*Publ. Math. Debrecen*
**85/1-2** (2014), 219–231
DOI: 10.5486/PMD.2014.5918

# Quasi Ahlfors–David regularity of Moran sets

By YU SUN (Wuhan), MANLI LOU (Guangzhou), FAN LÜ (Wuhan)
and LIFENG XI (Ningbo)

**Abstract.** For Moran fractals in Euclidean spaces, under suitable assumption, we obtain their quasi Ahlfors–David regularity and quasi-Lipschitz equivalence.

## 1. Introduction

**1.1. Moran set.** Recall some notions in fractals. A map $S:\mathbb{R}^n\to\mathbb{R}^n$ is called a similitude with ratio $r$, if $|S(x)-S(y)|=r|x-y|$ for all $x,y\in\mathbb{R}^n$. For two subsets $A$ and $B$ of $\mathbb{R}^n$, we say that $A$ is geometrically similar to $B$ with ratio $r$, if there is a similitude $S$ of ratio $r$ such that $S(A)=B$. For subset $A$, let $|A|$ be the diameter of the set, and $\bar{A}$ the closure of $A$.

The notion of Moran set is introduced by Wen in [16].

Given integers $\{n_k\}_{k\geq 1}$ with $n_k\geq 2$ and ratios $\{c_{k,i}\}_{k\geq 1,\ 1\leq i\leq n_k}\subset(0,1)$, let $\tilde{\Sigma}^*=\bigcup_{k=0}^{\infty}\prod_{i=1}^{k}\{1,\dots,n_i\}$ be the collection of finite words with $k$-th letter in $\{1,\dots,n_k\}$ for each $k$, suppose $\{V_{i_1\cdots i_k}\}_{i_1\cdots i_k\in\tilde{\Sigma}^*}$ are non-empty open sets satisfying

$$V_{i_1\cdots i_{k-1}i_k}\subset V_{i_1\cdots i_{k-1}} \text{ and } V_{i_1\cdots i_{k-1}i_k}\cap V_{i_1\cdots i_{k-1}j_k}=\varnothing \quad \text{if } i_k\neq j_k, \tag{1.1}$$

$V_{i_1\cdots i_{k-1}i_k}$ is geometrically similar to $V_{i_1\cdots i_{k-1}}$ with ratio

$$|V_{i_1\cdots i_{k-1}i_k}|/|V_{i_1\cdots i_{k-1}}|=c_{k,i_k}. \tag{1.2}$$

---

*Mathematics Subject Classification:* 28A80.

*Key words and phrases:* Moran set, Ahlfors–David regularity, Lipschitz equivalence.
The research is supported by NSFC (Nos. 11371329,11101159, 11301092, 11225101), NCET, NSF of Zhejiang Province (Nos. LR13A1010001, LY12F02011) and NSF of Guangdong Province (No. S2011040005741).

A *Moran set* $F$ is defined by

$$F = \bigcap_{k=1}^{\infty} \bigcup_{i_1 \cdots i_k \in \tilde{\Sigma}^*} \bar{V}_{i_1 \cdots i_k}. \tag{1.3}$$

For convenience, we write $d_k = \min_{1 \le i \le n_k} c_{k,i}$, $D_k = \max_{1 \le i \le n_k} c_{k,i}$ and suppose $s_k$ satisfies $\prod_{j=1}^{k} \left( \sum_{i=1}^{n_j} (c_{j,i})^{s_k} \right) = 1$. Let $s_* = \underline{\lim}_{k \to \infty} s_k$ and $s^* = \overline{\lim}_{k \to \infty} s_k$.

*Remark 1.* The self-similar set satisfying the open set condition is a Moran set. In fact, suppose $E = \cup_{i=1}^{m} S_i(E)$ is a self-similar set (see [7]), where $\{S_i\}_{i=1}^{m}$ are contracting similitudes with ratios $\{r_i\}_{i=1}^{m}$ such that the open set condition holds, i.e., there is a non-empty open set $U$ such that

$$\cup_i S_i(U) \subset U \text{ and } S_i(U) \cap S_j(U) = \varnothing \quad \text{for any } i \neq j. \tag{1.4}$$

Let $n_k \equiv m$ and $c_{k,i_k} = r_{i_k}$, $\Sigma^* = \cup_{k=0}^{\infty} \{1, \ldots, m\}^k$ and $\emptyset$ the empty word. We write $U_\emptyset = U$ and $U_{i_1 \cdots i_k} = S_{i_1} \circ \cdots \circ S_{i_k}(U)$ for word $i_1 \cdots i_k \in \Sigma^*$, then (1.4) implies $U_{i_1 \cdots i_{k-1} i_k} \subset U_{i_1 \cdots i_{k-1}}$ and $U_{i_1 \cdots i_{k-1} i_k} \cap U_{i_1 \cdots i_{k-1} j_k} = \varnothing$ for any $i_1 \cdots i_k \in \Sigma^*$ and $i_k \neq j_k$and $U_{i_1 \cdots i_{k-1} i_k}$ is geometrically similar to $U_{i_1 \cdots i_{k-1}}$ with ratio $|U_{i_1 \cdots i_{k-1} i_k}|/|U_{i_1 \cdots i_{k-1}}| = r_{i_k}$. It follows from (1.4) that $\cup_i S_i(\bar{U}) \subset \bar{U}$, hence the self-similar set

$$E = \bigcap_{k=1}^{\infty} \bigcup_{i_1 \cdots i_k \in \Sigma^*} \bar{U}_{i_1 \cdots i_k}. \tag{1.5}$$

