

Characterization of finite simple group A_{p+3} by its order and degree pattern

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Abstract. It is proved that some finite groups are OD-characterizable, i.e. they are uniquely determined by order and degree pattern. In [R. KOGANI-MOGHADAM and A. R. MOGHADDAMFAR, Groups with the same order and degree pattern, *Science China Mathematics*, 2012], the authors posed the following conjecture:

Conjecture. *All alternating groups A_m with $m \neq 10$ are OD-characterizable.*

Up to now it has been proved that this conjecture is correct for $m = p, p+1, p+2$, where p is a prime number. Also it has been proved that the conjecture is true for A_{106} and A_{112} . In this paper, by an example we show that this conjecture is not true in general and so we reformulate this conjecture as follows:

Conjecture. *If $m \neq 10$ is even, then all alternating groups A_m are OD-characterizable.*

Recently, the OD-characterization of A_{p+3} , where $p \neq 7$ and $p < 100$ has been proved. In this paper we continue this work and we prove that if $p \neq 7$ is a prime number, then the alternating group A_{p+3} is OD-characterizable. We note that this is the first work that verify an infinite family of alternating groups with connected prime graphs.

1. Introduction

In this paper every group is finite. If n is a natural number, then we denote by $\pi(n)$, the set of all prime divisors of n . If G is a finite group, then $\pi(|G|)$ is denoted by $\pi(G)$. Also for a prime number p , n_p denotes the p -part of n , i.e. $n_p = p^k$ if $p^k \mid n$ but $p^{k+1} \nmid n$. If $p \in \pi(G)$, then the set of all Sylow p -subgroups

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of G is denoted by $\text{Syl}_p(G)$. The prime graph $GK(G)$ of a group G is defined as a graph with vertex set $\pi(G)$ in which two distinct primes $p, q \in \pi(G)$ are adjacent (and we write $p \sim q$) if G contains an element of order pq . A subset ρ of vertices of $GK(G)$ is called an independent subset of $GK(G)$, if there does not exist any edge between any two elements of ρ . The maximal number of vertices in the independent subsets of $GK(G)$ is denoted by $t(G)$ and we denote by $\rho(G)$ an independent subset of $GK(G)$ with $t(G)$ elements. Also the maximal number of vertices in the independent subsets of $GK(G)$ containing 2 is denoted by $t(2, G)$. If n is a natural number and p is a prime divisor of n , then we denote the p -part of n by n_p .

Definition 1.1 ([4]). Let G be a group and $\pi(G) = \{p_1, p_2, \dots, p_k\}$, where $p_1 < p_2 < \dots < p_k$. Then the degree pattern of G is defined as follows:

$$D(G) := (\deg(p_1), \deg(p_2), \dots, \deg(p_k)),$$

where $\deg(p_i)$, $1 \leq i \leq k$, is the degree of vertex p_i in $GK(G)$.

Definition 1.2. ([4]) Let G be a finite group. We say that G is k -fold OD-characterizable if there exist exactly k non-isomorphic groups $H_1, H_2, \dots, H_k$ such that $|H_i| = |G|$ and $D(H_i) = D(G)$, for $1 \leq i \leq k$. When $k = 1$, the finite group G is called OD-characterizable.

The k -fold OD-characterization of some finite groups are considered by some authors (see [3], [4], [9], [6], [10], [11]). If G is a finite group such that $D(G) = D(A_{10})$ and $|G| = |A_{10}|$, then $G \cong A_{10}$ or $G \cong J_2 \times \mathbb{Z}_3$ (see [11]). Therefore A_{10} is not OD-characterizable. In [4], the authors put the following conjecture:

Conjecture 1.3. *All the alternating groups A_m with $m \neq 10$ are OD-characterizable.*

Until now, the above conjecture is valid for $n = p, p + 1, p + 2$, where p is an odd prime number. We note that in these cases the prime graph of A_n is disconnected and p is an isolated vertex. In this paper, by an example, we show that the above conjecture is not true in general and so we reformulate this conjecture as follows:

