

Groups, partitions and representation functions

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Abstract. Let X be a semigroup written additively and $h \geq 2$ a fixed integer. Let x be an element of X and $A_1, \dots, A_h$ be nonempty subsets of X . Let $R_{A_1+\dots+A_h}(x)$ denote the number of solutions of the equation $a_1 + \dots + a_h = x$, where $a_i \in A_i$. In this paper for $X = \mathbb{N}$ we give a necessary and sufficient condition such that the equality $R_{A_1+A_2}(n) = R_{X \setminus A_1 + X \setminus A_2}(n)$ holds from a certain point on. We study similar questions when $X = \mathbb{Z}_m$ and in general when $X = G$, where G is a finite additive group.

1. Introduction

Let X be a semigroup, written additively. Let $A_1, \dots, A_h$ be nonempty subsets of X and let x be an element of X . We denote by $|A|$ the cardinality of the set A . We define the ordered representation function

$$R_{A_1+\dots+A_h}(x) = |\{(a_1, \dots, a_h) \in A_1 \times \dots \times A_h : a_1 + \dots + a_h = x\}|.$$

If $A_i = A$ for $i = 1, \dots, h$, then we write

$$R_{A,h}^{(1)}(x) = |\{(a_1, \dots, a_h) : a_i \in A, a_1 + \dots + a_h = x\}|.$$

Let X be an abelian semigroup, written additively. For $A \subset X$, let A^h denote the set of all h -tuples of A . Two h -tuples $(a_1, \dots, a_h) \in A^h$ and $(a'_1, \dots, a'_h) \in A^h$ are equivalent if there is a permutation $\alpha : \{1, \dots, h\} \rightarrow \{1, \dots, h\}$ such that

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$a_{\alpha(i)} = a'_i$ for $i = 1, \dots, h$. Two other representation functions arise often and naturally in additive number theory. The unordered representation function $R_{A,h}^{(2)}(x)$ counts the number of equivalence classes of h -tuples $(a_1, \dots, a_h)$ such that $a_1 + \dots + a_h = x$. The unordered restricted representation function $R_{A,h}^{(3)}(x)$ counts the number of equivalence classes of h -tuples $(a_1, \dots, a_h)$ of pairwise distinct elements of A such that $a_1 + \dots + a_h = x$. It is easy to see that the definitions of the unordered and the unordered restricted representation functions make sense only in Abelian groups.

Alternative definitions for $R_{A,2}^{(2)}(x)$ and $R_{A,2}^{(3)}(x)$ are the following. Denote by

$$D_A(x) = |\{a : a \in A, a + a = x\}|$$

then

$$R_{A,2}^{(2)}(x) = \frac{1}{2}R_{A,2}^{(1)}(x) + \frac{1}{2}D_A(x) \quad (1)$$

and

$$R_{A,2}^{(3)}(x) = \frac{1}{2}R_{A,2}^{(1)}(x) - \frac{1}{2}D_A(x). \quad (2)$$

Let $\mathbb{N}$ be the set of nonnegative integers. Let $X = \mathbb{N}$. Sárközy asked if there exist two sets A and B with $|A \Delta B| = \infty$ such that $R_{A,2}^{(i)}(n) = R_{B,2}^{(i)}(n)$, for $i = 1, 2, 3$ and for all sufficiently large n . For $i = 2$ DOMBI [3] proved that the answer is positive and for $i = 1$ the answer is negative. For $i = 3$ CHEN and WANG [1] proved that the set of natural numbers can be partitioned into two subsets A and B such that $R_{A,2}^{(3)}(n) = R_{B,2}^{(3)}(n)$ for every n large enough. LEV [5] and independently SÁNDOR [6] characterized all subsets $A \subset \mathbb{N}$ such that $R_{A,2}^{(2)}(n) = R_{\mathbb{N} \setminus A, 2}^{(2)}(n)$ or $R_{A,2}^{(3)}(n) = R_{\mathbb{N} \setminus A, 2}^{(3)}(n)$ for big enough n . The precise theorems are the following.

