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Corrigendum to "Chern connection of a pseudo-Finsler metric as a family of affine connections"

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Abstract. In this note we give the correct statements of [2, Proposition 3.3 and Theorem 3.4] and a formula of the Chern curvature in terms of the curvature tensor R^V of the affine connection ∇^V and the Chern tensor P.

1. Curvature of two parameter maps

Throughout this note, we will use the same notation and conventions as in [2] with a small exception: in local calculation $(\Omega, (u^i)_{i=1}^n)$ denotes a chart on the base manifold M, and $(\pi^{-1}(\Omega), (x^i)_{i=1}^n, (y^i)_{i=1}^n)$ is the induced chart on TM. Then $x^i = u^i \circ \pi$, $y^i(v) = v(u^i)$ for every $v \in T_pM$, $p \in \Omega$. We agree to abbreviate composite mappings like $f \circ g$ as $f(g)$. Vector fields on Ω can naturally be regarded as local sections of the pull-back bundle $\pi^*(TM)$; we use such harmless identifications all the time.

Proposition 3.3 and Theorem 3.4 in [2] are not correct. In particular, the problem in Proposition 3.3 is that $R^V(V,U)W$ depends also on $D_\gamma^{\dot\gamma} U(\gamma(a))$, so it is not independent of the extension of u. The corrected versions of such results

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are given in Theorems 2.2 and 1.1, respectively. In order to formulate the correct results we need to associate a curvature operator to every two-parameter map.

Let us begin with some definitions. Given a pseudo-Finsler manifold (M, L) and an L-admissible vector field V on $\Omega \subset M$, we define a $(1,3)$ tensor P_V by

$$P_V(X,Y,Z) = \frac{\partial}{\partial t}\left(\nabla_X^{V+tZ} Y\right)|_{t=0},$$

where X, Y, Z are arbitrary smooth vector fields on Ω. Let us observe that the tensor P_V is symmetric in its first two arguments because ∇^V is torsion-free. Moreover, in coordinates P_V is given by

$$P_V(X,Y,Z) = X^j Y^k Z^l P^i_{\ jkl}(V)\frac{\partial}{\partial u^k},$$

where $P^i_{\ jkl}(V) = \frac{\partial \Gamma^i_{\ jk}}{\partial y^l}(V)$. In particular, it is clear that the value of $P_V(X,Y,Z)$ at a point $p \in \Omega$ depends only on $V(p)$ and not on the extension V used to compute it. From the homogeneity of ∇^V, namely, from the property $\nabla^{\lambda V} = \nabla^V$ for every $\lambda > 0$, it follows easily that

$$P_V(X,Y,V) = 0. \tag{1}$$

In [4, equation (7.23)], this tensor is called the *Chern curvature*. To avoid confusion with the Chern curvature in [2, equation (15)] we will refer to it as *Chern tensor*. Observe that there is a misprint in [2, equation (15)]. The right formula for Chern curvature is

$$R_v(V(q),U(q))W(q) = V^k U^l W^j(q) R_j{}^i{}_{kl}(v)\frac{\partial}{\partial u^i}(q), \quad q := \pi(v).$$

Let us also define the curvature of any L-admissible two parameter map

$$\Lambda : [a,b] \times (-\varepsilon,\varepsilon) \to M, \quad (t,s) \to \Lambda(t,s).$$

(Here 'L-admissible' means that $\Lambda_t(t,s) \in A$ for every $(t,s) \in [a,b] \times (-\varepsilon,\varepsilon)$). To avoid problems with differentiability, we assume that this map can be extended to a smooth map defined in an open subset $(\bar{a},\bar{b}) \times (-\varepsilon,\varepsilon) \subset \mathbb{R}^2$, with $[a,b] \subset (\bar{a},\bar{b})$. Then for every smooth vector field W along Λ we define the curvature operator

$$R^\Lambda(W) := D^{\Lambda_t}_{\gamma_s} D^{\Lambda_t}_{\beta_t} W - D^{\Lambda_t}_{\beta_t} D^{\Lambda_t}_{\gamma_s} W,$$

(see the notation in [2, Section 3.1]). In the following theorem we will relate the curvature of a two parameter map with the Chern curvature and the tensor P_V. Then we obtain the correct version of [2, Theorem 3.4].

