

Co-commutators with generalized derivations in prime and semiprime rings

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Abstract. Let R be a prime ring of characteristic different from 2 with Utumi quotient ring U and extended centroid C , F and G two nonzero generalized derivations of R , I an ideal of R and $f(x_1, \dots, x_n)$ be a multilinear polynomial over C which is not central valued on R . If

$$F^2(f(x_1, \dots, x_n))f(x_1, \dots, x_n) - f(x_1, \dots, x_n)G^2(f(x_1, \dots, x_n)) = 0$$

for all $x_1, \dots, x_n \in I$, then one of the following holds:

- (1) $F(x) = xa$ and $G(x) = xb$ for all $x \in R$ with $a^2 = b^2 \in C$;
- (2) $F(x) = xa$ and $G(x) = bx$ for all $x \in R$ with $a^2 = b^2$;
- (3) $F(x) = ax$ and $G(x) = xb$ for all $x \in R$ with $a^2 = b^2 \in C$;
- (4) $F(x) = ax$ and $G(x) = xb$ for all $x \in R$ with $a^2 = b^2$ and $f(x_1, \dots, x_n)^2$ is central valued on R ;
- (5) $F(x) = ax$ and $G(x) = bx$ for all $x \in R$, with $a^2 = b^2 \in C$.

We finally extend the result to a semiprime ring R in case $F^2(x)x - xG^2(x) = 0$ for all $x \in R$.

Throughout this paper R always denotes an associative prime ring with center $Z(R)$, extended centroid C and U its Utumi quotient ring. The Lie commutator of x and y is denoted by $[x, y]$ and defined by $[x, y] = xy - yx$ for $x, y \in R$. An

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additive mapping $d : R \rightarrow R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. An additive subgroup L of R is said to be a Lie ideal of R if $[L, R] \subseteq L$. An additive mapping $F : R \rightarrow R$ is called a generalized derivation if there exists a derivation $d : R \rightarrow R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$. Evidently, any derivation is a generalized derivation. Thus, the generalized derivation covers both the concepts of derivation and left multiplier mapping. The left multiplier mapping means an additive mapping $F : R \rightarrow R$ such that $F(xy) = F(x)y$ holds for all $x, y \in R$. We denote by s_4 the standard identity in four variables.

A well known result of POSNER [22] states that if d is a derivation of R such that $d(x)x - xd(x) \in Z(R)$ for all $x \in R$, then either $d = 0$ or R is commutative. Several authors generalized Posner's theorem. For instance, BREŠAR proved in [3] that if d and δ are two derivations of R such that $d(x)x - x\delta(x) \in Z(R)$ for all $x \in R$, then either $d = \delta = 0$ or R is commutative. Later LEE and WONG [19] consider the situation $d(x)x - x\delta(x) \in Z(R)$ for all x in some noncentral Lie ideal L of R and obtained the result that either $d = \delta = 0$ or R satisfies s_4 . In [15], LEE and SHIUE consider the situation $d(x)x - x\delta(x) \in C$ for all $x \in \{f(x_1, \dots, x_n) \mid x_1, \dots, x_n \in R\}$, where $f(x_1, \dots, x_n)$ is any polynomial over C and obtained that either $d = \delta = 0$, or $\delta = -d$ and $f(x_1, \dots, x_n)^2$ is central valued on RC , except when $\text{char}(R) = 2$ and $\dim_C RC = 4$.

In [8], the first author and Sharma studied the case when R is a prime ring of $\text{char}(R) \neq 2$, d a derivation of R , $f(x_1, \dots, x_n)$ a multilinear polynomial over C and I a right ideal of R such that $d^2(x)x - xd^2(x) = 0$ for all $x \in \{f(x_1, \dots, x_n) \mid x_1, \dots, x_n \in I\}$, then either $[f(x_1, \dots, x_n), x_{n+1}]x_{n+2}$ is satisfied by I , or there exists $b \in U$ such that $d(x) = [b, x]$ for all $x \in R$, with $b^2 = 0$ and $bI = (0)$.

Recently, in [1] ARGAC and the second author of this paper, studied the case $F(x)x - xG(x) = 0$ for all $x \in \{f(x_1, \dots, x_n) \mid x_1, \dots, x_n \in I\}$, where F and G two generalized derivations of R and I is an ideal of R and then determined the structure of the maps.

In the present paper, we shall study the situation $F^2(x)x - xG^2(x) = 0$ for all $x \in \{f(x_1, \dots, x_n) \mid x_1, \dots, x_n \in I\}$, where F and G are two generalized derivations of R and I is an ideal of R .

Lemma 1 ([1, Lemma 1]). *Let R be a noncommutative prime ring, $a, b \in U$, $p(x_1, \dots, x_n)$ be any polynomial over C , which is not an identity for R . If $ap(r) - p(r)b = 0$ for all $r = (r_1, \dots, r_n) \in R^n$, then one of the following holds:*

- (1) $a = b \in C$,
- (2) $a = b$ and $p(x_1, \dots, x_n)$ is central valued on R ,

(3) $\text{char}(R) = 2$ and R satisfies s_4

Lemma 2 ([1, Lemma 3]). *Let R be a noncommutative prime ring with Utumi quotient ring U and extended centroid C , and $f(x_1, \dots, x_n)$ be a multilinear polynomial over C , which is not central valued on R . Suppose that there exist $a, b, c, q \in U$ such that $(af(r) + f(r)b)f(r) - f(r)(cf(r) + f(r)q) = 0$ for all $r = (r_1, \dots, r_n) \in R^n$. Then one of the following holds:*

- (1) $a, q \in C$ and $q - a = b - c = \alpha \in C$;
- (2) $f(x_1, \dots, x_n)^2$ is central valued on R and there exists $\alpha \in C$ such that $q - a = b - c = \alpha$;
- (3) $\text{char}(R) = 2$ and R satisfies s_4 .

In particular, from above Lemma, we have the followings:

Lemma 3. *Let R be a noncommutative prime ring of characteristic different from 2 with Utumi quotient ring U and extended centroid C , and $f(x_1, \dots, x_n)$ be a multilinear polynomial over C , which is not central valued on R . Suppose that there exist $c, q \in U$ such that $f(r)(cf(r) + f(r)q) = 0$ for all $r = (r_1, \dots, r_n) \in R^n$. Then $q = -c \in C$.*

Lemma 4. *Let R be a noncommutative prime ring of characteristic different from 2 with Utumi quotient ring U and extended centroid C , and $f(x_1, \dots, x_n)$ be a multilinear polynomial over C , which is not central valued on R . Suppose that there exist $c \in U$ such that $f(r)cf(r) = 0$ for all $r = (r_1, \dots, r_n) \in R^n$. Then $c = 0$.*

Lemma 5. *Let R be a noncommutative prime ring of characteristic different from 2 with Utumi quotient ring U and extended centroid C , and $f(x_1, \dots, x_n)$ be a multilinear polynomial over C , which is not central valued on R . Suppose that there exist $a, b \in U$ such that $(af(r) + f(r)b)f(r) = 0$ for all $r = (r_1, \dots, r_n) \in R^n$. Then $-a = b \in C$.*

Lemma 6 (Lemma 1 in [6]). *Let C be an infinite field and $m \geq 2$. If $A_1, \dots, A_k$ are not scalar matrices in $M_m(C)$ then there exists some invertible matrix $P \in M_m(C)$ such that any matrices $PA_1P^{-1}, \dots, PA_kP^{-1}$ have all non-zero entries.*

Proposition 1. *Let $R = M_m(C)$ be the ring of all $m \times m$ matrices over the infinite field C , $f(x_1, \dots, x_n)$ a non-central multilinear polynomial over C and $a, a', b, b', c, q, q' \in R$. If*

$$a'f(r)^2 + 2af(r)bf(r) + f(r)b'f(r) - 2f(r)cf(r)q - f(r)^2q' = 0$$

for all $r = (r_1, \dots, r_n) \in R^n$, then either a or b and either c or q are central.