Theorem (LEV, SÁNDOR, 2004). *Let $X = \mathbb{N}$. Let N be a positive integer. The equality $R_{A,2}^{(2)}(n) = R_{\mathbb{N} \setminus A, 2}^{(2)}(n)$ holds for $n \geq 2N - 1$ if and only if $|A \cap [0, 2N - 1]| = N$ and $2m \in A \Leftrightarrow m \in A, 2m + 1 \in A \Leftrightarrow m \notin A$ for $m \geq N$.*

Theorem (LEV, SÁNDOR, 2004). *Let $X = \mathbb{N}$. Let N be a positive integer. The equality $R_{A,2}^{(3)}(n) = R_{\mathbb{N} \setminus A, 2}^{(3)}(n)$ holds for $n \geq 2N - 1$ if and only if $|A \cap [0, 2N - 1]| = N$ and $2m \in A \Leftrightarrow m \notin A, 2m + 1 \in A \Leftrightarrow m \in A$ for $m \geq N$.*

TANG [4] gave an elementary proofs of LEV and SÁNDOR's results. In [7], [8], [9], [10]. CHEN and YANG studied related problems about weighted representation functions. Similar statement to the above theorems can not be formulated for the representation function $R_{A,2}^{(1)}(n)$ because $R_{A,2}^{(1)}(n)$ is odd if and only if $\frac{n}{2} \in A$, therefore either $R_{A,2}^{(1)}(2m)$ or $R_{\mathbb{N} \setminus A, 2}^{(1)}(2m)$ is odd. A nontrivial result is the following in this direction.

Theorem 1. *Let $X = \mathbb{N}$. The equality $R_{A+B}^{(1)}(n) = R_{\mathbb{N} \setminus A + \mathbb{N} \setminus B}^{(1)}(n)$ holds from a certain point on if and only if $|\mathbb{N} \setminus (A \cup B)| = |A \cap B| < \infty$.*

The modular questions were solved by CHEN and YANG [3].

Theorem (CHEN, YANG, 2012). *Let $X = \mathbb{Z}_m$. The equality $R_{A,2}^{(1)}(n) = R_{\mathbb{Z}_m \setminus A,2}^{(1)}(n)$ holds for all $n \in \mathbb{Z}_m$ if and only if m is even and $|A| = m/2$.*

Theorem (CHEN, YANG, 2012). *Let $X = \mathbb{Z}_m$. For $i \in \{2, 3\}$, the equality $R_A^{(i)}(n) = R_{\mathbb{Z}_m \setminus A}^{(i)}(n)$ holds for all $n \in \mathbb{Z}_m$ if and only if m is even and $t \in A \Leftrightarrow t + m/2 \notin A$ for $t = 0, 1, \dots, m/2 - 1$.*

We extend the first theorem to arbitrary finite group G and the second theorem to finite Abelian group.

Theorem 2. *Let $X = G$ be a finite group. Then*

- (i) *If there exists a $g \in G$ for which the equality $R_{A+B}(g) = R_{G \setminus A + G \setminus B}(g)$ holds, then $|A| + |B| = |G|$.*
- (ii) *If $|A| + |B| = |G|$, then the equality $R_{A+B}(g) = R_{G \setminus A + G \setminus B}(g)$ holds for all $g \in G$.*

We generalize Chen and Yang's theorems in the following way.

Theorem 3. *Let $X = G$ be a finite group and $h \geq 2$ a fixed integer.*

- (i) *If the equality $R_{A,h}^{(1)}(g) = R_{G \setminus A,h}^{(1)}(g)$ holds for all $g \in G$, then $|G|$ is even and $|A| = |G|/2$.*
- (ii) *If h is even and $|A| = |G|/2$ then $R_{A,h}^{(1)}(g) = R_{G \setminus A,h}^{(1)}(g)$ holds for all $g \in G$.*

The case when h is odd is still open.

