)]_\infty = D_\mu w_\infty(\Lambda)$.

ii) *If* $\mu \in \Gamma$, *then* $A \in ([C_1, C(\mu)]_\infty, s_\eta)$ *if and only if*

$$\sum_{i=1}^{\infty} 2^i \max_{2^i \leq m \leq 2^{i+1}-1} |a_{nm}|\mu_m = \eta_n O(1) \quad (n \to \infty).$$

Proof. i) We have $\Delta_\rho = D_{1/\mu} \Delta D_\mu$ with $\rho_n = \mu_{n-1}/\mu_n$ for all n, and $\Delta : D_\mu w_\infty(\Lambda) \longmapsto D_\mu w_\infty(\Lambda)$ is bijective, so $D_\mu w_\infty(\Lambda)(\Delta) = D_\mu w_\infty(\Lambda)$. We have $X \in [C(\Lambda), C(\mu)]_\infty$ if and only if $C(\mu)X \in w_\infty(\Lambda)$, that is

$$X \in C(\mu)^{-1} w_\infty(\Lambda);$$

and since $C(\mu)^{-1} = \Delta(\mu) = \Delta D_\mu$, we have $X \in \Delta D_\mu w_\infty(\Lambda) = D_\mu w_\infty(\Lambda)$. Thus we have shown Part i).

ii) Here $\lambda_n = n$ for all n, so $\overline{\lim}_{n\to\infty}(\lambda_{n-1}/\lambda_n) = 1$ and condition (3.7) is satisfied, since $\mu \in \Gamma$. By Part i), we have $[C_1, C(\mu)]_\infty = D_\mu w_\infty$ and $A \in ([C_1, C(\mu)]_\infty, s_\eta)$ if and only if $D_{1/\eta} A D_\mu \in (w_\infty(\Lambda), \ell_\infty)$. We conclude, using the characterization of $(w_\infty(\Lambda), \ell_\infty)$, given in [11, Remark, p. 33]. $\square$

Now we write $\Delta_\rho(\mu) = \Delta_\rho D_\mu$ and

$$[C(\Lambda), \Delta_\rho(\mu)]_\infty = \{X \in \omega : C(\Lambda)(|\Delta_\rho(\mu)X|) \in \ell_\infty\}.$$

Then we have

$$[C(\Lambda), \Delta_\rho(\mu)]_\infty = \left\{X \in \omega : \sup_n \left(\frac{1}{\lambda_n}\sum_{k=1}^{n} |\mu_k x_k - \mu_{k-1}\rho_{k-1}x_{k-1}|\right) < \infty\right\}.$$

We obtain the following result.

Proposition 3.18. *Let $\Lambda, \mu, \rho \in U^+$.*

i) *We assume that* (3.7) *holds. Then* $[C(\Lambda), \Delta_\rho(\mu)]_\infty = D_{1/\mu} w_\infty(\Lambda)$;

ii) $A \in ([C(\Lambda), \Delta_\rho(\mu)]_\infty, s_\eta)$ *if and only if*

$$\sum_{i=1}^{\infty} 2^i \max_{2^i \le m \le 2^{i+1}-1} \frac{|a_{nm}|}{\mu_n} = \eta_n O(1) \quad (n \to \infty).$$

Proof. Here $C(\Lambda)(|\Delta_\rho(\mu)X|) \in \ell_\infty$ if and only if $D_\mu X \in w_\infty(\Lambda)(\Delta_\rho)$ and, by (3.7), we get $w_\infty(\Lambda)(\Delta_\rho) = w_\infty(\Lambda)$.

(ii) We obtain Part ii), reasoning as in Proposition 3.17 ii). □

4. On an infinite tridiagonal matrix considered as an operator from $w_\infty(\Lambda)$ into itself and applications

In this section, we give some results on the infinite tridiagonal matrix $M(\gamma, a, \eta)$, considered as an operator from χ into $D_a\chi$ where χ is any of the sets $w_\infty(\Lambda)$, s_Λ,s_Λ^0, $s_\Lambda^{(c)}$, or $\ell_p(\Lambda)$.

Note that the previous results on Δ_ρ^+ and Δ_ρ cannot be considered as a consequence of the next ones.

Let $\gamma = (\gamma_n)_{n\ge1}$, $\eta = (\eta_n)_{n\ge1}$ and $a = (a_n)_{n\ge1}$ be given sequences, and $a \in U$. We consider the infinite tridiagonal matrix

$$M(\gamma, a, \eta) = \begin{pmatrix} a_1 & \eta_1 & & & \\ \gamma_1 & a_2 & \eta_2 & & \mathbf{O} \\ & . & . & . & \\ \mathbf{O} & & \gamma_{n-1} & a_n & \eta_n \\ & & & . & . \end{pmatrix}.$$

In the following we use the sets

$$\ell_p = \left\{ X = (x_n)_{n\geq 1} : \sum_{n=1}^{\infty} |x_n|^p < \infty \right\} \quad \text{for } 1 \leq p < \infty$$

and $\ell_p(\tau) = D_\tau \ell_p$ for $\tau \in U^+$.