PROOF. By our assumption R satisfies the generalized identity

$$a'f(x_1, \dots, x_n)^2 + 2af(x_1, \dots, x_n)bf(x_1, \dots, x_n) + f(x_1, \dots, x_n)b'f(x_1, \dots, x_n) - 2f(x_1, \dots, x_n)cf(x_1, \dots, x_n)q - f(x_1, \dots, x_n)^2q' = 0. \quad (1)$$

We assume first that $a \notin Z(R)$ and $b \notin Z(R)$. Now we shall show that this case leads a contradiction.

Since $a \notin Z(R)$ and $b \notin Z(R)$, by Lemma 6 there exists a C -automorphism ϕ of $M_m(C)$ such that $a_1 = \phi(a)$, $b_1 = \phi(b)$ have all non-zero entries. Clearly a_1 , b_1 , $c_1 = \phi(c)$, $a'_1 = \phi(a')$, $b'_1 = \phi(b')$, $q_1 = \phi(q)$ and $q'_1 = \phi(q')$ must satisfy the condition (). Without loss of generality we may replace a , a' , b , b' , c , q , q' with a_1 , a'_1 , b_1 , b'_1 , c_1 , q_1 , q'_1 respectively.

Here e_{kl} denotes the usual matrix unit with 1 in (k, l) -entry and zero elsewhere. Since $f(x_1, \dots, x_n)$ is not central, by [17] (see also [20]), there exist $u_1, \dots, u_n \in M_m(C)$ and $\gamma \in C - \{0\}$ such that $f(u_1, \dots, u_n) = \gamma e_{kl}$, with $k \neq l$. Moreover, since the set $\{f(r_1, \dots, r_n) : r_1, \dots, r_n \in M_m(C)\}$ is invariant under the action of all C -automorphisms of $M_m(C)$, then for any $i \neq j$ there exist $r_1, \dots, r_n \in M_m(C)$ such that $f(r_1, \dots, r_n) = e_{ij}$. Hence by () we have

$$2ae_{ij}be_{ij} + e_{ij}b'e_{ij} - 2e_{ij}ce_{ij}q = 0 \quad (2)$$

and then left multiplying by e_{ij} , it follows $2e_{ij}ae_{ij}be_{ij} = 0$, which is a contradiction, since a and b have all non-zero entries. Thus we conclude that either a or b are central.

Similarly we can prove that c or q are central. □

Proposition 2. *Let $R = M_m(C)$ be the ring of all matrices over the field C with $\text{char}(R) \neq 2$ and $f(x_1, \dots, x_n)$ a non-central multilinear polynomial over C and $a, a', b, b', c, q, q' \in R$. If*

$$a'f(r)^2 + 2af(r)bf(r) + f(r)b'f(r) - 2f(r)cf(r)q - f(r)^2q' = 0$$

for all $r = (r_1, \dots, r_n) \in R^n$, then either a or b and either c or q are central.

PROOF. If one assumes that C is infinite, then the conclusions follow by Proposition 1.

Now let $\bar{C}$ be finite and K be an infinite field which is an extension of the field C . Let $\bar{R} = M_m(K) \cong R \otimes_C K$. Notice that the multilinear polynomial

$f(x_1, \dots, x_n)$ is central-valued on R if and only if it is central-valued on $\overline{R}$. Consider the generalized polynomial

$$\begin{aligned} P(x_1, \dots, x_n) = & a'f(x_1, \dots, x_n)^2 + 2af(x_1, \dots, x_n)bf(x_1, \dots, x_n) \\ & + f(x_1, \dots, x_n)b'f(x_1, \dots, x_n) \\ & - 2f(x_1, \dots, x_n)cf(x_1, \dots, x_n)q - f(x_1, \dots, x_n)^2q' \end{aligned} \quad (3)$$

which is a generalized polynomial identity for R .

Moreover, it is a multi-homogeneous of multi-degree $(2, \dots, 2)$ in the indeterminates $x_1, \dots, x_n$.

Hence the complete linearization of $P(x_1, \dots, x_n)$ is a multilinear generalized polynomial $\Theta(x_1, \dots, x_n, y_1, \dots, y_n)$ in $2n$ indeterminates, moreover

$$\Theta(x_1, \dots, x_n, x_1, \dots, x_n) = 2^n P(x_1, \dots, x_n).$$

Clearly the multilinear polynomial $\Theta(x_1, \dots, x_n, y_1, \dots, y_n)$ is a generalized polynomial identity for R and $\overline{R}$ too. Since $\text{char}(C) \neq 2$ we obtain $P(r_1, \dots, r_n) = 0$ for all $r_1, \dots, r_n \in \overline{R}$ and then conclusion follows from Proposition 1. $\square$

Lemma 7. *Let R be a noncommutative prime ring of characteristic different from 2 with Utumi quotient ring U and extended centroid C , and $f(x_1, \dots, x_n)$ be a multilinear polynomial over C , which is not central valued on R . Suppose that for some $a, b, c, q \in U$, $F(x) = ax + xb$ and $G(x) = cx + xq$ for all $x \in R$ such that*

$$F^2(f(x_1, \dots, x_n))f(x_1, \dots, x_n) - f(x_1, \dots, x_n)G^2(f(x_1, \dots, x_n)) = 0$$

for all $x_1, \dots, x_n \in R$. Then one of the following holds:

- (1) $F(x) = x(a+b)$ and $G(x) = x(c+q)$ for all $x \in R$ with $(a+b)^2 = (c+q)^2 \in C$;
- (2) $F(x) = x(a+b)$ and $G(x) = (c+q)x$ for all $x \in R$ with $(a+b)^2 = (c+q)^2$;
- (3) $F(x) = (a+b)x$ and $G(x) = x(c+q)$ for all $x \in R$ with $(a+b)^2 = (c+q)^2 \in C$;
- (4) $F(x) = (a+b)x$ and $G(x) = x(c+q)$ for all $x \in R$ with $(a+b)^2 = (c+q)^2$ and $f(x_1, \dots, x_n)^2$ is central valued on R ;
- (5) $F(x) = (a+b)x$ and $G(x) = (c+q)x$ for all $x \in R$, with $(a+b)^2 = (c+q)^2 \in C$.

PROOF. By hypothesis, we have

$$\begin{aligned} h(x_1, \dots, x_n) = & a^2f(x_1, \dots, x_n)^2 + 2af(x_1, \dots, x_n)bf(x_1, \dots, x_n) \\ & + f(x_1, \dots, x_n)(b^2 - c^2)f(x_1, \dots, x_n) - 2f(x_1, \dots, x_n)cf(x_1, \dots, x_n)q \\ & - f(x_1, \dots, x_n)^2q^2 = 0 \end{aligned} \quad (4)$$

for all $x_1, \dots, x_n \in R$. Since R and U satisfy same generalized polynomial identity (see [4]), U satisfies $h(x_1, \dots, x_n) = 0$. Suppose that $h(x_1, \dots, x_n)$ is a trivial GPI for U . Let $T = U *_C C\{x_1, x_2, \dots, x_n\}$, the free product of U and $C\{x_1, \dots, x_n\}$, the free C -algebra in noncommuting indeterminates $x_1, x_2, \dots, x_n$. Then, $h(x_1, \dots, x_n)$ is zero element in $T = U *_C C\{x_1, \dots, x_n\}$. This implies that $\{a^2, a, 1\}$ is a linearly dependent over C . Let $\alpha a^2 + \beta a + \gamma = 0$. If $\alpha = 0$, then $\beta \neq 0$, and hence $a \in C$. If $\alpha \neq 0$, then $a^2 = \lambda a + \mu$ for some $\lambda, \mu \in C$. In this case our identity reduces to

$$\begin{aligned} & (\lambda a + \mu)f(x_1, \dots, x_n)^2 + 2af(x_1, \dots, x_n)bf(x_1, \dots, x_n) \\ & + f(x_1, \dots, x_n)(b^2 - c^2)f(x_1, \dots, x_n) - 2f(x_1, \dots, x_n)cf(x_1, \dots, x_n)q \\ & - f(x_1, \dots, x_n)^2q^2 = 0. \end{aligned} \quad (5)$$

If $a \notin C$, then

$$\lambda af(x_1, \dots, x_n)^2 + 2af(x_1, \dots, x_n)bf(x_1, \dots, x_n) = 0 \quad (6)$$

that is

$$af(x_1, \dots, x_n)(\lambda + 2b)f(x_1, \dots, x_n) = 0. \quad (7)$$

This implies $\lambda + 2b = 0$. Since $\text{char}(R) \neq 2$, this implies $b \in C$. Thus we conclude that either $a \in C$ or $b \in C$.