Problem. *Let $h > 1$ be a fixed odd positive integer. Let G be an Abelian group and $A \subset G$ be a nonempty subset. Does there exist a $g \in G$ such that $R_{A,h}^{(1)}(g) \neq R_{G \setminus A,h}^{(1)}(g)$?*

When h is odd we can only prove the following weaker result.

Theorem 4. *Let $X = \mathbb{Z}_m$ and $h > 2$ be a fixed odd integer. If $A \subset \mathbb{Z}_m$ such that $|A| = m/2$ then there exists a $g \in \mathbb{Z}_m$ such that $R_{A,h}^{(1)}(g) \neq R_{\mathbb{Z}_m \setminus A,h}^{(1)}(g)$.*

It would be interesting to characterize all that partitions of a finite Abelian group G such that $R_{A_i,h}^{(1)}(g) = R_{A_j,h}^{(1)}(g)$ for every $g \in G$.

Problem. *Let G be an Abelian group and $h \geq 2$. Characterize all the partitions of G into pairwise disjoint sets $A_1, A_2, \dots, A_h$ such that for every $g \in G$ and for every $1 \leq i, j \leq h$, $R_{A_i,h}^{(1)}(g) = R_{A_j,h}^{(1)}(g)$.*

For the two other representation functions we have the following result.

Theorem 5. *Let $X = G$ be a finite Abelian group. For $i \in \{2, 3\}$ the equality $R_A^{(i)}(g) = R_{G \setminus A, 2}^{(i)}(g)$ holds for every $g \in G$, if and only if $D_A(g) = D_{G \setminus A}(g)$ for every $g \in G$.*

2. Proofs

PROOF OF THEOREM 1. Let $A(x) = \sum_{a \in A} x^a$ be the generating function of the set A and let $B(x) = \sum_{b \in B} x^b$ be the generating function of the set B . It is easy to see that

$$A(x)B(x) = \left(\sum_{a \in A} x^a \right) \left(\sum_{b \in B} x^b \right) = \sum_{n=0}^{\infty} R_{A+B}(n)x^n.$$

Since

$$\frac{1}{1-x} - A(x) = \sum_{n=0}^{\infty} x^n - \sum_{a \in A} x^a = \sum_{a \in \mathbb{N} \setminus A} x^a,$$

and similarly

$$\frac{1}{1-x} - B(x) = \sum_{n=0}^{\infty} x^n - \sum_{b \in B} x^b = \sum_{b \in \mathbb{N} \setminus B} x^b,$$

it follows that

$$\left(\frac{1}{1-x} - A(x) \right) \left(\frac{1}{1-x} - B(x) \right) = \sum_{n=0}^{\infty} R_{\mathbb{N} \setminus A + \mathbb{N} \setminus B}(n)x^n.$$

Hence the condition $R_{A+B}(n) = R_{\mathbb{N} \setminus A + \mathbb{N} \setminus B}(n)$ holds from a certain point on is equivalent to

$$A(x)B(x) - \left(\frac{1}{1-x} - A(x) \right) \left(\frac{1}{1-x} - B(x) \right) = p(x),$$

where $p(x)$ is a polynomial with integral coefficients. This is equivalent to

$$A(x) + B(x) = \frac{1}{1-x} + p(x)(1-x). \quad (3)$$

Let

$$\frac{1}{1-x} + p(x)(1-x) = \sum_{n=0}^{\infty} x^n + \sum_{n=0}^N d_n x^n = \sum_{n=0}^{\infty} c_n x^n,$$

where $c_n = 0, 1$ or 2 . As $\sum_{n=0}^N d_n = 0$, it follows that the equation (3) holds if and only if $c_n = 1$ except for finitely many integer n and the number of n for which $c_n = 0$ is equal to the number of n for which $c_n = 2$. This is equivalent to the condition $|\mathbb{N} \setminus (A \cup B)| = |A \cap B| < \infty$. $\square$