**1.2. Ahlfors–David regularity.** A compact set $E$ is said to be Ahlfors–David $s$-regular [11], [12], if there is a Borel measure $\nu$ supported on $E$ such that

$$C^{-1} r^s \le \nu(B(x, r)) \le C r^s \tag{1.6}$$

for any $x \in E$ and $r \le |E|$. For self-similar set $E$ satisfying the open set condition as above, as shown in [7], the self-similar measure $\nu$ satisfies (1.6) where $s = \dim_H E = \dim_B E$ is the solution of the equation $(r_1)^s + \cdots + (r_m)^s = 1$.

For Ahlfors–David $s$-regular set $E$, by Theorem 5.7 of [11], we have

$$\overline{\dim}_B E = \underline{\dim}_B E = \dim_H E = s.$$

It was shown in [16] that, if $c_* = \inf d_k > 0$ for Moran set $F$ as above, then

$$\dim_H F = s_* \quad \text{and} \quad \overline{\dim}_B F = s^*.$$

Therefore, if $s_* < s^*$, then $F$ can not be Ahlfors–David regular.

*Example 1.* Let $n_k \equiv 2$ and $c_k \in \{1/3, 1/5\}$. Then $c_* > 0$. Take a sequence $\{c_k\}_k$ such that $a = \underline{\lim}_{k\to\infty} q_k < \overline{\lim}_{k\to\infty} q_k = b$, where $q_k = \frac{\#\{i\le k: c_k = 1/3\}}{k}$. Then $\underline{\lim}_{k\to\infty} s_k = \underline{\lim}_{k\to\infty} \frac{k\log 2}{-(\log c_1\cdots c_k)} = \underline{\lim}_{k\to\infty} \frac{\log 2}{q_k \log 3 + (1-q_k)\log 5} = \frac{\log 2}{a\log 3 + (1-a)\log 5}$ and $\overline{\lim}_{k\to\infty} s_k = \frac{\log 2}{b\log 3 + (1-b)\log 5}$, which means $\dim_H F < \overline{\dim}_B F$.

In the following example, $\dim_H F = \dim_B F$ but $F$ is not Ahlfors–David regular.

*Example 2.* Let $s = \log 3/\log 5$ and $c_k = \frac{1}{5}(1+\frac{1}{2k})$ for all $k$, then $\lim_{k\to\infty} c_k = 1/5$ and $\lim_{k\to\infty} 3^k(c_1\cdots c_k)^s = \lim_{k\to\infty} \big[\prod_{i=1}^k \big(1+\frac{1}{2i}\big)\big]^s = \infty$. Set $n_k \equiv 3$. Fix $I_\phi = [0,1]$ for empty word $\phi$. Given an interval $I_{i_1\cdots i_{k-1}} = [c,d]$ with $i_1\cdots i_{k-1} \in \{1,2,3\}^{k-1}$, we put $I_{i_1\cdots i_{k-1}1} = [c, c+c_k(d-c)]$, $I_{i_1\cdots i_{k-1}2} = \big[\frac{c+d}{2} - \frac{c_k(d-c)}{2}, \frac{c+d}{2} + \frac{c_k(d-c)}{2}\big]$ and $I_{i_1\cdots i_{k-1}3} = [d-(d-c)c_k, d]$. Let

$$F = \cap_{k=1}^{\infty} \cup_{i_1\cdots i_k \in \{1,2,3\}^k} I_{i_1\cdots i_k}.$$

Using result in [16], we have $\dim_H F = \dim_B F = s$. We will show that $F$ is not Ahlfors–David $s$-regular. On the contrary, there is a constant $\alpha > 0$ such that for the corresponding Ahlfors–David $s$-regular measure $\nu$, we have $(c_1\cdots c_k)^s \le \alpha\nu(I_{i_1\cdots i_k})$ for all $i_1\cdots i_k \in \{1,2,3\}^k$. Therefore,

$$3^k(c_1\cdots c_k)^s \le \alpha \sum_{i_1\cdots i_k\in\{1,2,3\}^k} \nu(I_{i_1\cdots i_k}) = \alpha\nu([0,1]) < \infty.$$

Letting $k\to\infty$, we obtain a contradiction.

However, the Moran set in Example 2 is quasi Ahlfors–David $s$-regular.