Similarly we can prove that either $c \in C$ or $q \in C$.

Next suppose that $h(x_1, \dots, x_n)$ is a non-trivial GPI for U . In case C is infinite, we have $h(x_1, \dots, x_n) = 0$ for all $x_1, \dots, x_n \in U \otimes_C \overline{C}$, where $\overline{C}$ is the algebraic closure of C . Since both U and $U \otimes_C \overline{C}$ are prime and centrally closed [9, Theorems 2.5 and 3.5], we may replace R by U or $U \otimes_C \overline{C}$ according to C finite or infinite. Then R is centrally closed over C and $h(x_1, \dots, x_n) = 0$ for all $x_1, \dots, x_n \in R$. By Martindale's theorem [21], R is then a primitive ring with nonzero socle $\text{soc}(R)$ and with C as its associated division ring. Then, by Jacobson's theorem [11, p. 75], R is isomorphic to a dense ring of linear transformations of a vector space V over C . Assume first that V is finite dimensional over C , that is, $\dim_C V = m$. By density of R , we have $R \cong M_m(C)$. Since $f(r_1, \dots, r_n)$ is not central valued on R , R must be noncommutative and so $m \geq 2$. In this case, by Lemma 6, we get that a or b and c or q are in C . If V is infinite dimensional over C , then for any $e^2 = e \in \text{soc}(R)$ we have $eRe \cong M_t(C)$ with $t = \dim_C Ve$. We want to show that in this case also a or b and c or q are in C . To prove this, let at least one of a and b and at least one of c and q are not in C . Then at least one of a and b and at least one of c and q does not centralize the nonzero ideal $\text{soc}(R)$. Hence there exist $h_1, h_2, h_3, h_4 \in \text{soc}(R)$ such that either $[a, h_1] \neq 0$ or $[b, h_2] \neq 0$

and $[c, h_3] \neq 0$ or $[q, h_4] \neq 0$. By Litoff's theorem [10], there exists idempotent $e \in \text{soc}(R)$ such that $ah_1, h_1a, bh_2, h_2b, ch_3, h_3c, qh_4, h_4q, h_1, h_2, h_3, h_4 \in eRe$. We have $eRe \cong M_k(C)$ with $k = \dim_C Ve$. Since R satisfies generalized identity

$$\begin{aligned} & e\{a^2f(ex_1e, \dots, ex_ne)^2 + 2af(ex_1e, \dots, ex_ne)bf(ex_1e, \dots, ex_ne) \\ & + f(ex_1e, \dots, ex_ne)(b^2 - c^2)f(ex_1e, \dots, ex_ne) \\ & - 2f(ex_1e, \dots, ex_ne)cf(ex_1e, \dots, ex_ne)q - f(ex_1e, \dots, ex_ne)^2q^2\}e = 0, \end{aligned} \quad (8)$$

the subring eRe satisfies

$$\begin{aligned} & ea^2ef(x_1, \dots, x_n)^2 + 2eae f(x_1, \dots, x_n)ebe f(x_1, \dots, x_n) \\ & + f(x_1, \dots, x_n)e(b^2 - c^2)ef(x_1, \dots, x_n) \\ & - 2f(x_1, \dots, x_n)ecef(x_1, \dots, x_n)eqe - f(x_1, \dots, x_n)^2eq^2e = 0. \end{aligned} \quad (9)$$

Then by the above finite dimensional case, either eae or ebe and either ece or eqe are central elements of eRe . Thus $ah_1 = (eae)h_1 = h_1eae = h_1a$ or $bh_2 = (ebe)h_2 = h_2(ebe) = h_2b$ and $ch_3 = (ece)h_3 = h_3(ece) = h_3c$ or $qh_4 = (eqe)h_4 = h_4eqe = h_4q$, a contradiction.

Thus up to now, we have proved that either a or b and c or q are in C . Thus we have the following four cases:

Case-I: $a, c \in C$.

In this case, we have

$$f(x_1, \dots, x_n)(a^2 + b^2 + 2ab)f(x_1, \dots, x_n) - f(x_1, \dots, x_n)^2(c^2 + 2cq + q^2) = 0 \quad (10)$$

that is

$$f(x_1, \dots, x_n)\{(a^2 + b^2 + 2ab)f(x_1, \dots, x_n) - f(x_1, \dots, x_n)^2(c^2 + 2cq + q^2)\} = 0 \quad (11)$$

for all $x_1, \dots, x_n \in R$. Then by Lemma 3, we have $a^2 + b^2 + 2ab = c^2 + 2cq + q^2 \in C$, that is $(a + b)^2 = (c + q)^2 \in C$. Hence we obtain that $F(x) = x(a + b)$ and $G(x) = x(c + q)$ with $(a + b)^2 = (c + q)^2 \in C$, which is our conclusion (1).

Case-II: $a, q \in C$.

In this case, we have,

$$f(x_1, \dots, x_n)(a^2 + b^2 + 2ab - q^2 - 2cq - c^2)f(x_1, \dots, x_n) = 0 \quad (12)$$

for all $x_1, \dots, x_n \in R$. Then by Lemma 4, $a^2 + b^2 + 2ab - q^2 - 2cq - c^2 = 0$, that is $(a + b)^2 = (c + q)^2$. Thus we have $F(x) = x(a + b)$ and $G(x) = (c + q)x$ with $(a + b)^2 = (c + q)^2$.

Case-III: $b, c \in C$.

In this case, we have

$$(a^2 + 2ab + b^2)f(x_1, \dots, x_n)^2 - f(x_1, \dots, x_n)^2(c^2 + 2cq + q^2) = 0 \quad (13)$$

for all $x_1, \dots, x_n \in R$. Then by Lemma 1, we have any one of the following two cases:

- $(a + b)^2 = (c + q)^2 \in C$. Then $F(x) = (a + b)x$ and $G(x) = x(c + q)$ for all $x \in R$.
- $(a + b)^2 = (c + q)^2$ and $f(x_1, \dots, x_n)^2$ is central valued on R . Then $F(x) = (a + b)x$ and $G(x) = x(c + q)$ for all $x \in R$.

Case-IV: $b, q \in C$.

In this case, we have

$$(a^2 + 2ab + b^2)f(x_1, \dots, x_n)^2 - f(x_1, \dots, x_n)(c^2 + 2cq + q^2)f(x_1, \dots, x_n) = 0 \quad (14)$$

that is

$$\{(a^2 + 2ab + b^2)f(x_1, \dots, x_n) - f(x_1, \dots, x_n)(c^2 + 2cq + q^2)\}f(x_1, \dots, x_n) = 0 \quad (15)$$

for all $x_1, \dots, x_n \in R$. Then by Lemma 5, we have $a^2 + 2ab + b^2 = c^2 + 2cq + q^2 \in C$, which is $(a + b)^2 = (c + q)^2 \in C$. Thus $F(x) = (a + b)x$ and $G(x) = (c + q)x$ for all $x \in R$, with $(a + b)^2 = (c + q)^2 \in C$. $\square$

Lemma 8. *Let R be a noncommutative prime ring of characteristic different from 2 and $f(x_1, \dots, x_n)$ a multilinear polynomial over C . If for any $i = 1, \dots, n$,*

$$\left[\sum_{i=0}^n f(x_1, \dots, t_i, \dots, x_n), f(x_1, \dots, x_n) \right] = 0 \quad (16)$$

for all $t_i, x_1, \dots, x_n \in R$, then the polynomial $f(x_1, \dots, x_n)$ is central-valued on R .