PROOF OF THEOREM 2. For a given $S \subset G$ we denote by χ_S its characteristic function, that is $\chi_S(g) = 1$ if $g \in S$ and $\chi_S(g) = 0$ if $g \notin S$ for every $g \in G$. It follows that $\chi_{G \setminus S}(g) = 1$ if $g \in G \setminus S$, i.e., $g \notin S$ and $\chi_{G \setminus S}(g) = 0$ if $g \in S$, i.e., $g \in S$. Thus we have

$$\chi_{G \setminus S} = 1 - \chi_S. \quad (4)$$

It is easy to see that

$$\begin{aligned} R_{A+B}(g) &= \sum_{\substack{c+d=g \\ c,d \in G}} \chi_A(c) \chi_B(d) = \sum_{c \in G} \chi_A(c) \chi_B(-c+g), \\ R_{G \setminus A + G \setminus B}(g) &= \sum_{\substack{c+d=g \\ c,d \in G}} \chi_{G \setminus A}(c) \chi_{G \setminus B}(d) = \sum_{c \in G} \chi_{G \setminus A}(c) \chi_{G \setminus B}(-c+g). \end{aligned}$$

It follows from (4) that for every $g \in G$ we have

$$\begin{aligned} R_{A+B}(g) - R_{G \setminus A + G \setminus B}(g) &= \sum_{c \in G} \chi_A(c) \chi_B(-c+g) - \sum_{c \in G} \chi_{G \setminus A}(c) \chi_{G \setminus B}(-c+g) \\ &= \sum_{c \in G} \chi_A(c) \chi_B(-c+g) - \sum_{c \in G} (1 - \chi_A(c))(1 - \chi_B(-c+g)) \\ &= \sum_{c \in G} \chi_A(c) + \sum_{c \in G} \chi_B(-c+g) - \sum_{c \in G} 1 = |A| + |B| - |G|. \end{aligned}$$

Hence if there exists a $g \in G$ such that $R_{A+B}(g) = R_{G \setminus A + G \setminus B}(g)$ then $|A| + |B| = |G|$ and if $|A| + |B| = |G|$ then $R_{A+B}(g) = R_{G \setminus A + G \setminus B}(g)$ for every $g \in G$. $\square$

PROOF OF THEOREM 3.

$$R_{A,h}^{(1)}(g) = R_{G \setminus A, h}^{(1)}(g) \quad (5)$$

holds for all $g \in G$. It is clear that

$$\sum_{g \in G} R_{A,h}^{(1)}(g) = |A|^h,$$

and

$$\sum_{g \in G} R_{G \setminus A, h}^{(1)}(g) = |G \setminus A|^h.$$

It follows from (5) that $|A|^h = |G \setminus A|^h$ which implies $|A| = |G|/2$, which proves (i).

In the next step we prove (ii). Assume that h is even. For $h = 2$ the statement follows immediately from Theorem 2. Let $h > 2$. Then we have

