

On generalised pseudo symmetric manifolds

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Dedicated to the memory of Professor Béla Barna

1. Introduction

The notion of a pseudo symmetric manifold was introduced by the author in an earlier paper [1]. A non-flat Riemannian manifold (M^n, g) ($n \geq 2$) was called pseudo symmetric if its curvature tensor R satisfies the condition

$$(1) \quad \begin{aligned} (\nabla_X R)(Y, Z, W) &= 2A(X)R(Y, Z, W) + A(Y)R(X, Z, W) \\ &+ A(Z)R(Y, X, W) + A(W)R(Y, Z, X) + g(R(Y, Z, W), X)P; \\ X, Y, Z, P &\in \chi(M^n) \end{aligned}$$

where A is a non-zero 1-form, ∇ denotes the operator of covariant differentiation with respect to the metric tensor g and P is a vector field given by

$$(2) \quad g(X, P) = A(X) \quad \forall X.$$

The 1-form A was called the associated 1-form of the manifold and such an n -dimensional manifold was denoted by $(PS)_n$. The vector field P defined by (2) is called the basic vector field corresponding to the associated 1-form A .

The object of this paper is to study a type of non-flat Riemannian manifold (M^n, g) ($n > 2$) whose curvature tensor R satisfies the condition

$$(3) \quad \begin{aligned} (\nabla_X R)(Y, Z, W) &= 2A(X)R(Y, Z, W) + B(Y)R(X, Z, W) \\ &+ C(Z)R(Y, X, W) + D(W)R(Y, Z, X) + g(R(Y, Z, W), X)P \end{aligned}$$

where A, B, C, D are non-zero 1-forms and ∇ and P have the meaning already mentioned.

Such a manifold shall be called a generalised pseudo symmetric manifold, A, B, C, D shall be called its associated 1-forms and an n -dimensional manifold of this kind shall be denoted by $G(PS)_n$. Let

$$(4) \quad \begin{aligned} g(X, \lambda) &= B(X), g(X, \mu) = C(X) \quad \text{and} \\ g(X, \nu) &= D(X) \quad \forall X \in \chi(M) \end{aligned}$$

Then $P, \lambda, \mu, \nu \in \chi(M)$ shall be called the basic vector fields of $G(PS)_n$ corresponding to the associated 1-forms A, B, C, D , respectively. If, in particular, $B = C = D = A$, then the manifold defined by (3) reduces to a pseudo symmetric manifold defined by (1). This justifies the name “Generalized pseudo symmetric manifold” and the use of the symbol $G(PS)_n$. It may be mentioned in this connection that following my paper [1], TAMÁSSY and BINH [2] studied a type of Riemannian manifold (M, g) whose curvature tensor R satisfies the condition

$$(5) \quad \begin{aligned} (\nabla_X R)(Y, Z, W) &= \alpha(X)R(Y, Z, W) + \beta(Y)R(X, Z, W) \\ &+ \gamma(Z)R(Y, X, W) + \delta(W)R(Y, Z, X) + g(R(Y, Z, W), X)F \end{aligned}$$

where $\alpha, \beta, \gamma, \delta$ are 1-forms and F any vector field. They called such a manifold weakly symmetric. (5) is a little weaker assumption than (3). (5) gives (3) if α and F are related by $g(X, F) = \alpha \forall X$. Though the definition of a $G(PS)_n$ is similar to that of a weakly symmetric manifold mentioned above, our study of a $G(PS)_n$ is different from that of Tamássy and Binh.

In this paper the question whether a $G(PS)_n$ can be of constant curvature has been answered. Considering an Einstein $G(PS)_n$ it is shown that such a manifold is necessarily of zero scalar curvature under a certain condition. Further, an interesting result of paper [1] for a conformally flat $(PS)_n$ has been generalised for a conformally flat $G(PS)_n$. Finally, it is shown that if a $G(PS)_n$ admits a parallel vector field which is not orthogonal to the basic vector field P , then the manifold cannot be conformally flat.