PROOF. Let a be a noncentral element of R . Then replacing t_i with $[a, x_i]$, we have that

$$\left[\sum_{i=0}^n f(x_1, \dots, [a, x_i], \dots, x_n), f(x_1, \dots, x_n) \right] = 0 \quad (17)$$

which gives,

$$[a, f(x_1, \dots, x_n)]_2 = 0 \quad (18)$$

for all $x_1, \dots, x_n \in R$ implying $f(x_1, \dots, x_n)$ is central-valued on R [16, Theorem]. $\square$

Theorem 1. *Let R be a prime ring of characteristic different from 2 with Utumi quotient ring U and extended centroid C , F and G two nonzero generalized derivations of R , I an ideal of R and $f(x_1, \dots, x_n)$ a multilinear polynomial over C which is not central valued on R . If*

$$F^2(f(x_1, \dots, x_n))f(x_1, \dots, x_n) - f(x_1, \dots, x_n)G^2(f(x_1, \dots, x_n)) = 0$$

for all $x_1, \dots, x_n \in I$, then one of the following holds:

- (1) $F(x) = xa$ and $G(x) = xb$ for all $x \in R$ with $a^2 = b^2 \in C$;
- (2) $F(x) = xa$ and $G(x) = bx$ for all $x \in R$ with $a^2 = b^2$;
- (3) $F(x) = ax$ and $G(x) = xb$ for all $x \in R$ with $a^2 = b^2 \in C$;
- (4) $F(x) = ax$ and $G(x) = xb$ for all $x \in R$ with $a^2 = b^2$ and $f(x_1, \dots, x_n)^2$ is central valued on R ;
- (5) $F(x) = ax$ and $G(x) = bx$ for all $x \in R$, with $a^2 = b^2 \in C$.

PROOF. In [14, Theorem 3], Lee proved that every generalized derivation g on a dense right ideal of R can be uniquely extended to a generalized derivation of U and thus can be assumed to be defined on the whole U with the form $g(x) = ax + d(x)$ for some $a \in U$ and d is a derivation of U . In light of this, we may assume that there exist $a, b \in U$ and derivations d, δ of U such that $F(x) = ax + d(x)$ and $G(x) = bx + \delta(x)$. Since I , R and U satisfy the same generalized polynomial identities (see [4]) as well as the same differential identities (see [17]), without loss of generality, to prove our results, we may assume $F^2(f(x_1, \dots, x_n))f(x_1, \dots, x_n) - f(x_1, \dots, x_n)G^2(f(x_1, \dots, x_n)) = 0$ for all $x_1, \dots, x_n \in U$, where d, δ are two derivations on U .

If F and G both are inner generalized derivations of R , then by Lemma 7 we obtain our conclusions. Thus we assume that not both of F and G are inner. Hence U satisfies

$$\begin{aligned} & \{F(a)f(x_1, \dots, x_n) + 2ad(f(x_1, \dots, x_n)) + d^2(f(x_1, \dots, x_n))\}f(x_1, \dots, x_n) \\ & - f(x_1, \dots, x_n)\{G(b)f(x_1, \dots, x_n) + 2b\delta(f(x_1, \dots, x_n)) \\ & + \delta^2(f(x_1, \dots, x_n))\} = 0. \end{aligned} \quad (19)$$

By assumption, d and δ can not be both inner derivations of U . Assume that d and δ are C -dependent modulo inner derivations of U , say $d = \lambda\delta + ad_p$, where $\lambda \in C$, $p \in U$ and $ad_p(x) = [p, x]$ for all $x \in R$. Then δ can not be inner derivation of U . Moreover,

$$\begin{aligned} d^2(x) &= d(\lambda\delta(x) + [p, x]) = d(\lambda)\delta(x) + \lambda d\delta(x) + [d(p), x] + [p, d(x)] \\ &= d(\lambda)\delta(x) + \lambda^2\delta^2(x) + 2\lambda[p, \delta(x)] + [d(p), x] + [p, [p, x]]. \end{aligned} \quad (20)$$

From (19), we obtain that U satisfies

$$\begin{aligned} & \{F(a)f(x_1, \dots, x_n) + 2\lambda a\delta(f(x_1, \dots, x_n)) + 2a[p, f(x_1, \dots, x_n)] \\ & \quad + d(\lambda)\delta(f(x_1, \dots, x_n)) + \lambda^2\delta^2(f(x_1, \dots, x_n)) + 2\lambda[p, \delta(f(x_1, \dots, x_n))] \\ & \quad + [d(p), f(x_1, \dots, x_n)] + [p, [p, f(x_1, \dots, x_n)]]\}f(x_1, \dots, x_n) \\ & \quad - f(x_1, \dots, x_n)\{G(b)f(x_1, \dots, x_n) \\ & \quad + 2b\delta(f(x_1, \dots, x_n)) + \delta^2(f(x_1, \dots, x_n))\} = 0. \end{aligned} \quad (21)$$

This gives

$$\begin{aligned} & \left\{F(a)f(x_1, \dots, x_n) + (2\lambda a + d(\lambda))\left(f^\delta(x_1, \dots, x_n) + \sum_i f(x_1, \dots, \delta(x_i), \dots, x_n)\right) \right. \\ & \quad + 2a[p, f(x_1, \dots, x_n)] + \lambda^2\left(f^{\delta^2}(x_1, \dots, x_n) + 2\sum_i f^\delta(x_1, \dots, \delta(x_i), \dots, x_n) \right. \\ & \quad \left. + \sum_i f(x_1, \dots, \delta^2(x_i), \dots, x_n) + \sum_{i \neq j} f(x_1, \dots, \delta(x_i), \dots, \delta(x_j), \dots, x_n)\right) \\ & \quad + 2\lambda\left[p, f^\delta(x_1, \dots, x_n) + \sum_i f(x_1, \dots, \delta(x_i), \dots, x_n)\right] + [d(p), f(x_1, \dots, x_n)] \\ & \quad \left. + [p, [p, f(x_1, \dots, x_n)]]\right\}f(x_1, \dots, x_n) - f(x_1, \dots, x_n)\left\{G(b)f(x_1, \dots, x_n) \right. \\ & \quad + 2b\left(f^\delta(x_1, \dots, x_n) + \sum_i f(x_1, \dots, \delta(x_i), \dots, x_n)\right) \\ & \quad + f^{\delta^2}(x_1, \dots, x_n) + 2\sum_i f^\delta(x_1, \dots, \delta(x_i), \dots, x_n) + \sum_i f(x_1, \dots, \delta^2(x_i), \dots, x_n) \\ & \quad \left. + \sum_{i \neq j} f(x_1, \dots, \delta(x_i), \dots, \delta(x_j), \dots, x_n)\right\} = 0. \end{aligned} \quad (22)$$