Conjecture 1.4. *If $m \neq 10$ is even, then all alternating groups A_m are OD-characterizable.*

In [1], the OD-characterization of alternating group A_{p+3} is considered, where $p \neq 7$ and $p < 100$ is an odd prime number. In this paper, we continue their work and we prove the validity of Conjecture 1.4 for A_{p+3} , where p is an odd prime number. In fact we prove the following result:

Main Theorem 1.5. *If $p \neq 7$ is an odd prime, then A_{p+3} is OD-characterizable.*

We note that in this case the prime graph of A_{p+3} is a connected graph, unless $p + 2$ is a prime number. This paper is the first paper giving the OD-characterization of an infinite family of alternating groups A_n , where $GK(A_n)$ is connected.

In [5], the following problem about OD-characterization of finite simple groups is posed:

Problem 1.6. *Is there a simple group which is k -fold OD-characterizable for $k \geq 3$?*

Finally we introduce some finite simple groups where these groups are k -fold OD-characterizable, where $k \geq 3$, as an answer to the above problem.

2. Preliminary results

Lemma 2.1 ([8, Proposition 1.1]). *Let $G = A_n$ be an alternating group of degree n .*

- (1) *Let $r, s \in \pi(G)$ be odd primes. Then r, s are nonadjacent if and only if $r + s > n$.*
- (2) *Let $r \in \pi(G)$ be an odd prime. Then $2, r$ are nonadjacent if and only if $r + 4 > n$.*

Lemma 2.2 ([4, Lemma 2.15]). *Let S be a symmetric (resp., an alternating) group and let G be a finite group with $\pi(G) = \pi(S)$ and $D(G) = D(S)$. Then the prime graph $GK(G)$ coincides with $GK(S)$.*

Lemma 2.3. ([12]) *Let q and $s \geq 2$ be some natural numbers. Then one of the following holds:*

- (a) *there exists a prime p such that p divides $q^s - 1$ and p does not divide $q^t - 1$ for all natural numbers $t < s$;*
- (b) *$s = 6$ and $q = 2$;*
- (c) *$s = 2$ and $q = 2^t - 1$, for some t .*

A prime p satisfying condition (a) of Lemma 2.3, is said to be a primitive prime divisor of $q^s - 1$. Also a primitive prime divisor of $q^n - 1$ is denoted by $r_n(q)$ or briefly r_n .

Lemma 2.4 ([2, Lemma 2.12]). *Let n , s and q be positive integers and x be an odd prime. If q is odd, then $\left(\frac{q^{2n}-1}{q^2-1}\right)_2 = n_2$. If $x \mid (q^s - 1)$ and $s \mid n$, then $\left(\frac{q^n-1}{q^s-1}\right)_x$ divides $(n/s)_x$.*

Lemma 2.5. *Let $q > 1$ be an integer.*

- (i) *If $n \geq 5$ is a prime, then $3 \nmid (q^n - 1)/(q - 1)$ and $3 \nmid (q^n + 1)/(q + 1)$.*
- (ii) *If $n = 2^\beta$ for some $\beta > 0$, then $3 \nmid (q^n + 1)$.*

PROOF. (i) If $3 \mid q$, obviously we get the result. If $3 \mid (q - 1)$, then by Lemma 2.4, $((q^n - 1)/(q - 1))_3 = n_3 = 1$. If $3 \mid (q + 1)$, then $3 \nmid (q^n - 1)$, since n is odd. Also $(q^n + 1) \mid (q^{2n} - 1)$ and by the above discussion $3 \nmid (q^{2n} - 1)/(q^2 - 1)$.