**1.3. Quasi Ahlfors–David regularity.** We say that $F$ is quasi Ahlfors–David $s$-regular [15], if there is a Borel measure $\mu$ supported on $F$ such that

$$\frac{\log\mu(B(x,r))}{\log r} \to s \text{ uniformly for } x\in F \text{ as } r\to 0. \tag{1.7}$$

Notice that Ahlfors–David regularity implies quasi Ahlfors–David regularity.

Can we find the conditions for a Moran set to be quasi Ahlfors–David regular? Now, we shall pose some assumptions upon the Moran sets:

(A1): $\lim_{k\to\infty} \frac{\log d_k}{\log(D_1D_2\cdots D_{k-1})} = 0$;

(A2): $\sup_k \frac{\log(n_1n_2\cdots n_k)}{-\log(D_1D_2\cdots D_k)} < \infty$;

(A3): $\lim_{k\to\infty} s_k = s \in (0,\infty)$.

*Remark 2.* (A1) means $\frac{\log|\bar{V}_{i_1\cdots i_{k-1}i_k}|}{\log|\bar{V}_{i_1\cdots i_{k-1}}|} \to 1$ uniformly as $k \to \infty$, this is a natural assumption. We pose (A2) as a technical need.

*Remark 3.* If $c_* = \inf_k d_k > 0$, then (A1)–(A2) hold since $\sup_k n_k \leq 1/(c_*)^n$ and $0 < c_* \leq d_k \leq D_k \leq \sqrt[n]{1-(c_*)^n}$ by (1.1)–(1.2).

*Remark 4.* (A3) means $\dim_H F = \dim_B F = s$ in some sense as in [16]. If $c_* = \inf_k d_k > 0$, it follows from [16] that $\dim_H F = s_*$ and $s^* = \overline{\dim}_B F$. Using (A3), we have $\dim_H F = \dim_B F = s$.

*Remark 5.* For self-similar set $E$ satisfying the open set condition as above, we have $D_k \equiv \max_i r_i$, $d_k \equiv \min_i r_i$, $n_k \equiv m$, $c_{k,i} = r_i$ and $s_k \equiv s$ with

$$(r_1)^s + \cdots + (r_m)^s = 1.$$

It is obvious that assumptions (A1)–(A3) hold. The Ahlfors–David regularity of $E$ supports the rationality of these assumptions.

Under these assumption, we get the quasi Ahlfors–David regularity.

**Theorem 1.** *Suppose $F$ is a Moran set satisfying* (A1)–(A3). *Then $F$ is quasi Ahlfors–David $s$-regular.*

Besides Example 2, we give the following planar Moran set satisfying (A1)–(A3).

*Example 3* ([3]). Consider a plane fractal defined as follows. Given two sequences $\{l_k\}_{k\geq 1}$ and $\{n_k\}_{k\geq 1}$ with $n_k \leq l_k^2$ and

$$\lim_{k\to\infty} \frac{\log(n_1\cdots n_k)}{\log(l_1\cdots l_k)} = s.$$

Take a unit square $[0,1]^2$ as the initial set. In the first step, we divide $[0,1]^2$ into $(l_1)^2$ equal squares with side length $1/l_1$, then we select $n_1$ ones. By induction, assume that we get a square $Q_{i_1\cdots i_{k-1}}$ ($i_1\cdots i_{k-1} \in \tilde{\Sigma}^*$) of side length $(l_1\cdots l_{k-1})^{-1}$, we divide it into $(l_k)^2$ equal squares with side length $(l_1\cdots l_{k-1}l_k)^{-1}$, and then we select $n_k$ ones from them, denoted by $\{Q_{i_1\cdots i_{k-1}i_k}\}_{i_k=1}^{n_k}$. Again and again, we get a limit set which is a Moran set, where $c_{k,i} \equiv (l_k)^{-1}$ and $V_{i_i\cdots i_k}$ is the interior of $Q_{i_1\cdots i_k}$. It is easy to check that assumptions (A1)–(A3) hold.

**1.4. Quasi Lipschitz equivalence of Moran sets.** We say that two subsets $A$ and $B$ of Euclidean spaces are Lipschitz equivalent, if there is a bijection from $A$ to $B$ such that for all $x, y \in A$,

$$C^{-1}|x-y| \leq |f(x)-f(y)| \leq C|x-y|$$

for some constant $C > 0$.

Even for self-similar sets, their Lipschitz equivalences are very difficult to study, for example see Cooper and Pignataro [1], David and Semmes [3], Deng and He [4], Deng, Wen, Xiong and Xi [5], Falconer and Marsh [6], Lau and Luo [10], Llorente and Mattila [9], Rao, Ruan and Wang [13], Xi and Xiong [17], [19], [20], [21]. As we known, two dust-like self-similar sets with the same dimension may not be Lipschitz equivalent. But they are quasi Lipschitz equivalent.

The notion of quasi-Lipschitz equivalence is introduced by Xi [18].

We say that two subsets $A$ and $B$ of Euclidean spaces are quasi-Lipschitz equivalent, if there is a bijection from $A$ to $B$ such that for all $x, y \in A$,

$$\frac{\log|f(x)-f(y)|}{\log|x-y|} \to 1 \text{ uniformly as } |x-y| \to 0.$$

It is proved in [18] that two self-conformal sets are quasi-Lipschitz equivalent if and only if they have the same dimension. This result was developed by Wang and Xi [14], [15], Xiong and Xi [22].