Then by Kharchenko's theorem [12], we have that U satisfies

$$\begin{aligned} & \left\{F(a)f(x_1, \dots, x_n) + (2\lambda a + d(\lambda))\left(f^\delta(x_1, \dots, x_n) + \sum_i f(x_1, \dots, y_i, \dots, x_n)\right) \right. \\ & \quad + 2a[p, f(x_1, \dots, x_n)] + \lambda^2\left(f^{\delta^2}(x_1, \dots, x_n) + 2\sum_i f^\delta(x_1, \dots, y_i, \dots, x_n) \right. \\ & \quad \left. + \sum_i f(x_1, \dots, t_i, \dots, x_n) + \sum_{i \neq j} f(x_1, \dots, y_i, \dots, y_j, \dots, x_n)\right) \\ & \quad \left. + 2\lambda\left[p, f^\delta(x_1, \dots, x_n) + \sum_i f(x_1, \dots, y_i, \dots, x_n)\right] + [d(p), f(x_1, \dots, x_n)] \right\} \end{aligned}$$

$$\begin{aligned}
& + [p, [p, f(x_1, \dots, x_n)]] \Big\} f(x_1, \dots, x_n) - f(x_1, \dots, x_n) \Big\{ G(b)f(x_1, \dots, x_n) \\
& + 2b(f^\delta(x_1, \dots, x_n) + \sum_i f(x_1, \dots, y_i, \dots, x_n)) \\
& + f^{\delta^2}(f(x_1, \dots, x_n)) + 2 \sum_i f^\delta(x_1, \dots, y_i, \dots, x_n) + \sum_i f(x_1, \dots, t_i, \dots, x_n) \\
& + \sum_{i \neq j} f(x_1, \dots, y_i, \dots, y_j, \dots, x_n) \Big\} = 0. \tag{23}
\end{aligned}$$

In particular, U satisfies the blended component

$$\begin{aligned}
& \Big\{ \lambda^2 \sum_i f(x_1, \dots, t_i, \dots, x_n) \Big\} f(x_1, \dots, x_n) \\
& - f(x_1, \dots, x_n) \Big\{ \sum_i f(x_1, \dots, t_i, \dots, x_n) \Big\} = 0. \tag{24}
\end{aligned}$$

In particular, when $t_1 = x_1$ and $t_2 = \dots = t_n = 0$, we have from above that

$$(\lambda^2 - 1)f(x_1, \dots, x_n)^2 = 0. \tag{25}$$

This implies that $\lambda^2 = 1$. Then (24) becomes

$$\begin{aligned}
& \Big\{ \sum_i f(x_1, \dots, t_i, \dots, x_n) \Big\} f(x_1, \dots, x_n) \\
& - f(x_1, \dots, x_n) \Big\{ \sum_i f(x_1, \dots, t_i, \dots, x_n) \Big\} = 0. \tag{26}
\end{aligned}$$

In this case by Lemma 8, $f(x_1, \dots, x_n)$ is central valued on R , a contradiction.

The situation when $\delta = \lambda d + ad_q$ is similar.

Next assume that d and δ are C -independent modulo inner derivations of U . Let $f^d(x_1, \dots, x_n)$ and $f^{d^2}(x_1, \dots, x_n)$ be the polynomials obtained from $f(x_1, \dots, x_n)$ replacing each coefficients α_σ with $d(\alpha_\sigma)$ and $d^2(\alpha_\sigma)$ respectively. Then we have

$$d(f(x_1, \dots, x_n)) = f^d(x_1, \dots, x_n) + \sum_i f(x_1, \dots, d(x_i), \dots, x_n)$$

and

$$\begin{aligned}
d^2(f(x_1, \dots, x_n)) &= f^{d^2}(x_1, \dots, x_n) + 2 \sum_i f^d(x_1, \dots, d(x_i), \dots, x_n) \\
&+ \sum_i f(x_1, \dots, d^2(x_i), \dots, x_n) + \sum_{i \neq j} f(x_1, \dots, d(x_i), \dots, d(x_j), \dots, x_n). \tag{27}
\end{aligned}$$

Then we have from (19) that U satisfies

$$\begin{aligned}
& \left\{ F(a)f(x_1, \dots, x_n) + 2af^d(x_1, \dots, x_n) + 2a \sum_i f(x_1, \dots, d(x_i), \dots, x_n) \right. \\
& + f^{d^2}(x_1, \dots, x_n) + 2 \sum_i f^d(x_1, \dots, d(x_i), \dots, x_n) + \sum_i f(x_1, \dots, d^2(x_i), \dots, x_n) \\
& + \sum_{i \neq j} f(x_1, \dots, d(x_i), \dots, d(x_j), \dots, x_n) \left. \right\} f(x_1, \dots, x_n) \\
& - f(x_1, \dots, x_n) \left\{ G(b)f(x_1, \dots, x_n) + 2bf^\delta(x_1, \dots, x_n) \right. \\
& + 2b \sum_i (f(x_1, \dots, \delta(x_i), \dots, x_n)) + f^{\delta^2}(x_1, \dots, x_n) \\
& + 2 \sum_i f^\delta(x_1, \dots, \delta(x_i), \dots, x_n) + \sum_i f(x_1, \dots, \delta^2(x_i), \dots, x_n) \\
& \left. + \sum_{i \neq j} f(x_1, \dots, \delta(x_i), \dots, \delta(x_j), \dots, x_n) \right\} = 0. \tag{28}
\end{aligned}$$

Since neither d nor δ is inner, by Kharchenko's theorem [12], we have from above that U satisfies

$$\begin{aligned}
& \left\{ F(a)f(x_1, \dots, x_n) + 2af^d(x_1, \dots, x_n) + 2a \sum_i f(x_1, \dots, y_i, \dots, x_n) \right. \\
& + f^{d^2}(x_1, \dots, x_n) + 2 \sum_i f^d(x_1, \dots, y_i, \dots, x_n) \\
& + \sum_i f(x_1, \dots, t_i, \dots, x_n) + \sum_{i \neq j} f(x_1, \dots, y_i, \dots, y_j, \dots, x_n) \left. \right\} f(x_1, \dots, x_n) \\
& - f(x_1, \dots, x_n) \left\{ G(b)f(x_1, \dots, x_n) + 2bf^\delta(x_1, \dots, x_n) \right. \\
& + 2b \sum_i (f(x_1, \dots, y_i, \dots, x_n)) + f^{\delta^2}(x_1, \dots, x_n) + 2 \sum_i f^\delta(x_1, \dots, y_i, \dots, x_n) \\
& \left. + \sum_i f(x_1, \dots, t_i, \dots, x_n) + \sum_{i \neq j} f(x_1, \dots, y_i, \dots, y_j, \dots, x_n) \right\} = 0. \tag{29}
\end{aligned}$$

In particular U satisfies the blended component

$$\begin{aligned}
& \left\{ \sum_i f(x_1, \dots, t_i, \dots, x_n) \right\} f(x_1, \dots, x_n) \\
& - f(x_1, \dots, x_n) \left\{ \sum_i f(x_1, \dots, t_i, \dots, x_n) \right\} = 0. \tag{30}
\end{aligned}$$

Then by Lemma 8, $f(x_1, \dots, x_n)$ is central valued on R , a contradiction. $\square$

As a reduction of previous Theorem we also have:

Theorem 2. *Let R be a prime ring of characteristic different from 2 with Utumi quotient ring U and extended centroid C , F and G two nonzero generalized derivations of R , $a, b \in U$ and f, g derivations of R such that $F(x) = ax + f(x)$, $G(x) = bx + g(x)$, for all $x \in R$. If $F^2(x)x - xG^2(x) = 0$ for all $x \in R$, then $a^2 = b^2$ and either R is commutative or one of the following holds:*

- (1) $F(x) = xa$ and $G(x) = xb$ for all $x \in R$ with $a^2 \in C$;
- (2) $F(x) = xa$ and $G(x) = bx$ for all $x \in R$;
- (3) $F(x) = ax$ and $G(x) = xb$ for all $x \in R$ with $a^2 \in C$;
- (4) $F(x) = ax$ and $G(x) = bx$ for all $x \in R$, with $a^2 \in C$.

PROOF. By applying Theorem 1, we have that either R is commutative or one of the following holds:

- $F(x) = xa$ and $G(x) = xb$ for all $x \in R$ with $a^2 = b^2 \in C$;
- $F(x) = xa$ and $G(x) = bx$ for all $x \in R$ with $a^2 = b^2$;
- $F(x) = ax$ and $G(x) = xb$ for all $x \in R$ with $a^2 = b^2 \in C$;
- $F(x) = ax$ and $G(x) = bx$ for all $x \in R$, with $a^2 = b^2 \in C$.