We notice that $D_a s_\tau = D_{|a|} s_\tau = s_{|a|\tau}$, $D_a s_\tau^0 = D_{|a|} s_\tau^0 = s_{|a|\tau}^0$ and $D_a \ell_p(\tau) = \ell_p(|a|\tau)$ for $\tau \in U^+$.

We suppose that $K_1 = \sup_{n\geq 2} |\gamma_{n-1}/a_n| < 1$ and $K_2 = \sup_n |\eta_n/a_n| > 0$.

The following theorem holds.

Theorem 4.1. *Let $\Lambda \in U^+$ be an exponentially bounded sequence and assume that*

$$\sup_n \left(\frac{\lambda_{n+1}}{\lambda_n} \right) < \frac{1-K_1}{K_2}. \tag{4.1}$$

Then, for any given $B \in D_a w_\infty(\Lambda)$, the equation $M(\gamma, a, \eta)X = B$ has a unique solution in $w_\infty(\Lambda)$ given by $X = \sum_{i=0}^{\infty} (I - M(\gamma, a, \eta))^i B$.

Proof. We consider $D_{1/a} M(\gamma, a, \eta) = M(\zeta, e, \rho)$ with $\zeta_{n-1} = \gamma_{n-1}/a_n$ for $n \geq 2$ and $\rho_n = \eta_n/a_n$ for all n. Then we have

$$M(\zeta, e, \rho) = M(\zeta, \rho) = \begin{pmatrix} 1 & \rho_1 & & & \\ \zeta_1 & 1 & \rho_2 & & \mathbf{O} \\ & . & . & . & \\ \mathbf{O} & & \zeta_{n-1} & 1 & \rho_n \\ & & & . & . \end{pmatrix}.$$

(Note that $M(\zeta, \rho) = (\Delta_{-2\zeta} + \Delta_{-2\rho}^+)/2$). Here we have

$$(I - M(\zeta, \rho))\, X = -(\rho_1 x_2, \zeta_1 x_1 + \rho_2 x_3, \dots, \zeta_{n-1} x_{n-1} + \rho_n x_{n+1}, \dots)^T.$$

Then

$$\|(I - M(\zeta, \rho))X\|_{w_\infty(\Lambda)}^- = \sup_n \left(\frac{1}{\lambda_n} \sum_{k=1}^{n} |\zeta_{k-1} x_{k-1} + \rho_k x_{k+1}| \right)$$

with $x_0 = 0$ and

$$\frac{1}{\lambda_n} \sum_{k=1}^{n} |\zeta_{k-1} x_{k-1} + \rho_k x_{k+1}| \leq \frac{\lambda_{n-1}}{\lambda_n} \sup_{k\leq n-1} |\zeta_k| \frac{1}{\lambda_{n-1}} \sum_{k=1}^{n-1} |x_k|$$

$$+ \frac{\lambda_{n+1}}{\lambda_n} \sup_{k\leq n} |\rho_k| \frac{1}{\lambda_{n+1}} \sum_{k=2}^{n+1} |x_k| \leq \left[\sup_n \left(\frac{\lambda_{n-1}}{\lambda_n} \right) K_1 + \sup_n \left(\frac{\lambda_{n+1}}{\lambda_n} \right) K_2 \right] \|X\|_{w_\infty(\Lambda)}^-.$$

Since Λ is a non-decreasing sequence, $\sup_n(\lambda_{n-1}/\lambda_n) \leq 1$, and using (4.1), we easily conclude that

$$\begin{aligned}\|I - M(\zeta,\rho)\|^-_{(w_\infty(\Lambda), w_\infty(\Lambda))} &= \sup_{X\neq 0} \frac{\|(I - M(\zeta,\rho))X\|^-_{w_\infty(\Lambda)}}{\|X\|^-_{w_\infty(\Lambda)}} \\ &\leq K_1 + \sup_n \left(\frac{\lambda_{n+1}}{\lambda_n}\right) K_2 < 1.\end{aligned}$$

Finally, for any given $B \in w_\infty(\Lambda)$, the equation $M(\gamma, a, \eta)X = B$ is equivalent to $M(\zeta,\rho)X = D_{1/a}B$. So, for any given $B \in D_a w_\infty(\Lambda)$, the last equation has a unique solution in $w_\infty(\Lambda)$. $\square$