Theorem 1.1. *Let (M,L) be a pseudo-Finsler manifold. Consider an L-admissible smooth curve $\gamma:[a,b]\to M$, an L-admissible two parameter map $\Lambda:[a,b]\times(-\varepsilon,\varepsilon)\to M$ such that $\Lambda(\cdot,0)=\gamma$ and a smooth vector field W along Λ. With the above notation,*

$$R^{\Lambda}(W)=R_{\dot\gamma}(\dot\gamma,U)W+P_{\dot\gamma}(U,W,D^{\dot\gamma}_{\gamma}\dot\gamma)-P_{\dot\gamma}(\dot\gamma,W,D^{\dot\gamma}_{\gamma}U),\tag{2}$$

where U is the variational vector field of Λ along γ, namely, $U(t)=\Lambda_s(t,0)$.

Proof. Observe that using the formula for $D^{\Lambda_t}_{\gamma_s}D^{\Lambda_t}_{\beta_t}W-D^{\Lambda_t}_{\beta_t}D^{\Lambda_t}_{\gamma_s}W$ in the proof of [2, Proposition 3.3] and formulas (13) and (14) in [2], we get

$$R^{\Lambda}(W)=\Bigg[W^iU^j\dot\gamma^p\frac{\partial\Gamma^k_{\ ij}}{\partial x^p}(\dot\gamma)+W^iU^j\Lambda^p_{tt}\frac{\partial\Gamma^k_{\ ij}}{\partial y^p}(\dot\gamma)-W^i\dot\gamma^jU^p\frac{\partial\Gamma^k_{\ ij}}{\partial x^p}(\dot\gamma)$$
$$-W^i\dot\gamma^j\Lambda^p_{ts}\frac{\partial\Gamma^k_{\ ij}}{\partial y^p}(\dot\gamma)+W^iU^j\dot\gamma^m\left(\Gamma^l_{\ ij}(\dot\gamma)\Gamma^k_{\ lm}(\dot\gamma)-\Gamma^l_{\ im}(\dot\gamma)\Gamma^k_{\ lj}(\dot\gamma)\right)\Bigg]\frac{\partial}{\partial u^k}(\gamma).$$

Since

$$\Lambda^p_{tt}=(D^{\dot\gamma}_{\gamma}\dot\gamma)^p-\dot\gamma^i\dot\gamma^j\Gamma^p_{\ ij}(\dot\gamma),\qquad \Lambda^p_{ts}=(D^{\dot\gamma}_{\gamma}U)^p-\dot\gamma^iU^j\Gamma^p_{\ ij}(\dot\gamma),$$

we find that

$$\begin{aligned}R^{\Lambda}(W)=\Bigg[&W^iU^j\dot\gamma^p\frac{\partial\Gamma^k_{\ ij}}{\partial x^p}(\dot\gamma)-W^iU^j\dot\gamma^m\dot\gamma^n\Gamma^p_{\ mn}(\dot\gamma)\frac{\partial\Gamma^k_{\ ij}}{\partial y^p}(\dot\gamma)\\&-W^i\dot\gamma^jU^p\frac{\partial\Gamma^k_{\ ij}}{\partial x^p}(\dot\gamma)+W^i\dot\gamma^j\dot\gamma^mU^n\Gamma^p_{\ mn}(\dot\gamma)\frac{\partial\Gamma^k_{\ ij}}{\partial x^p}(\dot\gamma)\\&+W^iU^j(D^{\dot\gamma}_{\gamma}\dot\gamma)^p\frac{\partial\Gamma^k_{\ ij}}{\partial y^p}-W^i\dot\gamma^j(D^{\dot\gamma}_{\gamma}U)^p\frac{\partial\Gamma^k_{\ ij}}{\partial y^p}(\dot\gamma)\\&+W^iU^j\dot\gamma^m\left(\Gamma^l_{\ ij}(\dot\gamma)\Gamma^k_{\ lm}(\dot\gamma)-\Gamma^l_{\ im}(\dot\gamma)\Gamma^k_{\ lj}(\dot\gamma)\right)\Bigg]\frac{\partial}{\partial u^k}(\gamma).\end{aligned}\tag{3}$$

Finally, (7) follows easily from the last relation and definitions taking into account that $\dot\gamma^i\Gamma^k_{\ ij}(\dot\gamma)=N^k_{\ j}(\dot\gamma)$. □

As a consequence of Theorem 1.1, we can define

$$R^{\gamma}(\dot\gamma,U)W:=R^{\Lambda}(\tilde W)$$

for any vector fields U and W along γ, where Λ is any L-admissible two parameter map such that $\Lambda(\cdot,0)=\gamma$ and $\Lambda_s(t,0)=U(t)$, and $\tilde W$ is any extension of W to Λ. The last theorem ensures that R^{γ} does not depend on the choice of two parameter map, neither on the extension of W.