Therefore, in order to prove the present Theorem, we now assume that R is commutative and then show that $a^2 = b^2$.

By our assumption, R satisfies the following differential identity

$$a^2x + 2af(x) + f(a)x + f^2(x) - b^2x - 2bg(x) - g(b)x - g^2(x) = 0. \quad (31)$$

Notice that, if both f and g are inner derivations of R , then relation (31) implies that $(a^2 - b^2)R = (0)$ and so we are done. Hence we assume that at least one of f and g is not an inner derivation of R .

First, we assume that f and g are linearly C -independent modulo inner derivations, that is, f and g are linearly C -independent modulo 0. Thus, by Kharchenko's Theorem [12], we have from (31) that R satisfies

$$a^2x_1 + 2ax_2 + f(a)x_1 + x_3 - b^2x_1 - 2bx_4 - g(b)x_1 - x_5 = 0. \quad (32)$$

In particular, we have $2ax_2 = 2bx_4$. Since $\text{char}(R) \neq 2$, it follows $a = b$ and so $a^2 = b^2$.

Next we assume that f and g are linearly C -dependent modulo 0, that is, there exists $0 \neq \lambda \in C$ such that $f(x) = \lambda g(x)$, for all $x \in R$. Then by (31), R

satisfies

$$\begin{aligned} a^2x + 2\lambda ag(x) + \lambda g(a)x + \lambda g(\lambda)g(x) \\ + \lambda^2g^2(x) - b^2x - 2bg(x) - g(b)x - g^2(x) = 0. \end{aligned} \quad (33)$$

In light of previous comments, we may also assume $g \neq 0$, i.e. g is not inner. Again by Kharchenko's Theorem [12], R satisfies

$$a^2x_1 + 2\lambda ax_2 + \lambda g(a)x_1 + \lambda g(\lambda)x_2 + \lambda^2x_3 - b^2x_1 - 2bx_2 - g(b)x_1 - x_3 = 0. \quad (34)$$

For $x_1 = x_2 = 0$, we have $\lambda^2x_3 = x_3$ for all $x_3 \in R$, that is, $\lambda^2x = x$ for all $x \in R$. In particular $\lambda^2g(x) = g(x)$ for all $x \in R$. This implies that

$$g(x) = g(\lambda^2x) = g(\lambda^2)x + \lambda^2g(x) = g(\lambda^2)x + g(x),$$

which implies

$$0 = g(\lambda^2)x = 2\lambda g(\lambda)x \quad \forall x \in R.$$

This gives $\lambda g(\lambda) = 0$. Thus (34) reduces to

$$a^2x_1 + 2\lambda ax_2 + \lambda g(a)x_1 - b^2x_1 - 2bx_2 - g(b)x_1 = 0. \quad (35)$$

In particular, R satisfies $2\lambda ax_2 - 2bx_2 = 0$. Therefore $b = \lambda a$, so that $b^2 = (\lambda a)^2 = \lambda^2 a^2 = a^2$, and the proof is complete. $\square$

Remark 1. By [14, Theorem 3], every generalized derivation F on R can be uniquely extended to a generalized derivation of U and there exist $a \in U$ and a derivation $f : U \rightarrow U$ such that $F(x) = ax + f(x)$, for all $x \in U$. Starting from this, we may trivially write $F(x) = ax + xa - xa + f(x) = xa + [a, x] + f(x) = xa + f'(x)$, where $f' : U \rightarrow U$ is a derivation defined as $f'(x) = f(x) - [x, a]$, for all $x \in U$. Thus, for any generalized derivation F there exist two derivations f and f' of U (associated with F), such that both $F(x) = ax + f(x)$ and $F(x) = xa + f'(x)$.

In light of this, we may state Theorem 2 as follows: *Let f, f' be the associated derivations with F and g, g' the associated ones with G . Denote $F(x) = ax + f(x) = xa + f'(x)$ and $G(x) = bx + g(x) = xb + g'(x)$, for all $x \in R$. If $F^2(x)x - xG^2(x) = 0$ for all $x \in R$, then $a^2 = b^2$ and either R is commutative or one of the following holds:*

- (1) $f' = 0, g' = 0, F(x) = xa$ and $G(x) = xb$ for all $x \in R$ with $a^2 \in C$;
- (2) $f' = 0, g = 0, F(x) = xa$ and $G(x) = bx$ for all $x \in R$;
- (3) $f = 0, g' = 0, F(x) = ax$ and $G(x) = xb$ for all $x \in R$ with $a^2 \in C$;

(4) $f = 0$, $g = 0$, $F(x) = ax$ and $G(x) = bx$ for all $x \in R$, with $a^2 \in C$.

We finally examine the case of 2-torsion free semiprime rings. The first result we obtained is the following:

Lemma 9. *Let R be a semiprime ring of characteristic different from 2 with Utumi quotient ring U , extended centroid C and center $Z(R) \neq (0)$. Let F and G be two nonzero generalized derivations of R respectively defined as $F(x) = ax + f(x)$, $G(x) = bx + g(x)$, for f and g derivations of R and $a, b \in U$. If $F^2(x)x - xG^2(x) = 0$, for all $x \in R$, then $a^2 = b^2$.*

PROOF. Since R is semiprime, by Proposition 2.5.1 in [2], the derivations f and g can be uniquely extended on U . Since U and R satisfy the same differential identities (see [17]), then $F^2(x)x - xG^2(x) = 0$, for all $x \in U$. Let B be the complete boolean algebra of idempotents in C and M be any maximal ideal of B .

Since U is a B -algebra orthogonal complete (see for instance (2) of Fact 1 in [5], page 42), MU is a prime ideal of U , which is both f -invariant and g -invariant. Denote $\bar{U} = U/MU$ and $\bar{f}$ the derivation induced by f on $\bar{U}$ and $\bar{g}$ the derivation induced by g on $\bar{U}$. In particular, $\bar{U}$ is a prime ring and so, by Theorem 2, $\bar{a}^2 - \bar{b}^2 = 0$ in $\bar{U}$. This implies that, for any maximal ideal M of B , $a^2 - b^2 \in MU$, therefore $a^2 - b^2 \in \bigcap_M MU = 0$. $\square$

Theorem 3. *Let R be a semiprime ring of characteristic different from 2 with Utumi quotient ring U , extended centroid C and center $Z(R) \neq (0)$. Let F and G be two nonzero generalized derivations of R respectively defined as $F(x) = ax + f(x)$, $G(x) = bx + g(x)$, for f and g derivations of R and $a, b \in U$. If $F^2(x)x - xG^2(x) = 0$, for all $x \in R$, then $a^2 = b^2$ and either R contains a non-zero central ideal or $f(Z(R)) = f'(Z(R)) = g(Z(R)) = g'(Z(R)) = (0)$, $f(a) = f'(a) = g(b) = g'(b) = 0$, $ff'(R) = f'f(R) = gg'(R) = g'g(R) = (0)$ and $Z(R)(G^2(x) - a^2x) = Z(R)(F^2(x) - xa^2) = (0)$ for all $x \in R$, or one of the following holds:*

- (1) $F(x) = xa$ and $G(x) = xb$ for all $x \in R$ with $a^2 \in C$;
- (2) $F(x) = xa$ and $G(x) = bx$ for all $x \in R$;
- (3) $F(x) = ax$ and $G(x) = xb$ for all $x \in R$ with $a^2 \in C$;
- (4) $F(x) = ax$ and $G(x) = bx$ for all $x \in R$, with $a^2 \in C$.

PROOF. By Lemma 9, $a^2 = b^2$. In the sequel we assume that R does not contain any non-zero central ideal.

In light of Remark 1 we may write $F(x) = ax + f(x) = xa + f'(x)$ and $G(x) = bx + g(x) = xb + g'(x)$, for all $x \in R$.