Li *et al.* [8] studied the quasi-Lipschitz equivalence for some homogeneous Moran sets in line, a special class of Moran sets. For any Moran set in this class, we have $c_{k,1} = \cdots = c_{k,n_k} = c_k$ and an additional assumption $\lim_{k\to\infty} \frac{\log n_1 \cdots n_k}{\log c_1 \cdots c_k} = s \in (0,1)$. We see that the condition $\lim_{k\to\infty} \frac{\log n_1 \cdots n_k}{\log c_1 \cdots c_k} = s$ is like (A3). But the condition $s < 1$ plays an important role which is like the separation condition for self-similar sets.

In this paper, we study the general Moran sets under the separation assumption:

(A4): For any $i_1 \cdots i_{k-1} i_k \in \tilde{\Sigma}^*$,

$$\frac{\log d(\bar{V}_{i_1\cdots i_{k-1}i_k}, F\backslash \bar{V}_{i_1\cdots i_{k-1}i_k})}{\log|\bar{V}_{i_1\cdots i_{k-1}}|} \to 1 \text{ uniformly as } k \to \infty.$$

Here $d(A, B)$ is the least distance between compact sets $A$ and $B$.

For $s > 0$, a class $\mathcal{A}_s$ of Moran sets is defined by

$$\mathcal{A}_s = \begin{cases} \{F : F \text{ is a Moran set satisfying (A1)–(A4)}\} & \text{if } s \geq 1, \\ \{F : F \text{ is a Moran set satisfying (A1)–(A3)}\} & \text{if } s < 1. \end{cases}$$

**Theorem 2.** *For every $s > 0$, any two Moran sets in $\mathcal{A}_s$ are quasi-Lipschitz equivalent.*

We pose the last assumption:

(A5): There exists a constant $c > 0$ such that for any $i_1 \cdots i_{k-1} i_k \in \tilde{\Sigma}^*$,

$$\frac{|\bar{V}_{i_1 \cdots i_{k-1} i_k}|}{|\bar{V}_{i_1 \cdots i_{k-1}}|} = c_{k,i_k} \geq c \quad \text{and} \quad \frac{\min_{j_k \neq i_k} d(\bar{V}_{i_1 \cdots i_{k-1} i_k}, \bar{V}_{i_1 \cdots i_{k-1} j_k})}{|\bar{V}_{i_1 \cdots i_{k-1}}|} \geq c.$$

Then (A5) implies (A1)–(A2) (see Remark 3) and (A4).

Using Theorem 2, we have

**Theorem 3.** *Fix $s > 0$. Suppose $F_1$ and $F_2$ are Moran sets satisfying* (A3) *(with the same parameter $s$) and* (A5)*, then they are quasi-Lipschitz equivalent.*

*Remark 6.* In fact, (A3) and (A5) hold for self-similar sets satisfying the strong separation condition. Then we obtain the result in [18]: two dust-like self-similar sets are quasi-Lipschitz equivalent if and only if they have the same dimension.

The paper is organized as follows. Section 2 is the preliminaries, including the construction of the measure which plays an important role in this paper. Section 3 is the proof of Theorem 1. In Section 4, we get Theorem 2 based on the main result in [15]. In fact, [15] proved that if two compact sets are quasi Ahlfors–David $s$-regular and quasi uniformly disconnected, then they are quasi-Lipschitz equivalent.

## 2. Preliminaries

For $\sigma = \sigma_1 \cdots \sigma_k \in \tilde{\Sigma}_*$, let $|\sigma| (= k)$ denote its length. Write

$$(\sigma_1 \cdots \sigma_k) * \sigma_{k+1} = \sigma_1 \cdots \sigma_k \sigma_{k+1}.$$

We say that $\bar{V}_\sigma$ is of rank $k$, if $|\sigma| = k$. Without loss of generality, we assume the diameter

$$|\bar{V}_\emptyset| = 1.$$

If $\sigma = \sigma_1 \cdots \sigma_k$, then every $\bar{V}_\sigma$ is similar to $|\bar{V}_\emptyset|$ with ratio $c_{1,\sigma_1} c_{2,\sigma_2} \cdots c_{k,\sigma_k}$ and

$$|\bar{V}_\sigma| = c_{1,\sigma_1} c_{2,\sigma_2} \cdots c_{k,\sigma_k}. \tag{2.1}$$

**2.1. Construction of measure.** Fix $s > 0$. As in [2] by DAI *et al.*, we can define a probability measure supported on $F$ depending on $s$.

Let $\mu(\bar{V}_\emptyset) = 1$, where $\emptyset$ is the empty word.