Corollary 4.2. *Let $\lambda \in U^+$ be an exponentially bounded sequence and assume that* (4.1) *holds. Then*

i) *$M(\gamma,a,\eta)$ is bijective from $w_\infty(\Lambda)$ to $D_a w_\infty(\Lambda)$ and $M(\gamma, a, \eta)^{-1} \in (D_a w_\infty(\Lambda), w_\infty(\Lambda))$;*

ii) *$M(\gamma, a, \eta)$ is bijective from s_Λ into $s_{|a|\Lambda}$ and $M(\gamma, a, \eta)^{-1} \in (s_{|a|\Lambda}, s_\Lambda)$;*

iii) *$M(\gamma, a, \eta)$ is bijective from $\overset{\circ}{s}_\Lambda$ into $\overset{\circ}{s}_{|a|\Lambda}$ and $M(\gamma, a, \eta)^{-1} \in (\overset{\circ}{s}_{|a|\Lambda}, \overset{\circ}{s}_\Lambda)$;*

iv) *for $a \in U^+$, the condition $\lim_{n\to\infty}(\gamma_{n-1}\lambda_{n-1} + \eta_n\lambda_{n+1})\lambda_n^{-1}a_n^{-1} = l \neq 0$, implies that $M(\gamma, a, \eta)$ is bijective from $s^{(c)}_\Lambda$ into $s^{(c)}_{a\Lambda}$, and $M(\gamma, a, \eta)^{-1} \in (s^{(c)}_{a\Lambda}, s^{(c)}_\Lambda)$.*

v) *Let $p \geq 1$ be a real. If $\widetilde{K}_{p,\Lambda} = K_1 + K_2' < 1$ with $K_1 = \sup_n(|\gamma_{n-1}/a_n|)$ and $K_2' = \sup_n(|\eta_n/a_n|\lambda_{n+1}/\lambda_n)$, then $M(\gamma, a, \eta)$ is bijective from $\ell_p(\Lambda)$ into $\ell_p(|a|\Lambda)$, and $M(\gamma, a, \eta)^{-1} \in (\ell_p(|a|\Lambda), \ell_p(\Lambda))$.*

Proof. By [5, Theorem 6.1] 4.1 and Condition (4.1), we have

$$\xi(\Lambda) = \sup_{n\geq 1}\left[\frac{1}{|a_n|}\left(|\gamma_{n-1}|\frac{\lambda_{n-1}}{\lambda_n} + |\eta_n|\frac{\lambda_{n+1}}{\lambda_n}\right)\right] < 1,$$

and $\widetilde{K}_{p,\Lambda} = K_1 + K_2' < 1$, since

$$\xi(\Lambda) \leq \widetilde{K}_{p,\Lambda} \leq K_1 + \sup_n \left(\frac{\lambda_{n+1}}{\lambda_n}\right) K_2 < 1. \qquad \square$$

We can also explicitly calculate the solution of an infinite tridiagonal system when $\gamma_n = \gamma$ and $\eta_n = \eta$ for all n. Indeed the next result was shown in [6], where

$$M(\zeta,\rho) = M(\zeta, e, \rho).$$

Proposition 4.3. *Let ζ, ρ be positive reals with $0 < \zeta + \rho < 1$. Then*

i) $M(\zeta,\rho) : X \longmapsto M(\zeta,\rho)X$ *is bijective from χ into itself, for $\chi \in \{s_1, c_0, c\}$.*

ii) a) *Let χ be any of the sets s_1, or c_0, or c, and put*

$$u = \left(1 - \sqrt{1 - 4\zeta\rho}\right)/2\zeta \quad \textit{and} \quad v = \left(1 - \sqrt{1 - 4\zeta\rho}\right)/2\zeta.$$

Then, for any given $B \in \chi$, the equation $M(\zeta,\rho)X = B$ has a unique solution $X^{\circ} = (\overset{\circ}{x}_n)_{n\geq 1}$ in χ given by

$$\overset{\circ}{x}_n = \left(\frac{uv+1}{uv-1}\right)(-1)^n v^n \sum_{m=1}^{\infty} [1 - (uv)^{-l}](-1)^m u^m b_m \quad \textit{for all } n,$$

with $l = \min\{n, m\}$.

b) *The inverse $[M(\zeta,\rho)]^{-1} = (a'_{nm})_{n,m\geq 1}$ is given by*

$$a'_{nm} = \left(\frac{uv+1}{uv-1}\right)(-v)^{n-m}[(uv)^l - 1] \quad \textit{for all } n, m \geq 1 \textit{ and } l = \min\{n, m\}.$$

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