(ii) Obviously the result holds if $3 \nmid q$. Otherwise, $q \equiv \pm 1 \pmod{3}$. Hence we have $q^n \equiv 1 \pmod{3}$ and so $3 \nmid (q^n + 1)$. $\square$

Lemma 2.6 ([8, Lemma 1.2]). *With the notations in [8], we have:*

- (1) *Every maximal torus T of $G = A_{n-1}^\varepsilon(q)$ ($\varepsilon \in \{+, -\}$) has the order*

$$\frac{1}{(n, q - \varepsilon 1)(q - \varepsilon 1)} (q^{n_1} - (\varepsilon 1)^{n_1})(q^{n_2} - (\varepsilon 1)^{n_2}) \dots (q^{n_k} - (\varepsilon 1)^{n_k})$$

for an appropriate partition $n_1 + n_2 + \dots + n_k = n$ of n . Moreover, for every partition, there exists a torus of corresponding order.

- (2) *Every maximal torus T of G , where $G = B_n(q)$ or $G = C_n(q)$, has the order*

$$\frac{1}{(2, q - 1)} (q^{n_1} - 1)(q^{n_2} - 1) \dots (q^{n_k} - 1)(q^{l_1} + 1)(q^{l_2} + 1) \dots (q^{l_m} + 1)$$

for an appropriate partition $n_1 + n_2 + \dots + n_k + l_1 + l_2 + \dots + l_m = n$ of n . Moreover, for every partition, there exists a torus of corresponding order.

- (3) *Every maximal torus T of $G = D_n^\varepsilon(q)$ has the order*

$$\frac{1}{(4, q^n - \varepsilon 1)} (q^{n_1} - 1)(q^{n_2} - 1) \dots (q^{n_k} - 1)(q^{l_1} + 1)(q^{l_2} + 1) \dots (q^{l_m} + 1)$$

for an appropriate partition $n_1 + n_2 + \dots + n_k + l_1 + l_2 + \dots + l_m = n$ of n , where m is even if $\varepsilon = +$ and m is odd if $\varepsilon = -$. Moreover, for every partition, there exists a torus of corresponding order.

3. Proof of the Main Theorem

First we prove some result about the OD-characterization of an alternating group with degree m , where m is a natural number greater than 106. Then we consider the special case $m = p + 3$, where p is an odd prime number. So we have the following lemma.

Lemma 3.1. *Let G be a finite group such that $|G| = |A_m|$ and $D(G) = D(A_m)$, where $m \geq 106$. Also let:*

$$T := \left\{ t \text{ is a prime number} \mid \frac{m}{2} < t \leq m \right\}.$$

Then the following assertions hold:

- (1) *T is an independent subset of $GK(G)$. If $t \in T$, then $|G|_t = t$ and also $|T| \geq 9$.*
- (2) *There exists a nonabelian simple group S such that:*

$$S \leq \bar{G} := \frac{G}{N} \leq \text{Aut}(S),$$

where N is the largest normal subgroup of G with $\pi(N) \cap T = \emptyset$. Also we have $T \subseteq \pi(S)$ and $\pi(\bar{G}/S) \cap T = \emptyset$.

- (3) *The simple group S is isomorphic to a classical simple group of Lie type or an alternating group.*

PROOF. (1) Since $|G| = |A_m|$ and $D(G) = D(A_m)$, by Lemma 2.2, we get that $GK(G) = GK(A_m)$. Also by the definition of T and Lemma 2.1, it is obvious that T is an independent subset of $GK(G)$. Since $|G| = m!/2$, if $t \in T$, then $|G|_t = t$.

Now since $m \geq 106$, by [7, Corollary 3], the number of prime numbers t such that $m/2 < t \leq m$, is bigger than $(3m/2)/(5 \ln(m/2))$. So by the definition of T , we get that

$$|T| > \frac{3m/2}{5 \ln(m/2)} \geq \frac{3(53)}{5 \ln(53)} > 8,$$

which implies that $|T| \geq 9$.

- (2) Let N be the largest normal subgroup of G such that $\pi(N) \cap T = \emptyset$. Also let S be a minimal normal subgroup of $\bar{G} := G/N$.