By induction, for every $k \geq 1$ and $\bar{V}_\sigma$ of rank $(k-1)$, we define

$$\mu(\bar{V}_{\sigma * i}) = \frac{c_{k,i}^s}{\sum_{j=1}^{n_k} c_{k,j}^s} \mu(\bar{V}_\sigma) \quad \text{for } 1 \leq i \leq n_k.$$

In fact, if $\sigma = \sigma_1 \cdots \sigma_k$, then

$$\mu(\bar{V}_\sigma) = \frac{|\bar{V}_\sigma|^s}{\prod_{i=1}^k \left( \sum_{j=1}^{n_i} c_{i,j}^s \right)}. \tag{2.2}$$

More and more, we get a probability measure supported on $F$. In fact, we use the condition (1.1) during the above construction of measure, as in the way by HUTCHINSON [7] for the open set condition.

**2.2. Disconnectedness and quasi-Lipschitz equivalence.** As above, we introduce the Moran set on the analogy of the self-similar sets satisfying the open set condition. Now, we will give some property of disconnectedness on the analogy of the self-similar sets satisfying the strong separation condition (SSC in short), here SSC holds for $E = \cup_{i=1}^m S_i(E)$, if

$$\min_{i \neq j} d(S_i(E), S_j(E)) > 0,$$

where $d(A, B)$ is the distance between $A$ and $B$.

*Definition 1* ([15]). We say that a subset $K$ of Euclidean space is *quasi uniformly disconnected* if there is a function $\eta : (0, +\infty) \to (0, +\infty)$ with $\lim\limits_{t \to 0} \frac{\log \eta(t)}{\log t} = 1$ such that for any $x \in K$ and $r > 0$, there is a subset $B \subset K$ such that

$$K \cap B(x, \eta(r)) \subset B \subset B(x, r) \quad \text{and} \quad d(B, K \setminus B) > \eta(r). \tag{2.3}$$

We notice that if $E$ is a self-similar set satisfying SSC, then $E$ is quasi uniformly disconnected. In fact, we can take

$$\eta(r) = (\min_i r_i) \frac{\min_{i \neq j} d(S_i(E), S_j(E))}{|E|} \cdot r.$$

The following two lemmas come from [15] by WANG and XI.

**Lemma 1.** *Suppose that compact and quasi uniformly disconnected subsets $E_1$ and $E_2$ of Euclidean spaces are quasi Ahlfors–David $s$ -regular. Then $E_1$ and $E_2$ are quasi Lipschitz equivalent.*

**Lemma 2.** *If $E$ is quasi Ahlfors–David $s$-regular with $s \in (0,1)$, then $E$ is quasi uniformly disconnected.*

In fact,Mattila and Saaranen [12] proved that if $E$ is Ahlfors–David $s$-regular with $s \in (0,1)$, then $E$ is uniformly disconnected.

## 3. Quasi Ahlfors–David Regularity

In this section, we will prove Theorem 1. Now, fix

$$s = \lim_{k\to\infty} s_k,$$

where $\prod_{i=1}^{k}\left(\sum_{j=1}^{n_i} c_{i,j}^{s_k}\right) = 1$.

**3.1. Measure of $\bar{V}_\sigma$.** We will estimate the measure of $\bar{V}_\sigma$.

**Lemma 3.** *There is a non-increasing function $\delta : \mathbb{N} \to \mathbb{R}^+$ such that $\lim_{k\to\infty}\delta(k) = 0$ and for any basic interval $I_\sigma$,*

$$|\bar{V}_\sigma|^{s+\delta(|\sigma|)} \le \mu(\bar{V}_\sigma) \le |\bar{V}_\sigma|^{s-\delta(|\sigma|)}.$$

Proof. Suppose $|\sigma| = k$. By (2.2), we only need to show

$$\frac{\log\prod_{i=1}^{k}\left(\sum_{j=1}^{n_i} c_{i,j}^{s}\right)}{\log|\bar{V}_\sigma|} \to 0 \text{ as } k\to\infty. \tag{3.1}$$

In fact, using (2.1) and $D_k = \max_i c_{k,i}$, we have

$$\big|\log|\bar{V}_\sigma|\big| \ge |\log(D_1\cdots D_k)|. \tag{3.2}$$

On the other hand, since $\prod_{i=1}^{k}\left(\sum_{j=1}^{n_i} c_{i,j}^{s_k}\right) = 1$, we have

$$\begin{aligned}&\left|\log\prod_{i=1}^{k}\left(\sum_{j=1}^{n_i} c_{i,j}^{s}\right)\right|\\ &= \left|\log\prod_{i=1}^{k}\left(\sum_{j=1}^{n_i} c_{i,j}^{s}\right) - \log\prod_{i=1}^{k}\left(\sum_{j=1}^{n_i} c_{i,j}^{s_k}\right)\right| = |g(s)-g(s_k)|,\end{aligned}$$

where

$$g(x) = \sum_{i=1}^{k} \log\left(\sum_{j=1}^{n_i} c_{i,j}^{x}\right).$$

By Mean value theorem, there exists $\xi_k$ lying in the interval with endpoints $s$ and $s_k$ such that

$$|g(s) - g(s_k)| = (s_k - s) \sum_{i=1}^{k} \frac{\sum_{j=1}^{n_i} c_{i,j}^{\xi_k} \log c_{i,j}}{\sum_{j=1}^{n_i} c_{i,j}^{\xi_k}}. \tag{3.3}$$