Firstly we consider the case when both F and G are centralizers. Hence either $f(R) = (0)$ or $f'(R) = (0)$ and either $g(R) = (0)$ or $g'(R) = (0)$. In other words either $F(x) = ax$ or $F(x) = xa$ and either $G(x) = bx$ or $G(x) = xb$, for any $x \in R$.

Let P be a prime ideal of R , set $\bar{R} = R/P$ and write $\bar{r} = r + P \in \bar{R}$, for all $r \in R$. We notice that in any case $F(P) \subseteq P$ and $G(P) \subseteq P$. Thus $\bar{F} : \bar{R} \rightarrow \bar{R}$ and $\bar{G} : \bar{R} \rightarrow \bar{R}$ are generalized derivations of $\bar{R} = R/P$, for any ideal P . By the prime case it follows that one of the following cases must occur:

- (1) $F(x) = ax$ and $G(x) = bx$, for all $x \in R$. Thus $0 = a^2x^2 - xb^2x = [a^2x, x]$, for all $x \in R$ and by Lemma 3.2 in [7] and since R is not commutative, we have that $a^2 \in Z(R)$;
- (2) $F(x) = xa$ and $G(x) = bx$, for all $x \in R$. In this case we are done, since $a^2 = b^2$.
- (3) $F(x) = ax$ and $G(x) = xb$, for all $x \in R$. Thus $0 = a^2x^2 - x^2b^2 = [a^2, x^2]$ and by Main Theorem in [13] and since R does not contain any non-zero central ideal, it follows $a^2 \in Z(R)$;
- (4) $F(x) = xa$ and $G(x) = xb$, for all $x \in R$. In this case $0 = xa^2x - x^2b^2 = [xa^2, x]$, for all $x \in R$. Once again, by Lemma 3.2 in [7] we have that $a^2 \in Z(R)$.

By the previous argument, we now assume that either F or G is not a centralizer, without loss of generality we consider the case $g(R) \neq (0)$. We will prove that $f(Z(R)) = f'(Z(R)) = g(Z(R)) = g'(Z(R)) = (0)$, $f(a) = f'(a) = g(b) = g'(b) = 0$, $ff'(R) = f'f(R) = gg'(R) = g'g(R) = (0)$ and $Z(R)(G^2(x) - a^2x) = Z(R)(F^2(x) - xa^2) = (0)$ for all $x \in R$.

As above, let P be a prime ideal of R , set $\bar{R} = R/P$ and write $\bar{r} = r + P \in \bar{R}$ for all $r \in R$. We start from

$$\overline{F^2(\bar{r})\bar{r} - \bar{r}G^2(\bar{r})} = \bar{0}, \quad \forall \bar{r} = r + P \in \bar{R}, \quad x \in P \quad (36)$$

and by computations we get

$$\begin{aligned} & (a^2r + af(r) + 2af(x) + f(ar) + f^2(r) + f^2(x))r \\ & - r(b^2r + bg(r) + 2bg(x) + g(br) + g^2(r) + g^2(x)) \in P, \quad \forall x \in P, r \in R. \end{aligned} \quad (37)$$

Replace x with xy in (37), for any $y \in P$, then it follows

$$\begin{aligned} & (a^2r + af(r) + f(ar) + f^2(r) + 2f(x)f(y))r \\ & - r(b^2r + bg(r) + g(br) + g^2(r) + 2g(x)g(y)) \in P, \quad \forall x, y \in P, r \in R. \end{aligned} \quad (38)$$

By comparing (37) with (38) we get

$$(f^2(x) + 2af(x) - 2f(x)f(y))r - r(g^2(x) + 2bg(x) - 2g(x)g(y)) \in P, \\ \forall x, y \in P, r \in R. \quad (39)$$

We divide our argument into three cases:

Case 1. $f(P) \subseteq P$, $g(P) \not\subseteq P$.

In this case $\overline{g(P)}$ is a non-zero ideal of $\overline{R}$. Moreover by (39) we have that

$$r(g^2(x) + 2bg(x) - 2g(x)g(y)) \in P, \quad \forall x, y \in P, r \in R. \quad (40)$$

Replace y with ys in (40), for any $s \in R$, then it follows

$$r(g^2(x) + 2bg(x) - 2g(x)g(y)s) \in P, \quad \forall x, y \in P, r, s \in R. \quad (41)$$

On the other hand, replacing y with sy in (40) we also have

$$r(g^2(x) + 2bg(x) - 2g(x)sg(y)) \in P, \quad \forall x, y \in P, r, s \in R. \quad (42)$$

Hence by (41) and (42) we get $rg(x)[s, g(y)] \in P$, that is $\overline{Rg(P)[R, g(P)]} = (\overline{0})$. Since $\overline{g(P)}$ is a non-zero ideal of the prime ring $\overline{R}$, we conclude that $\overline{g(P)} \subseteq Z(\overline{R})$, that is $\overline{R}$ is commutative.

An analogous argument shows that if $g(P) \subseteq P$ and $f(P) \not\subseteq P$, then $\overline{R}$ is commutative (we omit it for brevity).

Case 2. $f(P) \not\subseteq P$ and $g(P) \not\subseteq P$.

Notice that both $\overline{f(P)}$ and $\overline{g(P)}$ are non-zero ideals of $\overline{R}$. Also in this case we start from (39) and replace here y with ys , for any $s \in R$. Thus

$$(f^2(x) + 2af(x) - 2f(x)f(y)s)r - r(g^2(x) + 2bg(x) - 2g(x)g(y)s) \in P, \\ \forall x, y \in P, r, s \in R. \quad (43)$$

On the other hand, replacing y with sy in (39) we have

$$(f^2(x) + 2af(x) - 2f(x)s f(y))r - r(g^2(x) + 2bg(x) - 2g(x)sg(y)) \in P, \\ \forall x, y \in P, r, s \in R \quad (44)$$

and comparing (43) with (44) it follows

$$f(x)[f(y), s]r - rg(x)[g(y), s] \in P, \quad \forall x, y \in P, r, s \in R \quad (45)$$

that is

$$\overline{f(x)[f(y), s]r - rg(x)[g(y), s]} = \bar{0}, \quad \forall \bar{r}, \bar{s} \in \bar{R}, \quad x, y, \in P. \quad (46)$$

Denote $\bar{u} = \overline{f(x)[f(y), s]}$ and $\bar{v} = \overline{g(x)[g(y), s]}$, so that $\bar{u}\bar{r} - \bar{r}\bar{v} = \bar{0}$, for any $\bar{r} \in \bar{R}$. In this case, it is well known that $\bar{u} = \bar{v} \in Z(\bar{R})$, which means

$$\overline{f(x)[f(y), s]} \in Z(\bar{R}), \quad \forall \bar{s} \in \bar{R}, \quad x, y, \in P. \quad (47)$$

For $s = rt$ in (47) and since $\overline{f(P)R} \subseteq \overline{f(P)}$, it follows that

$$\overline{f(x)[f(y), r]t} \in Z(\bar{R}), \quad \forall \bar{r}, \bar{t} \in \bar{R}, \quad x, y, \in P. \quad (48)$$

By the primeness of $\bar{R}$, either $\bar{t} \in Z(\bar{R})$ for any $\bar{t} \in \bar{R}$, or $\overline{f(x)[f(y), r]} = \bar{0}$, for any $\bar{r} \in \bar{R}$ and $x, y, \in P$. Therefore we may assume that

$$\overline{f(x)[f(y), t]} = \bar{0}, \quad \forall \bar{t} \in \bar{R}, \quad x, y, \in P. \quad (49)$$

By using Theorem 1 in [18], one has that either $\overline{f(x)} = \bar{0}$ or $\overline{f(y)} \in Z(\bar{R})$. In any case we have $\overline{f(P)} \subseteq Z(\bar{R})$. Since $\overline{f(P)}$ is a non-zero ideal of the prime ring $\bar{R}$, it follows that $\bar{R}$ is commutative.