By the properties of N , we get that $\pi(S) \cap T \neq \emptyset$. Also we know that $S = S_1 \times \cdots \times S_k$, where S_i 's are isomorphic to a simple group. So since $\pi(S) \cap T \neq \emptyset$, we get that for each $1 \leq i \leq k$, $\pi(S_i) \cap T \neq \emptyset$. On the other hand by (1), we

conclude that if $t \in \pi(S) \cap T$, then $|S|_t = t$. It follows that $k = 1$ and so S is a simple group.

By (1), we have $|T| \geq 9$. Also by the above discussion, $\pi(S) \cap T \neq \emptyset$. We claim that $T \subseteq \pi(S)$. On the contrary, let $t_1 \in \pi(S) \cap T$ and $t_2 \in T \setminus \pi(S)$. By the Frattini argument, if $\bar{Q} \in \text{Syl}_{t_1}(S)$, then $\bar{G} = S N_{\bar{G}}(\bar{Q})$. Thus since $t_2 \in T \setminus \pi(S)$, we get that $t_2 \in \pi(N_{\bar{G}}(\bar{Q}))$. So we may assume that $\bar{G}$ contains a subgroup $\bar{H}$ such that $|\bar{H}| = t_2$ and $\bar{Q} \rtimes \bar{H}$ is a subgroup of $\bar{G}$. Hence since t_1 and t_2 are nonadjacent in $GK(G)$, we conclude that $\bar{Q} \rtimes \bar{H}$ is a Frobenius subgroup of $\bar{G}$. This implies that $|\bar{H}| \mid (|\bar{Q}| - 1)$ and so $t_2 \mid (t_1 - 1)$. On the other hand, we have $m/2 < t_2 < t_1 \leq m$, which implies that $t_2 \nmid (t_1 - 1)$. So we get a contradiction and so $T \subseteq \pi(S)$. So since $|T| \geq 9$, we get that S is a nonabelian simple group. Also since for every $t \in T$, $|G|_t = t$, we deduce that $\pi(\bar{G}/S) \cap T = \emptyset$.

Let $C_{\bar{G}}(S) \neq 1$. Similarly to the above, we get that $\pi(C_{\bar{G}}(S)) \cap T \neq \emptyset$. On the other hand S is a nonabelian simple group, which implies that $S \cap C_{\bar{G}}(S) = 1$. Also by the above argument, $\pi(\bar{G}/S) \cap T = \emptyset$, which contradicts to $\pi(C_{\bar{G}}(S)) \cap T \neq \emptyset$. Therefore $C_{\bar{G}}(S) = 1$ and so we have:

$$S \leq \bar{G} = \frac{G}{N} \leq \text{Aut}(S).$$

(3) We note that by the classification of finite simple groups, the nonabelian simple group S is isomorphic to a sporadic simple group, an exceptional simple group of Lie type, a classical simple group of Lie type or an alternating group.

Let S be isomorphic to a sporadic simple group. By (2), we know that $\pi(S)$ contains every prime number t such that $m/2 < t \leq m$, where $m \geq 106$. This implies that there is at least a prime number $t > 100$ such that $t \in \pi(S)$. So by the orders of the sporadic simple groups we get a contradiction.

Now let S be isomorphic to an exceptional simple group of Lie type. By (2), we have $t(S) \geq |T| \geq 9$. Therefore by Table 9 in [8], it follows that $S \cong E_8(q)$. Also by this table, we get that

$$\rho(S) = \{r_5, r_7, r_8, r_9, r_{10}, r_{12}, r_{14}, r_{15}, r_{18}, r_{20}, r_{24}, r_{30}\}.$$

By (2), there exist at least 9 prime numbers $t_1, t_2, \dots, t_9$ in $\pi(S)$ which are nonadjacent to each other in $GK(S)$ and $|S|_{t_i} = t_i$, for $1 \leq i \leq 9$. The order of $E_8(q)$ shows that if r is a prime divisor of $|E_8(q)|$ and $r \notin A := \{r_7, r_9, r_{14}, r_{15}, r_{18}, r_{20}, r_{24}, r_{30}\}$, then $r^2 \mid |E_8(q)|$. Now since $|A| = 8$, we get a contradiction.