Write $\tilde{c_{i,j}} = \frac{c_{i,j}^{\xi_k}}{\sum_{j=1}^{n_i} c_{i,j}^{\xi_k}}$, then $\sum_{j=1}^{n_i} \tilde{c_{i,j}} = 1$ and

$$\begin{aligned} \sum_{i=1}^{k} \frac{\sum_{j=1}^{n_i} c_{i,j}^{\xi_k} \log c_{i,j}}{\sum_{j=1}^{n_i} c_{i,j}^{\xi_k}} &= \frac{1}{\xi_k} \cdot \sum_{i=1}^{k} \frac{\sum_{j=1}^{n_i} c_{i,j}^{\xi_k} \log c_{i,j}^{\xi_k}}{\sum_{j=1}^{n_i} c_{i,j}^{\xi_k}} \\ &= \frac{1}{\xi_k} \cdot \left(\sum_{i=1}^{k} \sum_{j=1}^{n_i} \tilde{c_{i,j}} \log \tilde{c_{i,j}} + \sum_{i=1}^{k} \log \sum_{j=1}^{n_i} c_{i,j}^{\xi_k}\right). \end{aligned} \tag{3.4}$$

Using the convexity of function $f(x) = -x\log x$ on $[0,1]$, we have

$$\Big|\sum_{j=1}^{n_i} \tilde{c_{i,j}} \log \tilde{c_{i,j}}\Big| \le \log n_i \quad \text{and} \quad \prod_{i=1}^{k} D_i^{\xi_k} \le \prod_{i=1}^{k} \sum_{j=1}^{n_i} c_{i,j}^{\xi_k} \le \prod_{i=1}^{k} n_i. \tag{3.5}$$

Then it follows from assumption (A2) and (3.3)-(3.5) that

$$\lim_{k\to\infty} \frac{\big|\log \prod_{i=1}^{k} \big(\sum_{j=1}^{n_i} c_{i,j}^{s}\big)\big|}{\log(D_1 \cdots D_k)} = 0. \tag{3.6}$$

Therefore, we get (3.1) by (3.2) and (3.6). □

**3.2. Proof of Theorem 1.** For $x \in F$ and $r$ small enough, now we shall estimate

$$\frac{\log \mu(B(x,r))}{\log r}.$$

Let $N_r = \min\{k : d_1 \cdots d_k \le r\} \to \infty$ as $r \to 0$. Let $\sigma^-$ be the father of $\sigma$, that is $\sigma = \sigma^- * i_{|\sigma|}$ for some letter $i_{|\sigma|}$.

(1) *Lower bound*:

Using assumption (A1) $\lim_{k\to\infty} \frac{\log d_k}{\log(D_1 D_2 \cdots D_{k-1})} = 0$, we have

$$\lim_{|\sigma|\to\infty} \frac{\log |\bar{V}_\sigma|}{\log |\bar{V}_{\sigma^-}|} = 1, \tag{3.7}$$

then there exists a non-increasing function $\alpha : \mathbb{N} \to (0, \infty)$ with $\lim_{k\to\infty} \alpha(k) = 0$ such that

$$|\bar{V}_{\sigma^-}| \geq |\bar{V}_\sigma| \geq |\bar{V}_{\sigma^-}|^{1+\alpha(|\sigma|)}. \tag{3.8}$$

Take a word $\sigma$ such that

$$|\bar{V}_\sigma| \leq r \quad \text{and} \quad |\bar{V}_{\sigma^-}| > r, \tag{3.9}$$

Then by Lemma 3, we have

$$\mu(B(x,r)) \geq \mu(\bar{V}_\sigma) \geq |\bar{V}_\sigma|^{s+\delta(|\sigma|)}. \tag{3.10}$$

It follows from (3.8), (3.9) and (3.10) that

$$\mu(B(x,r)) \geq |\bar{V}_{\sigma^-}|^{(s+\delta(|\sigma|))(1+\alpha(|\sigma|))} > r^{(s+\delta(N_r))(1+\alpha(N_r))}, \tag{3.11}$$

where

$$(s+\delta(N_r))(1+\alpha(N_r)) \to s \text{ as } r \to 0.$$

(2) *Upper bound*:

Let

$$\Omega_{x,r} = \{\sigma : B(x,r) \cap \bar{V}_\sigma \neq \varnothing \quad \text{and} \quad |\bar{V}_\sigma| \leq r, |\bar{V}_{\sigma^-}| > r\},$$

and $l_{x,r} := \min_{\sigma\in\Omega_{x,r}} |\bar{V}_\sigma| = \min_{\sigma\in\Omega_{x,r}} |V_\sigma|$.