1. Preliminaries

Let L be the symmetric endomorphism of the tangent space at each point of a $G(PS)_n$ corresponding to the Ricci tensor S of type $(0, 2)$. Then

$$(1.1) \quad g(LX, Y) = S(X, Y) \quad \forall X, Y \in \chi(M).$$

Further, let

$$(1.2) \quad \begin{aligned} \bar{A}(X) &= A(LX), \quad \bar{B}(X) = B(LX), \\ \bar{C}(X) &= C(LX), \quad \bar{D}(X) = D(LX). \end{aligned}$$

Then the 1-forms $\bar{A}$, $\bar{B}$, $\bar{C}$, $\bar{D}$ shall be called the auxiliary associated 1-forms of a $G(PS)_n$ corresponding to the forms A , B , C , D , respectively. Establishing the inner product of both sides of (3) with a vector $U \in \chi(M^n)$ and then contracting over Z and W , we get

$$(1.3) \quad (\nabla_X S)(Y, U) = 2A(X)S(Y, U) + B(Y)S(X, U) \\ + C(R(X, Y, U)) + D(R(X, U, Y)) + A(U)S(X, Y).$$

Next, contracting (1.3) over Y and U , we obtain

$$(1.4) \quad dr(X) = 2A(X)r + S(X, P) + S(X, \lambda) + S(X, \mu) + S(X, \nu),$$

where r is the scalar curvature. Using the notations $P + \lambda + \mu + \nu = \rho$ and $A + B + C + D = E$, we obtain $\bar{E}(X) = E(LX) = S(X, \rho)$ and then (1.4) takes the form

$$(1.4') \quad dr(X) = 2A(X)r + S(X, \rho) = 2A(X)r + \bar{E}(X).$$

From (1.4') we get

$$ddr(X, Y) = 2rdA(X, Y) + 2\bar{E}(X)A(Y) - 2\bar{E}(Y)A(X) + d\bar{E}(X, Y).$$

Since $ddr(X) = 0$, we obtain

$$(1.5) \quad rdA(X, Y) + [\bar{E}(X)A(Y) - \bar{E}(Y)A(X)] + \frac{1}{2}d\bar{E}(X, Y) = 0.$$

These formulas will be used in the sequel.

2. $G(PS)_n$ of non-zero constant scalar curvature

We suppose that in a $G(PS)_n$ the scalar curvature r is a constant different from zero. Then from (1.4') we get

$$2A(X)r + \bar{E}(X) = 0 \quad \text{or} \quad \bar{E}(X) = -2A(X)r.$$

From this it follows that

$$(2.1) \quad S(X, \rho) = -2rA(X).$$

If, in particular, $B = C = D = A$, then $S(X, \rho) = 4\bar{A}(X)$ and (2.1) takes the form $4\bar{A}(X) = -2rA(X)$, from which we get

Theorem 1. *In a $G(PS)_n$ of non-zero constant scalar curvature, in which $B = C = D = A$ we obtain.*

$$\bar{A}(X) = -\left(\frac{r}{2}\right)A(X).$$

This result has already been obtained in paper [1] of the author.

3. Einstein $G(P S)_n$ ($n > 2$)

In this section we suppose that a $G(P S)_n$ is an Einstein manifold. Then

$$(3.1) \quad S(X, Y) = \frac{r}{n}g(X, Y).$$

It is known [3] that in an Einstein manifold (M^n, g) ($n > 2$) r is constant. Hence in this case $dr(X) = 0$. Therefore from (1.4) it follows that

$$2A(X)r + S(X, P) + S(X, \lambda) + S(X, \mu) + S(X, \nu) = 0.$$

Or, in virtue of (3.1) we have

$$2A(X)r + \frac{r}{n}[A(X) + B(X) + C(X) + D(X)] = 0,$$

or

$$(3.2) \quad [(2n + 1)A(X) + B(X) + C(X) + D(X)]r = 0.$$

From (3.2) it follows that if

$$(3.3) \quad (2n + 1)A(X) + B(X) + C(X) + D(X) \neq 0, \quad \text{then } r = 0.$$