Therefore S is isomorphic to a classical simple group of Lie type or an alternating group. $\square$

Now by the previous lemma, we consider the special case $m = p + 3$, where p is a prime number.

PROOF OF THE MAIN THEOREM. Let G be a finite group such that $|G| = |A_{p+3}| = (p+3)!/2$ and $D(G) = D(A_{p+3})$, where p is a prime number. We note that if $p < 100$ and $p \neq 7$, the conjecture is proved in [1]. Also if $p' = p + 2$ is a prime number, then $A_{p+3} = A_{p'+1}$, which is considered in [4]. Therefore we suppose that $p \geq 101$ is a prime number such that $p + 2$ is not a prime so $p \geq 103$. Let

$$T := \{t \text{ is a prime number} \mid (p+3)/2 < t \leq p+3\}.$$

By Lemma 3.1, there exists a nonabelian simple group S such that:

$$S \leq \bar{G} := \frac{G}{N} \leq \text{Aut}(S),$$

where N is the largest normal subgroup of G with $\pi(N) \cap T = \emptyset$. Also by Lemma 3.1, $T \subseteq \pi(S)$, which implies that $p \in \pi(S)$ and S is isomorphic to a classical simple group of Lie type or an alternating group.

In the sequel we consider each possibility for S , separately:

- (1) Let $S \cong A_{n-1}(q)$. By Lemma 3.2 and Table 8 in [8], $t(S) = [(n+1)/2] \geq 9$ and so $n \geq 17$. Also $p \approx 2$ in $GK(S)$. Then by Table 6 in [8], p is a primitive prime divisor of $q^n - 1$ or $q^{n-1} - 1$.
- (1.a) Let p be a primitive prime divisor of $q^n - 1$. By Lemma 2.6, S contains an abelian subgroup of order $(q^n - 1)/((q - 1)(n, q - 1))$. Therefore every prime divisor of $(q^n - 1)/((q - 1)(n, q - 1))$ is adjacent to p in $GK(G)$. On the other hand p is only adjacent to 3 in $GK(G)$. This implies that $\pi((q^n - 1)/((q - 1)(n, q - 1))) \subseteq \{p, 3\}$.

Suppose that n is not a prime number. Also let m be the largest divisor of n such that $m < n$. Since $n \geq 17$, easily we get that $m > 4$ and $m \neq 6$. Hence by Lemma 2.3, there exists a primitive prime divisor u of $q^m - 1$. Since $m < n$, we get that $u \neq p$. Also $m > 4$ implies that $u \neq 3$. So we get a contradiction since $\pi(q^m - 1) \subseteq \pi(q^n - 1)$. Therefore n is an odd prime number.

By Lemma 2.5, we know that $3 \nmid (q^n - 1)/((q - 1)(n, q - 1))$. Therefore we have $\pi((q^n - 1)/((q - 1)(n, q - 1))) = \{p\}$. Since $|G|_p = |S|_p = p$, we conclude that $p = (q^n - 1)/((q - 1)(n, q - 1))$.

- (1.b) Let p be a primitive prime divisor of $q^{n-1} - 1$. By Lemma 2.6, S contains an abelian subgroup of order $(q^{n-1} - 1)/(n, q - 1)(q - 1)$. Similarly to the

above, this shows that $\pi((q^{n-1} - 1)/(n, q - 1)(q - 1)) \subseteq \{p, 3\}$ which implies that $n - 1 \geq 16$ is an odd prime number and $p = (q^{n-1} - 1)/(n, q - 1)(q - 1)$.

Now let $t \in T \subseteq \pi(S)$. Then in each case, we get that $t > (p + 3)/2 > q^{n-4}$. On the other hand, by the order of S , we know that t is a primitive prime divisor of $q^m - 1$, where $1 \leq m \leq n$. Therefore $q^m \geq t > q^{n-4}$, which implies that t is a primitive prime divisor of $q^n - 1$, $q^{n-1} - 1$, $q^{n-2} - 1$ or $q^{n-3} - 1$. This shows that $|T| \leq 4$, which is a contradiction since $|T| \geq 9$.