It follows from Lemma 3 that for $\sigma \in \Omega_{x,r}$,

$$\mu(\bar{V}_\sigma) \leq r^{s-\delta(N_r)}. \tag{3.12}$$

By (3.8), we have

$$l_{x,r} \geq r^{1+\alpha(N_r)} \text{ with } \alpha(N_r) \to 0 \quad \text{as} \quad r \to 0. \tag{3.13}$$

Denote by $\mathcal{L}^n$ the Lebesgue measure on $\mathbb{R}^n$. Since $\bigcup_{\sigma\in\Omega_{x,r}} V_\sigma$ is a disjoint union contained in $B(x,2r)$, we have

$$\begin{aligned}\mathcal{L}^n B(x,2r) \geq \mathcal{L}^n\Big(\bigcup_{\sigma\in\Omega_{x,r}} V_\sigma\Big) &= \sum_{\sigma\in\Omega_{x,r}} \mathcal{L}^n(V_\sigma) \\ &= \sum_{\sigma\in\Omega_{x,r}} \mathcal{L}^n(V_\emptyset) \cdot \frac{|V_\sigma|^n}{|V_\emptyset|^n} \geq \frac{\mathcal{L}^n(V_\emptyset)}{|V_\emptyset|^n} (\#\Omega_{x,r})(l_{x,r})^n.\end{aligned}$$

This means the cardinality

$$\#\Omega_{x,r} \leq C_1 \frac{r^n}{(l_{x,r})^n} \tag{3.14}$$

for some constant $C_1 > 0$.

For any $y \in F \cap B(x,r)$, we can find a word $\sigma$ with $y \in \bar{V}_\sigma$ such that $|\bar{V}_\sigma| \le r$ and $|\bar{V}_{\sigma^-}| > r$. That means

$$F \cap B(x,r) \subset \bigcup_{\sigma \in \Omega_{x,r}} \bar{V}_\sigma. \tag{3.15}$$

By (3.12)–(3.15), we have

$$\begin{aligned}\mu(B(x,r)) &= \mu(F \cap B(x,r)) \le \mu(\bigcup_{\sigma \in \Omega_{x,r}} \bar{V}_\sigma) \le \#\Omega_{x,r} \cdot \max_{\sigma \in \Omega_{x,r}} \mu(\bar{V}_\sigma) \\ &\le C_1 \frac{r^n}{(l_{x,r})^n} \max_{\sigma \in \Omega_{x,r}} \mu(\bar{V}_\sigma) \le C_1 \cdot r^{s-\delta(N_r)-n\alpha(N_r)},\end{aligned}$$

where

$$s - \delta(N_r) - n\alpha(N_r) \to s \quad \text{as} \quad r \to 0.$$

Then Theorem 1 follows from the estimates of lower and upper bounds.

## 4. Quasi-Lipschitz equivalence

In this section, we will prove Theorem 2.

For $E_1, E_2 \in \mathcal{A}_s$, Theorem 1 implies that they are quasi Ahlfors–David $s$-regular.

By Lemmas 1 and 2, we only need to show that for $s \ge 1$ any set in $F \in \mathcal{A}_s$ is quasi uniformly disconnected by using assumption (A4).

In fact, given $F \in \mathcal{A}_s$ with $|\bar{V}_\emptyset| = 1$, the assumption (A4) shows that for any $\bar{V}_\sigma$,

$$\lim_{|\sigma| \to \infty} \frac{\log d(\bar{V}_\sigma, F \backslash \bar{V}_\sigma))}{\log |\bar{V}_{\sigma^-}|} = 1. \tag{4.1}$$

Using assumption (A1) $\lim_{k\to\infty} \frac{\log d_k}{\log(D_1 D_2 \cdots D_{k-1})} = 0$, we have

$$\lim_{|\sigma| \to \infty} \frac{\log |\bar{V}_\sigma|}{\log |\bar{V}_{\sigma^-}|} = 1. \tag{4.2}$$

By (4.1) and (4.2), we can take a non-increasing function $\varepsilon : \mathbb{N} \to (0, \infty)$ with $\lim_{k\to\infty} \varepsilon(k) = 0$ such that

$$|\bar{V}_{\sigma^-}| \ge d(\bar{V}_\sigma, F \backslash \bar{V}_\sigma)) \ge |\bar{V}_{\sigma^-}|^{1+\varepsilon(|\sigma|)} \tag{4.3}$$

and

$$|\bar{V}_{\sigma^-}| \ge |\bar{V}_\sigma| \ge |\bar{V}_{\sigma^-}|^{1+\varepsilon(|\sigma|)}. \tag{4.4}$$

Take

$$\eta(r) = r^{1+\varepsilon(N_r)},$$

where $N_r = \min\{k : d_1 \cdots d_k \le r\} \to \infty$ as $r \to 0$ which implies

$$\lim_{r\to 0} \frac{\log \eta(r)}{\log r} = 1.$$

For $B(x,r)$ with $x \in F$ and $r$ small enough, take a word $\sigma$ such that

$$|\bar{V}_\sigma| \le r \quad \text{and} \quad |\bar{V}_{\sigma^-}| > r.$$

Then by (4.3)–(4.4), we have

$$F \cap B(x, \eta(r)) \subset \bar{V}_\sigma \subset B(x,r)$$

and

$$d(\bar{V}_\sigma, F\backslash \bar{V}_\sigma) \ge \eta(r).$$

Hence $F$ is quasi uniformly disconnected.