Lemma 1.2. *Given a pseudo-Finsler manifold (M, L) and its Chern tensor P, we have that $P_v(v, v, u) = 0$ for any $v \in A$ and $u \in T_{\pi(v)}M$.*

PROOF. It is enough to show that $y^i y^j \frac{\partial \Gamma^k{}_{ij}}{\partial y^p} = 0$ for any $k = 1, \dots, n$. Using that $y^i \Gamma^k{}_{ij} = N^k_j$ we get

$$y^i \frac{\partial \Gamma^k{}_{ij}}{\partial y^l} = \frac{\partial N^k_j}{\partial y^l} - \Gamma^k{}_{lj}. \tag{4}$$

Since the functions $N^i{}_j(v)$ are positive homogeneous of degree one, we have

$$y^l \frac{\partial N^k_j}{\partial y^l} = N^k_j. \tag{5}$$

Using (4) and $y^j \Gamma^k{}_{lj} = N^k_l$, it follows that

$$y^j y^i \frac{\partial \Gamma^k{}_{ij}}{\partial y^l} = y^j \frac{\partial N^k_j}{\partial y^l} - y^j \Gamma^k{}_{lj} = y^j \frac{\partial N^k_j}{\partial y^l} - N^k_l. \tag{6}$$

Introduce the spray coefficients $G^i = \gamma^i{}_{jk} y^j y^k$ and observe that $\frac{1}{2} \frac{\partial G^i}{\partial y^j} = N^i{}_j$ (see [1, equation (3.8.2)]). Using the last relation and (5), we get

$$y^j \frac{\partial N^k_j}{\partial y^l} = y^j \frac{1}{2} \frac{\partial^2 G^k}{\partial y^l \partial y^j} = y^j \frac{\partial N^k_l}{\partial y^j} = N^k_l.$$

Substituting this in (6) we finally conclude that $y^j y^i \frac{\partial \Gamma^k{}_{ij}}{\partial y^l} = 0$. □

Corollary 1.3. *For any L-admissible curve $\gamma : [a, b] \to M$ we have*

$$R^\gamma(\dot\gamma, U)\dot\gamma = R_{\dot\gamma(a)}(\dot\gamma(a), U)\dot\gamma + P_{\dot\gamma}(U, \dot\gamma, D^{\dot\gamma}_\gamma \dot\gamma). \tag{7}$$

Therefore, the value of $R^\gamma(\dot\gamma, U)\dot\gamma$ at $t = s_0 \in [a, b]$ depends only on $U(s_0)$ and γ, and not on the particular extension of U used to compute it.

PROOF. A straightforward consequence from (2) and Lemma 1.2. □

2. Relation with the curvature tensor R^V

Let us see how the curvature tensor R^V relates to the Chern curvature R_V and the Chern tensor P_V.

Theorem 2.1. *Let (M, L) be a pseudo-Finsler manifold, V an L-admissible vector field on an open subset $\Omega \subset M$, and let X, Y, Z be arbitrary smooth vector fields on Ω. Then*

$$R^V(X,Y)Z = R_V(X,Y)Z + P_V(Y,Z,\nabla^V_X V) - P_V(X,Z,\nabla^V_Y V). \tag{8}$$

Proof. By definition, using the same notation as in the proof of [2, Proposition 3.3], over Ω we have

$$R^V(X,Y)Z = \Bigg[Z^i Y^j X^p \frac{\partial \tilde{\Gamma}^k_{\ ij}}{\partial u^p} - Z^i X^j Y^p \frac{\partial \tilde{\Gamma}^k_{\ ij}}{\partial u^p} + Z^i Y^j X^m \left(\tilde{\Gamma}^l_{\ ij} \tilde{\Gamma}^k_{\ lm} - \tilde{\Gamma}^l_{\ im} \tilde{\Gamma}^k_{\ lj} \right) \Bigg] \frac{\partial}{\partial u^k}. \tag{9}$$

Now observe that $\frac{\partial \tilde{\Gamma}^k_{\ ij}}{\partial u^p} = \frac{\partial \Gamma^k_{\ ij}}{\partial x^p}(V) + \frac{\partial V^l}{\partial u^p}\frac{\partial \Gamma^k_{\ ij}}{\partial y^l}(V)$ and $(\nabla^V_X V)^k = X^p \frac{\partial V^k}{\partial u^p} + X^p V^l \Gamma^k_{\ pl}(V)$. Using this, we conclude that

$$\begin{aligned} X^p \frac{\partial \tilde{\Gamma}^k_{\ ij}}{\partial u^p} &= X^p \frac{\partial \Gamma^k_{\ ij}}{\partial x^p}(V) + X^p \frac{\partial V^l}{\partial u^p}\frac{\partial \Gamma^k_{\ ij}}{\partial y^l}(V) \\ &= X^p \frac{\partial \Gamma^k_{\ ij}}{\partial x^p}(V) + (\nabla^V_X V)^l \frac{\partial \Gamma^k_{\ ij}}{\partial y^l}(V) - X^m V^n \Gamma^l_{\ mn}(V) \frac{\partial \Gamma^k_{\ ij}}{\partial y^l}(V). \end{aligned}$$