- (2) Let $S \cong {}^2A_{n-1}(q)$. By Table 8 in [8], we have $t(S) = [(n + 1)/2] \geq 9$ and so $n \geq 17$. Also by Table 6 in [8], we get that p is a primitive prime divisor of $q^{2n} - 1$, $q^{2n-2} - 1$, $q^n - 1$, $q^{(n-1)/2} - 1$ or $q^{n/2} - 1$.
- (2.a) Let p be a primitive prime divisor of $q^{2n} - 1$. By Tables in [8], n is odd. Also by Lemma 2.6, S contains an abelian subgroup of order $(q^n + 1)/((q + 1)(n, q + 1))$. Similarly to the above, we get that n is an odd prime number and $p = (q^n + 1)/((q + 1)(n, q + 1))$.
- (2.b) Let p be a primitive prime divisor of $q^{2n-2} - 1$. By Tables 4 and 6 in [8], we get that n is an even number. In this case since S contains an abelian subgroup of order $(q^{n-1} + 1)/(n, q + 1)$, similarly to the above we conclude that $p = (q^{n-1} + 1)/(n, q + 1)$.
Now let $t \in T$. Then in Cases (2.a) and (2.b), we have $t > (p + 3)/2 > q^{n-4}$. Similarly to the above, we get that $|T| \leq 7$, which is a contradiction, since $|T| \geq 9$.
- (2.c) Let p be a primitive prime divisor of $q^n - 1$, $q^{n/2} - 1$ or $q^{(n-1)/2} - 1$. By Lemma 2.6, S contains an abelian subgroup of order $(q^n - 1)/((n, q + 1)(q + 1))$ if n is even and $(q^{n-1} - 1)/(n, q + 1)$ if n is odd. This implies that p is adjacent to some prime number different from 3 in $GK(S)$, which is a contradiction.
- (3) Let $S \cong B_n(q)$ or $S \cong C_n(q)$. By Table 8 in [8], we have $t(S) = [(3n + 5)/4] \geq 9$ and so $n \geq 11$. Also by Table 6 in [8], we get that p is a primitive prime divisor of $q^n - 1$ or $q^{2n} - 1$.
- (3.a) Let p be a primitive prime divisor of $q^n - 1$. Using the maximal torus of order $(q^n - 1)/(2, q - 1)(q - 1)$, we conclude that n is an odd prime number and $p = (q^n - 1)/((2, q - 1)(q - 1))$.
- (3.b) Let p be a primitive prime divisor of $q^{2n} - 1$. Using the maximal torus of order $(q^n + 1)/(2, q - 1)$, we get that n is an odd prime number or a power of 2.

Hence $p = (q^n + 1)/(2, q - 1)$ if $n = 2^\beta$ and $p = (q^n + 1)/((q + 1)(2, q - 1))$ if n is an odd prime, by Lemma 2.5.

Now let $t \in T$. By the above discussion, we get that $t > (p+3)/2 > q^{n-3}$. This implies that $|T| \leq 6$, which is a contradiction.

- (4) Let $S \cong D_n(q)$. By Table 8 in [8], we get that $(3n+3)/4 \geq t(S) \geq 9$ and so $n \geq 11$. Also since $p \approx 2$ in $GK(S)$, by Tables 4 and 6 in [8], we get that p is a primitive prime divisor of $q^n - 1$, $q^{n-1} - 1$ or $q^{2(n-1)} - 1$.
- (4.a) Let p be a primitive prime divisor of $q^n - 1$. Then by Tables 4 and 6 in [8], n is odd. Using the maximal torus of order $(q^n - 1)/(4, q^n - 1)(q - 1)$, we get that $n \geq 11$ is an odd prime number and $p = (q^n - 1)/(4, q^n - 1)(q - 1)$.
- (4.b) Let p be a primitive prime divisor of $q^{2(n-1)} - 1$. Using the maximal torus of order $(q^{n-1} + 1)(q + 1)/(4, q^n - 1)$, we conclude that $n - 1$ is an odd prime number or a power of 2.