Applying Lemma 1, we obtain Theorem 2.

## References


[1] D. Cooper and T. Pignataro, On the shape of Cantor sets, *J. Diff. Geom.* **28** (1988), 203–221.

[2] Y. X. Dai, Z. X. Wen, L. F. Xi and Y. Xiong, *Ann. Acad. Sci. Fenn. Math.* **36** (2011), 139–151.

[3] G. David and S. Semmes, Fractured Fractals and Broken Dreams: Self-Similar Geometry Through Metric and Measure, *Oxford University Press, Oxford*, 1997.

[4] G. T. Deng and X. G. H, Lipschitz equivalence of fractal sets, *R. Sci. China Math.* **55** (2012), 2095–2107.

[5] J. Deng, Z. Y. Wen, Y. Xiong, and L. F. Xi, Bilipschitz embedding of self-similar sets, *J. Anal. Math.* **114** (2011), 63–97.

[6] K. J. Falconer and D. T. Marsh, On the Lipschitz equivalence of Cantor sets, *Mathematika 39* (1992), 223–233.

[7] J. E. Hutchinson, Fractals and self-similarity, *Indiana Univ. Math. J.* **30** (1981), 713–747.

[8] H. Li, Q. Wang and L. F Xi, Classification of Moran fractals, *J. Math. Anal. Appl.* **378** (2011), 230–237.

[9] M. Llorente and P. Mattila, Lipschitz equivalence of subsets of self-conformal sets, *Nonlinearity* **23** (2010), 875–882.

[10] J. J. Luo and K. S. Lau, Lipschitz equivalence of self-similar sets and hyperbolic boundaries, *Adv. Math.* **235** (2013), 555–579.

[11] P. Mattila, Geometry of sets and measures in Euclidean spaces, Fractals and Rectifiability, *Cambridge University Press, Cambridge*, 1995.

[12] P. Mattila and P. Saaranen, *Ann. Acad. Sci. Fenn. Math.* **34** (2009), 487–502.

[13] H. Rao, H. J. Ruan and Y. Wang, Lipschitz equivalence of Cantor sets and algebraic properties of contraction ratios, *Trans. Amer. Math. Soc.* **364** (2012), 1109–1126.

[14] Q. Wang and L. F. Xi, Quasi-Lipschitz equivalence of Ahlfors–David regular sets, *Nonlinearity* **24** (2011), 941–950.

[15] Q. Wang and L. F. Xi, Quasi-Lipschitz equivalence of quasi Ahlfors–David regular sets, *Sci. China Math.* **54** (2011), 2573–2582.

[16] Z. Y. Wen, Moran sets and Moran classes, *Chinese Sci. Bull.* **46** (2001), 1849–1856.

[17] L. F. Xi, Lipschitz equivalence of self-conformal sets, *J. London Math. Soc. (2)* **70** (2004), 369–382.

[18] L. F. Xi, Quasi-Lipschitz equivalence of fractals, *Israel J. Math.* **160** (2007), 1–21.

[19] L. F. Xi, Lipschitz equivalence of dust-like self-similar sets, *Math. Z.* **266** (2010), 683–691.

[20] L. F. Xi and Y. Xiong, Self-similar sets with initial cubic patterns, *C. R. Math. Acad. Sci. Paris* **348** (2010), 15–20.

[21] L. F. Xi and Y. Xiong, Lipschitz equivalence of fractals generated by nested cubes, *Math. Z.* **271** (2012), 1287–1308.

[22] L. F. Xi and Y. Xiong, Studia Math., Vol. 194, 2009, 197–205.

YU SUN
SCHOOL OF MATHEMATICS
AND STATISTICS
HUAZHONG UNIVERSITY
OF SCIENCE AND TECHNOLOGY
430074, WUHAN
P.R. CHINA

*E-mail:* sunyu88sy@163.com

FAN LÜ
SCHOOL OF MATHEMATICS
AND STATISTICS
HUAZHONG UNIVERSITY
OF SCIENCE AND TECHNOLOGY
430074, WUHAN
P.R. CHINA

*E-mail:* lv.fan@smail.hust.edu.cn

MANLI LOU
DEPARTMENT OF MATHEMATICS
GUANGDONG POLYTECHNIC
NORMAL UNIVERSITY
GUANGZHOU, 510665
P.R. CHINA

*E-mail:* loumanli@126.com

LIFENG XI
DEPARTMENT OF COMPUTER SCIENCE
ZHEJIANG WANLI UNIVERSITY
NINGBO, 315100
P.R. CHINA

*E-mail:* xilifengningbo@yahoo.com

*(Received July 8, 2013; revised December 7, 2013)*