This leads to the following theorem:

Theorem 2. *An Einstein $G(P S)_n$ satisfying the condition (3.3) is of zero scalar curvature.*

If, in particular, $B = C = D = A$, then the $G(P S)_n$ reduces to a $(P S)_n$ and the expression $(2n + 1)A(X) + B(X) + C(X) + D(X)$ takes the form $2(n + 2)A(X)$ which is not zero, because $A(X) \neq 0$. Thus it follows that an Einstein $(P S)_n$ with $n > 2$ is of zero scalar curvature — a result already proved by the author in his paper [1]. It is known that a manifold of constant curvature is an Einstein manifold, but the converse is not, in general, true. The question therefore arises whether a $G(P S)_n$ can be of constant curvature.

Suppose that a $G(P S)_n$ is of constant curvature. Then we can write

$$(3.4) \quad R(X, Y, Z) = \kappa[g(Y, Z)X - g(X, Z)Y]$$

where κ is constant. Being of constant curvature, the $G(P S)_n$ under consideration is an Einstein manifold. Hence if (3.3) holds, then according to Theorem 2, $r = 0$. Therefore $\kappa = 0$, because from (3.4) we easily get $r = \kappa n(n - 1)$ by contraction.

Consequently, from (3.4) it follows that $R(X, Y, Z) = 0$, that is, the manifold is flat. But this is not admissible by the definition of a $G(P S)_n$. Therefore in answer to the question raised above we can state the following theorem:

Theorem 3. *A $G(PS)_n$ satisfying the condition (3.3) cannot be of constant curvature.*

Since a 3-dimensional Einstein manifold is of constant curvature ([3] p. 293), we can state the following corollary of Theorem 3.

Corollary. *An Einstein $G(PS)_n$ satisfying the condition $7A(X) + B(X) + C(X) + D(X) \neq 0$ does not exist.*

4. Conformally flat $G(PS)_n$ ($n \geq 3$)

It has been proved by the author elsewhere [1] that in a conformally flat $(PS)_n$, the associated 1-form A is proportional to the auxiliary associated 1-form $\bar{A}$. It is therefore natural to enquire about the nature of generalisation of this result for a conformally flat $G(PS)_n$. An answer to this enquiry is given in this section.

It is known ([4] p. 91) that in a conformally flat (M^n, g) ($n \geq 3$)

$$(4.1) \quad (\nabla_X S)(Y, Z) - (\nabla_Z S)(Y, X) \\ = \frac{1}{2(n-1)} [dr(X)g(Y, Z) - dr(Z)g(X, Y)].$$

In virtue of (1.4) the equation (4.1) can be written as follows:

$$(4.2) \quad 2(n-1)[(\nabla_X S)(Y, Z) - (\nabla_Z S)(Y, X) = 2r[A(X)g(Y, Z) - A(Z)g(X, Y)] \\ + [\bar{A}(X)g(Y, Z) - \bar{A}(Z)g(X, Y)] + [\bar{B}(X)g(Y, Z) - \bar{B}(Z)g(X, Y)] \\ + [\bar{C}(X)g(Y, Z) - \bar{C}(Z)g(X, Y)] + [\bar{D}(X)g(Y, Z) - \bar{D}(Z)g(X, Y)]$$

Again in virtue of (1.3)

$$(\nabla_X S)(Y, Z) - (\nabla_Z S)(Y, X) = A(X)S(Y, Z) - A(Z)S(Y, X) \\ + C(R(X, Z, Y)) + 2D(R(X, Z, Y)).$$

Therefore

$$(4.3) \quad 2(n-1)[(\nabla_X S)(Y, Z) - (\nabla_Z S)(Y, X)] \\ = 2(n-1)A(X)S(Y, Z) - 2(n-1)A(Z)S(Y, X) \\ + 2(n-1)C(R(X, Z, Y)) + 4(n-1)D(R(X, Z, Y)).$$