Hence we get that $\pi(q^{n-1} + 1) \subseteq \{2, 3, p\}$. Now we consider the following cases:

- (i) Let q be a power of 2 and $n - 1$ is a power of 2. Then $\pi(q^{n-1} + 1) \subseteq \{3, p\}$ and so by Lemma 2.5, we conclude that $\pi(q^{n-1} + 1) \subseteq \{p\}$. Hence similarly to the above we have $p = q^{n-1} + 1$.
- (ii) Let q be a power of 2 and $n - 1$ is an odd prime number. Then $\pi(q^{n-1} + 1) \subseteq \{3, p\}$ and so by Lemma 2.5, we get that $\pi((q^{n-1} + 1)/(q + 1)) \subseteq \{p\}$. Hence similarly to the above we get that $p = (q^{n-1} + 1)/(q + 1)$.
- (iii) Let q be an odd number and $n - 1$ is a power of 2. Then by Lemma 2.5, we get that $\pi(q^{n-1} + 1) \subseteq \{2, p\}$. Since $n - 1$ is a power of 2, we have $4 \mid (q^{n-1} - 1)$ and so $(q^{n-1} + 1)_2 = 2$. Therefore $\pi((q^{n-1} + 1)/2) \subseteq \{p\}$ and so $p = (q^{n-1} + 1)/2$.
- (iv) Let q be an odd number and $n - 1$ is an odd prime number. Then by Lemma 2.4, we get that $2 \nmid (q^{n-1} + 1)/(q + 1)$. This implies that $\pi((q^{n-1} + 1)/(q + 1)) \subseteq \{3, p\}$ and so by Lemma 2.5, we conclude that $\pi((q^{n-1} + 1)/(q + 1)) \subseteq \{p\}$. Hence similarly to the above we get that $p = (q^{n-1} + 1)/(q + 1)$.

Therefore in each case we get that $p \geq (q^{n-1} + 1)/(q + 1)$.

- (4.c) Let p be a primitive prime divisor of $q^{n-1} - 1$. Using the maximal torus of order $(q^{n-1} - 1)(q - 1)/(4, q^n - 1)$, we conclude that $n - 1$ is an odd prime number and $p = (q^{n-1} - 1)/(q - 1)$.

Now let $t \in T$. By the above discussion we have $t > (p + 3)/2 > q^{n-4}$. This implies that $|T| \leq 8$, which is a contradiction.

If $S \cong {}^2D_n(q)$, then similarly we get a contradiction and for convenience we omit the proof.

By the above discussion, we get that S is isomorphic to an alternating group A_m , where $m \geq 5$.

Hence we have $A_m \leq G/N \leq \text{Aut}(A_m)$. Since $p \in \pi(S)$, we get that $m \geq p \geq 103$. This implies that $A_m \leq G/N \leq S_m$ and so $|G| = |A_m| \cdot |N|$ or $|G| = 2|A_m| \cdot |N|$. Also since $|A_m| \mid |G| = (p+3)!/2$, we conclude that $p \leq m \leq p+3$. Therefore $A_m \cong A_p, A_{p+1}, A_{p+2}$ or A_{p+3} .

We claim that $S \cong A_{p+3}$. Otherwise let $S \cong A_m$, where $p \leq m \leq p+2$. In each case we can easily see that $(p+3)/2 \mid |N|$. Thus we have $|N| = k(p+3)/2$, where $k \mid 2(p+1)(p+2)$.

First we show that $(p+3)/2$ is a power of 2. Let $r \in \pi((p+3)/2)$ be an odd prime number and $((p+3)/2)_r = r^\alpha$, for some natural number α . We note that $r \neq 3$. Since $|N| = k(p+3)/2$, where $k \mid 2(p+1)(p+2)$, we get that $|N|_r = ((p+3)/2)_r = r^\alpha$.