$$\begin{aligned}
 R_{A,h}^{(1)}(g) &= |\{(a_1, a_2, \dots, a_h) : a_i \in A, a_1 + \dots + a_h = g\}| \\
 &= \sum_{\substack{g_1, \dots, g_{h/2} \in G \\ g_1 + \dots + g_{h/2} = g}} |\{(a_1, a_2, \dots, a_h) : a_i \in A, a_1 + a_2 = g_1, \dots \\
 &\quad \dots, a_{h-1} + a_h = g_{h/2}, a_1 + \dots + a_h = g\}| \\
 &= \sum_{\substack{g_1, \dots, g_{h/2} \in G \\ g_1 + \dots + g_{h/2} = g}} |\{(a_1, a_2) \in A : a_1 + a_2 = g_1\}| \cdots |\{(a_{h-1}, a_h) \in A : a_{h-1} + a_h = g_{h/2}\}| \\
 &= \sum_{\substack{g_1, \dots, g_{h/2} \in G \\ g_1 + \dots + g_{h/2} = g}} R_{A,2}^{(1)}(g_1) \cdots R_{A,2}^{(1)}(g_{h/2}) = \sum_{\substack{g_1, \dots, g_{h/2} \in G \\ g_1 + \dots + g_{h/2} = g}} R_{G \setminus A, 2}^{(1)}(g_1) \cdots R_{G \setminus A, 2}^{(1)}(g_{h/2}) \\
 &= \sum_{\substack{g_1, \dots, g_{h/2} \in G \\ g_1 + \dots + g_{h/2} = g}} |\{(a_1, a_2) \in G \setminus A : a_1 + a_2 = g_1\}| \cdots \\
 &\quad \cdots |\{(a_{h-1}, a_h) \in G \setminus A : a_{h-1} + a_h = g_{h/2}\}| \\
 &= \sum_{\substack{g_1, \dots, g_{h/2} \in G \\ g_1 + \dots + g_{h/2} = g}} |\{(a_1, a_2, \dots, a_h) : a_i \in G \setminus A, a_1 + a_2 = g_1, \dots \\
 &\quad \dots, a_{h-1} + a_h = g_{h/2}, a_1 + \dots + a_h = g\}| \\
 &= |\{(a_1, a_2, \dots, a_h) : a_i \in G \setminus A, a_1 + \dots + a_h = g\}| = R_{G \setminus A, h}^{(1)}(g). \quad \square
 \end{aligned}$$

PROOF OF THEOREM 4. We prove by contradiction. Assume that for every $g \in \mathbb{Z}_m$, $R_{A,h}^{(1)}(g) = R_{\mathbb{Z}_m \setminus A, h}^{(1)}(g)$. It is easy to see that

$$\begin{aligned}
 R_{A,h}^{(1)}(g) &= |\{(i_1, i_2, \dots, i_h) : a_{i_1} + \dots + a_{i_h} \equiv g \pmod{m}\}| \\
 &= \sum_{j=0}^{h-1} |\{(i_1, i_2, \dots, i_h) : a_{i_1} + \dots + a_{i_h} = g + jm\}|,
 \end{aligned}$$

thus we have

$$\sum_{g=0}^{m-1} R_{A,h}^{(1)}(g) z^g = \sum_{g=0}^{m-1} \left(\sum_{j=0}^{h-1} |\{(i_1, i_2, \dots, i_h) : a_{i_1} + \dots + a_{i_h} = g + jm\}| \right) z^g.$$

It is clear that

$$A^h(z) = \sum_{j=0}^{h-1} \sum_{g=0}^{m-1} |\{(i_1, i_2, \dots, i_h) : a_{i_1} + \dots + a_{i_h} = g + jm\}| z^{g+jm}$$

$$= \sum_{g=0}^{m-1} \sum_{j=0}^{h-1} |\{(i_1, i_2, \dots, i_h) : a_{i_1} + \dots + a_{i_h} = g + jm\}| z^{g+jm}.$$

It follows that there exist $p_1(z)$ and $p_2(z)$ polynomials with integral coefficients such that

$$A^h(z) = (1 - z^m)p_1(z) + \sum_{g=0}^{m-1} R_{A,h}^{(1)}(g)z^g,$$

and similarly

$$(1 + \dots + z^{m-1} - A(z))^h = (1 - z^m)p_2(z) + \sum_{g=0}^{m-1} R_{\mathbb{Z}_m \setminus A, h}^{(1)}(g)z^g.$$

Thus we have

$$(A(z))^h - \left(\frac{1-z^m}{1-z} - A(z) \right)^h = (1-z^m)(p_1(z) - p_2(z)) + \sum_{g=0}^{m-1} (R_{A,h}^{(1)}(g) - R_{\mathbb{Z}_m \setminus A, h}^{(1)}(g))z^g.$$