Taking into account that $V^n \Gamma^l_{\ mn}(V) = N^l_{\ m}(V)$, we get

$$X^p \frac{\partial \tilde{\Gamma}^k_{\ ij}}{\partial u^p} = X^p \frac{\partial \Gamma^k_{\ ij}}{\partial x^p}(V) + (\nabla^V_X V)^l \frac{\partial \Gamma^k_{\ ij}}{\partial y^l}(V) - X^m N^l_{\ m}(V) \frac{\partial \Gamma^k_{\ ij}}{\partial y^l}(V). \tag{10}$$

In the same way,

$$Y^p \frac{\partial \tilde{\Gamma}^k_{\ ij}}{\partial u^p} = Y^p \frac{\partial \Gamma^k_{\ ij}}{\partial x^p}(V) + (\nabla^V_Y V)^l \frac{\partial \Gamma^k_{\ ij}}{\partial y^l}(V) - Y^m N^l_{\ m}(V) \frac{\partial \Gamma^k_{\ ij}}{\partial y^l}(V). \tag{11}$$

Substituting (10) and (11) in (9), we obtain (8) in coordinates. □

Finally we can give the correct version of [2, Propositon 3.3].

Theorem 2.2. *Let $\gamma : [a,b] \to M$ be a smooth embedded L-admissible curve and V an L-admissible smooth vector field defined on an open subset $\Omega \subset M$. Assume that $\gamma([a,b]) \subset \Omega$ and V coincides with $\dot{\gamma}$ along γ. Then*

$$R^\gamma(\dot{\gamma}, U)W = \left(R^V(V, \tilde{U})\tilde{W} + P_V(V, \tilde{W}, [\tilde{U}, V]) \right)(\gamma), \tag{12}$$

where U and W are smooth vector fields along γ, and $\tilde{U}$, $\tilde{W}$ are extensions of U, W to Ω, resp.

Proof. This follows easily from (2) and (8) since ∇^V is torsion-free. $\square$

Remark 2.3. Observe that the expression in [2, Corollary 3.5] is valid. Indeed, more generally, for every $v \in A$ and $u, w \in T_{\pi(v)}M$, it holds that

$$K_v(u,w) = \frac{g_v((R^{\gamma_v}(\dot\gamma_v, U)W)(t_0), \dot\gamma_v(t_0))}{L(v)g_v(u,w) - g_v(v,u)g_v(v,w)}, \tag{13}$$

where γ_v is the geodesic such that $\dot\gamma_v(t_0) = v$ and U, W are arbitrary extensions of u, w along γ_v. Recall the $K_v(u,w)$, the predecessor of the flag curvature, is defined as

$$K_v(u,w) = \frac{g_v(R_v(v,u)w, v)}{L(v)g_v(u,w) - g_v(v,u)g_v(v,w)}.$$

To prove (13), we show that if γ is a geodesic, then

$$g_{\dot\gamma}(R^\gamma(\dot\gamma, U)W, \dot\gamma) = -g_{\dot\gamma}(R^\gamma(\dot\gamma, U)\dot\gamma, W) \tag{14}$$

where $g_{\dot\gamma}$ is given by the rule $g_{\dot\gamma}(X,Y) := g_{\dot\gamma(t)}(X(t), Y(t))$ for any two vector fields X, Y along γ. This holds trivially in the interior of the set where U is proportional to $\dot\gamma$, because in this case $[\tilde U, V]$ is proportional to V, and then applying (1) and the antisymmetry of R^V in its first two arguments to (12), we get $R^\gamma(\dot\gamma, U)W = R^\gamma(\dot\gamma, U)\dot\gamma = 0$. If $\dot\gamma$ and U are linearly independent, then we can choose extensions V and $\tilde U$ with $[\tilde U, V] = 0$, and (12) together with [2, Proposition 3.1] and [3, Lemma 3.10] conclude (14). By continuity we can extend (14) to the interval of definition of γ. As the right hand side of (14) does not depend on the extension U of u along γ, we can compute the left hand side assuming that $D^{\dot\gamma}_\gamma U = 0$, and using (2), we get (13).

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