So let $R \in \text{Syl}_r(N)$. Then by the Frattini argument we have $G = N_G(R)N$. This implies that $p \in \pi(N_G(R))$ and so there exists a subgroup of G , like $R \rtimes P$, where $P \in \text{Syl}_p(N_G(R))$. Since $r \neq 3$ and $r \not\sim p$ in $GK(G)$ we get that $R \rtimes P$ is a Frobenius subgroup of G .

Therefore we conclude that $|P| \mid (|R| - 1)$. Since $R \in \text{Syl}_r(N)$ and $P \in \text{Syl}_p(N_G(R))$, we have $|R| = |N|_r = ((p+3)/2)_r = r^\alpha$ and $|P| = |N_G(R)|_p = |G|_p = p$. Therefore $p \mid (r^\alpha - 1)$, which is a contradiction, since $r^\alpha \leq (p+3)/2 < p$.

Hence we conclude that $(p+3)/2$ is a power of 2. So let $(p+3)/2 = 2^\beta$, where $\beta > 1$ is a natural number. Since $|N| = k(p+3)/2$, where $k \mid 2(p+1)(p+2)$, we get that $|N|_2 = 2^\beta, 2^{\beta+1}$ or $2^{\beta+2}$. Similarly to the above if $Q \in \text{Syl}_2(N)$, then we conclude that G contains a Frobenius subgroup $Q \rtimes P$, where $P \in \text{Syl}_p(N_G(Q))$. So we have the following cases:

- (1) If $|N|_2 = 2^\beta$, then $p \mid (2^\beta - 1)$, which is impossible since $2^\beta = (p+3)/2 < p$.
- (2) If $|N|_2 = 2^{\beta+1}$, then $p \mid (2^{\beta+1} - 1)$. Since $2^\beta = (p+3)/2$, we get that $p \mid (p+2)$, which is a contradiction.
- (3) If $|N|_2 = 2^{\beta+2}$, then $p \mid (2^{\beta+2} - 1)$. Since $2^\beta = (p+3)/2$, we get that $p \mid (2p+5)$, which is a contradiction.

Therefore $S \cong A_{p+3}$ and so $A_{p+3} \leq G/N \leq S_{p+3}$. Since $|G| = |A_{p+3}|$, we conclude that $|N| = 1$ and so $G \cong A_{p+3}$. Therefore A_{p+3} is OD-characterizable. $\square$

Proposition 3.2. *Let $m = 3^a \cdot 5^b$ (or $3^a \cdot 5^b \cdot 7^c$, if $a > 0$) with $m > 3$. Suppose that neither $m-2$ nor $m-4$ is a prime. Let $G = A_m$. Let R be any finite group with $|R| = m$ and let $H = A_{m-1} \times R$. Then G and H have the same order and degree pattern. In particular, G is not OD-characterizable.*

PROOF. Clearly, $|G| = |H|$. Let $\pi(G) = \{p_1, p_2, \dots, p_k\}$ with $p_1 < p_2 < \dots < p_k$. By the hypothesis, $p_k \leq m-6$ and if $a > 0$, $p_k \leq m-8$. Thus, each

vertex in the prime graph of G and H is equal or connected to 2, 3 and 5 and also to 7 if $a > 0$. Now if $7 < p_i < p_j$, then p_i is connected to p_j in the prime graph of G if and only if $p_i + p_j \leq m$. Likewise, p_i and p_j are connected in the prime graph of H if and only if $p_i + p_j \leq m - 1$. Thus, connectivity differs if and only if $p_i + p_j = m$. But $p_i + p_j$ is even and m is odd. Hence G and H have the same degree pattern. $\square$

Remark 3.3. We note that if $m = 5^3$ or $m = 3^{14}$, then the assumptions of the above proposition satisfy and so Conjecture 1.1 is not true in general. Also this implies that there are some finite simple groups, like A_{125} and $A_{3^{14}}$, which are k -fold OD-characterizable, where $k \geq 3$, which is an answer to Problem 1.1.

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