Case 3. $f(P) \subseteq P$ and $g(P) \subseteq P$.

In this case we remark that f, f', g, g' induce canonical derivations $\bar{f}, \bar{f}', \bar{g}$ and $\bar{g}'$ on $\bar{R}$. It follows from the prime case and Remark 1 that either $\bar{R}$ is commutative, or the following holds simultaneously:

- (1) either $f(R) \subseteq P$ or $f'(R) \subseteq P$;
- (2) either $g(R) \subseteq P$ or $g'(R) \subseteq P$.

The argument contained in Cases 1, 2, 3 implies that:

$$\begin{aligned} [f(R)f'(R), R] &\subseteq \bigcap_i P_i = (0), \quad [f(R), f'(R)] \subseteq \bigcap_i P_i = (0), \\ [f(f'(R)), R] &\subseteq \bigcap_i P_i = (0). \end{aligned} \quad (50)$$

and also

$$\begin{aligned} [g(R)g'(R), R] &\subseteq \bigcap_i P_i = (0), \quad [g(R), g'(R)] \subseteq \bigcap_i P_i = (0), \\ [g(g'(R)), R] &\subseteq \bigcap_i P_i = (0). \end{aligned} \quad (51)$$

In the next step we prove that $f(R)f'(R) = (0)$. To do this, by contradiction we consider the case $f(R)f'(R) \neq (0)$.

For all $x, y, t \in R$ and since $f(R)f'(R) \subseteq Z(R)$ we have both $f(x)f'(yt) \in Z(R)$ and $f(xy)f'(t) \in Z(R)$ that is

$$f(x)f'(y)t + f(x)yf'(t) \in Z(R)$$

and

$$f(x)yf'(t) + xf(y)f'(t) \in Z(R).$$

Comparing these last two relations we get

$$f(x)f'(y)t - xf(y)f'(t) \in Z(R), \quad \forall x, y, t \in R. \quad (52)$$

In case $f(Z(R)) = (0)$, we have for all $y, t \in R$,

$$0 = f\left(f(y)f'(t)\right) = f^2(y)f'(t) + f(y)(ff')(t), \quad (53)$$

which implies $f(y)(ff')(t) \in Z(R)$, since $f^2(y)f'(t) \in Z(R)$. Moreover, by (50) we also have that $(ff')(R) \subseteq Z(R)$. Suppose that there exists $t \in R$ such that $0 \neq \alpha = (ff')(t) \in Z(R)$, then $\alpha f(y) \in Z(R)$ for all $y \in R$. Moreover, $f(\alpha) = 0$ implies $f(\alpha y) = \alpha f(y) \in Z(R)$, that is $f(\alpha R) \subseteq Z(R)$. Therefore R must contain a non-zero central ideal, which is a contradiction, unless when $(0) = f(\alpha R) = \alpha f(R)$. In this last case, since $\alpha \in f(R)$, it follows the contradiction $\alpha^2 = 0$. Thus the previous argument shows that $(ff')(R) = (0)$, and so by (53) we get

$$f^2(y)f'(t) = 0, \quad \forall y, t \in R. \quad (54)$$

Replace y with $yf'(t)y$ in (54) and then get $0 = f(y)f'(t)f(y)f'(t) = \beta^2$, for $\beta = f(y)f'(t) \neq 0$, which is again a contradiction.

Assume now $f(Z(R)) \neq (0)$ and let $0 \neq z \in Z(R)$ be such that $f(z) = \beta \neq 0$. Replacing x with zx in (52) and using again (52), we have $\beta x f'(y)t \in Z(R)$ for all $x, y, t \in R$, that is $R(\beta f'(R))R \subseteq Z(R)$. Therefore, R contains a non-zero central ideal which is a contradiction, unless when $\beta f'(R) = (0)$. On the other hand $f'(z) = [a, z] + f(z) = \beta$, thus $0 = \beta f'(z) = \beta^2$, a contradiction again.

In light of previous contradictions, it is proved that, if R does not contain any non-zero central ideal, then $f(R)f'(R) = (0)$ (similarly one can prove $f'(R)f(R) = (0)$).

Thus $f(R)Rf'(R) = (0)$ and $(0) = f(a)Rf'(a) = f(a)Rf(a)$ implying $f(a) = 0$ and also $f'(a) = 0$. Similarly, for any $z \in Z(R)$, $0 = f(z)Rf'(z) = f(z)Rf(z)$ implying $f(Z(R)) = (0)$ and also $f'(Z(R)) = (0)$. Moreover, for any $x, y \in R$, $0 = f(f(x)f'(y)) = f^2(x)f'(y) + f(x)ff'(y) = f(x)ff'(y)$. Replacing x with

$f'(y)$, we have that $(ff'(y))^2 = 0$ for all $y \in R$. Since by (50), $ff'(R) \subseteq Z(R)$, it follows that $ff'(R) = (0)$. Analogously $f'f(R) = (0)$.

In a similar way $g(b) = g'(b) = 0$, $g(Z(R)) = g'(Z(R)) = (0)$, $g(R)g'(R) = g'(R)g(R) = (0)$, $gg'(R) = (0)$ and $g'g(R) = (0)$.

By using all the previous conditions, we have

$$F^2(x) = (xa + f'(x))a + f'(xa + f'(x)) = xa^2 + 2f'(x)a + f'^2(x)$$

and

$$G^2(x) = b(bx + g(x)) + g(bx + g(x)) = a^2x + 2bg(x) + g^2(x)$$

so that, for all $x \in R$,

$$0 = F^2(x)x - xG^2(x) = 2f'(x)ax + f'^2(x)x - 2xbg(x) - xg^2(x). \quad (55)$$

Replace x with $x + y$ for any $y \in Z(R)$ in (55), we have

$$y(F^2(x) - G^2(x) + [a^2, x]) = 0. \quad (56)$$

Right multiplying (56) by x and since $F^2(x)x = xG^2(x)$, we have

$$Z(R)[G^2(x) - a^2x, x] = 0, \quad \forall x \in R. \quad (57)$$

Denote $\Delta(x) = G^2(x) - a^2x = 2bg(x) + g^2(x)$. By computations we get

$$\begin{aligned} \Delta(xy) &= G^2(xy) - a^2xy = G^2(x)y + 2G(x)g(y) + xg^2(y) - a^2xy \\ &= \Delta(x)y + 2G(x)g(y) + xg^2(y). \end{aligned} \quad (58)$$

Moreover, by using $G(x) = xb + g'(x)$ and $g'(R)g(R) = (0)$ in (58), it follows

$$\begin{aligned} \Delta(xy) &= \Delta(x)y + 2(xb + g'(x))g(y) + xg^2(y) \\ &= \Delta(x)y + 2xbg(y) + xg^2(y) = \Delta(x)y + x\Delta(y). \end{aligned} \quad (59)$$

This implies that Δ is a derivation of R satisfying $Z(R)[\Delta(x), x] = (0)$ for all $x \in R$. In particular, $\Delta(zx) = z\Delta(x)$ for all $z \in Z(R)$, then $[\Delta(zx), zx] = z^2[\Delta(x), x] = 0$, for all $x \in R$. This means that Δ is commuting on the non-zero ideal zR of R , for any $0 \neq z \in Z(R)$. Since R does not contain any non-zero central ideal, by [13] it follows $\Delta(zR) = (0)$, that is $z\Delta(R) = (0)$ and $Z(R)(G^2(x) - a^2x) = (0)$ for all $x \in R$, as required.

We finally remark that, starting again from (56) and left multiplying by x , we obtain $Z(R)[F^2(x) - xa^2, x] = (0)$ for all $x \in R$. The same above argument shows that $Z(R)(F^2(x) - xa^2) = (0)$ for all $x \in R$. $\square$

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