As $R_{A,h}^{(1)}(g) = R_{\mathbb{Z}_m \setminus A, h}^{(1)}(g)$ for every $g \in \mathbb{Z}_m$, we have

$$1 - z^m | (A(z))^h - \left(\frac{1-z^m}{1-z} - A(z) \right)^h |.$$

Since $\frac{1-z^m}{1-z} | 1 - z^m$ and h is odd, we have

$$\frac{1-z^m}{1-z} | 2(A(z))^h |.$$

As the polynomial $\frac{1-z^m}{1-z} = 1 + \dots + z^{m-1}$ has no multiple roots, we have

$$\frac{1-z^m}{1-z} | A(z),$$

which is a contradiction. $\square$

PROOF OF THEOREM 5. We only prove the case $i = 2$, because the proof of case $i = 3$ is very similar. For every fixed $g \in G$ we have

$$\begin{aligned} |A| &= |\{(a, y) : a \in A, y \in G, a + y = g\}| \\ &= |\{(a, y) : a \in A, y \in A, a + y = g\}| + |\{(a, y) : a \in A, y \in G \setminus A, a + y = g\}| \\ &= R_{A+A}(g) + R_{A+G \setminus A}(g) = 2R_{A,2}^{(2)}(g) - D_A(g) + R_{A+G \setminus A}(g), \end{aligned}$$

thus

$$R_{A,2}^{(2)}(g) = \frac{1}{2}|A| + \frac{1}{2}D_A(g) - \frac{1}{2}R_{A+G\setminus A}(g).$$

Replace A by $G \setminus A$ and using also the fact that $R_{A+G\setminus A}(g) = R_{G\setminus A+G\setminus(G\setminus A)}(g)$ we get

$$R_{G\setminus A,2}^{(2)}(g) = \frac{1}{2}|G \setminus A| + \frac{1}{2}D_{G\setminus A}(g) - \frac{1}{2}R_{A+G\setminus A}(g).$$

Hence for every $g \in G$ we have

$$\begin{aligned} R_{A,2}^{(2)}(g) - R_{G\setminus A,2}^{(2)}(g) &= \frac{1}{2}|A| + \frac{1}{2}D_A(g) - \left(\frac{1}{2}|G \setminus A| + \frac{1}{2}D_{G\setminus A}(g) \right) \\ &= |A| - \frac{1}{2}|G| + \frac{1}{2}(D_A(g) - D_{G\setminus A}(g)). \end{aligned}$$

Suppose that $R_{A,2}^{(2)}(g) = R_{G\setminus A,2}^{(2)}(g)$ for every $g \in G$. Then we have

$$\binom{|A|+1}{2} = \sum_{g \in G} R_{A,2}^{(2)}(g) = \sum_{g \in G} R_{G\setminus A,2}^{(2)}(g) = \binom{|G \setminus A|+1}{2},$$

therefore $|A| = |G \setminus A|$, that is $|A| = |G| - |A|$. Hence we get that $D_A(g) = D_{G\setminus A}(g)$ for every $g \in G$.

Finally, suppose that $D_A(g) = D_{G\setminus A}(g)$ for every $g \in G$. Then we have

$$\begin{aligned} |A| &= |\cup_{g \in G} \{a : a \in A, a + a = g\}| = \sum_{g \in G} |\{a : a \in A, a + a = g\}| \\ &= \sum_{g \in G} D_A(g) = \sum_{g \in G} D_{G\setminus A}(g) = |G \setminus A|, \end{aligned}$$

therefore $|A| = |G| - |A|$, thus we get that $R_{A,2}^{(2)}(g) = R_{G\setminus A,2}^{(2)}(g)$ for every $g \in G$, which completes the proof. $\square$

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