From (4.2) and (4.3) we get

$$\begin{aligned}
 & 2(n-1)A(X)S(Y, Z) - 2(n-1)A(Z)S(Y, X) \\
 & + 2(n-1)C(R(X, Z, Y)) + 4(n-1)D(R(X, Z, Y)) \\
 & = 2r[A(X)g(Y, Z) - A(Z)g(X, Y)] \\
 (4.4) \quad & + [\bar{A}(X)g(Y, Z) - \bar{A}(Z)g(X, Y)] \\
 & + [\bar{B}(X)g(Y, Z) - \bar{B}(Z)g(X, Y)] \\
 & + [\bar{C}(X)g(Y, Z) - \bar{C}(Z)g(X, Y)] \\
 & + [\bar{D}(X)g(Y, Z) - \bar{D}(Z)g(X, Y)]
 \end{aligned}$$

Again in a conformally flat (M^n, g) ($n > 2$)

$$\begin{aligned}
 R(X, Z, Y, W) &= \frac{1}{n-2}[S(Y, Z)g(X, W) - S(X, Y)g(Z, W) \\
 (4.5) \quad & + S(X, W)g(Y, Z) - S(Z, W)g(X, Y)] \\
 & + \frac{r}{(n-1)(n-2)}[g(X, Y)g(Z, W) - g(Y, Z)g(X, W)]
 \end{aligned}$$

where

$$(4.6) \quad R(X, Z, Y, W) = g[R(X, Z, Y), W]$$

In virtue of (4.5) we get

$$\begin{aligned}
 & 2(n-1)[C(R(X, Z, Y)) + 2D(R(X, Z, Y))] \\
 & = \frac{2(n-1)}{n-2}[S(Y, Z)\{C(X) + 2D(X)\} - S(X, Y)\{C(Z) + 2D(Z)\} \\
 (4.7) \quad & + g(Y, Z)\{\bar{C}(X) + 2\bar{D}(X)\} - g(X, Y)\{\bar{C}(Z) + 2\bar{D}(Z)\}] \\
 & + \frac{2r}{n-2}[g(X, Y)\{C(Z) + 2D(Z)\} - g(Y, Z)\{C(X) + 2D(X)\}].
 \end{aligned}$$

In virtue of (4.7) we can express (4.4) as follows:

$$\begin{aligned}
 & 2(n-1)S(Y, Z)[(n-2)A(X) + C(X) + 2D(X)] \\
 & - 2(n-1)S(X, Y)[(n-2)A(Z) + C(Z) + 2D(Z)] \\
 & + g(Y, Z)[2(n-1)\bar{C}(X) + 4(n-1)\bar{D}(X) \\
 (4.8) \quad & - (n-2)\{\bar{A}(X) + \bar{B}(X) + \bar{C}(X) + \bar{D}(X)\}] \\
 & - g(X, Y)[2(n-1)\bar{C}(Z) + 4(n-1)\bar{D}(Z) \\
 & - (n-2)\{\bar{A}(Z) + \bar{B}(Z) + \bar{C}(Z) + \bar{D}(Z)\}] \\
 & = 2r[g(Y, Z)\{(n-2)A(X) + 2C(X) + 4D(X)\} \\
 & - g(X, Y)\{(n-2)A(Z) + 2C(Z) + 4D(Z)\}]
 \end{aligned}$$

Putting $Y = P$ in (4.8) we obtain

$$\begin{aligned}
 & (n-1)[2C(X)\bar{A}(Z) - 2C(Z)\bar{A}(X) + 4D(X)\bar{A}(Z) - 4D(Z)\bar{A}(X)] \\
 & + (n-1)[2\bar{C}(X)A(Z) - 2\bar{C}(Z)A(X) + 4\bar{D}(X)A(Z) - 4\bar{D}(Z)A(X)] \\
 (4.9) \quad & - (n-2)[A(Z)\{(-2n+1)\bar{A}(X) + \bar{B}(X) + \bar{C}(X) + \bar{D}(X)\} \\
 & - A(X)\{(2n+1)\bar{A}(Z) + \bar{B}(Z) + \bar{C}(Z) + \bar{D}(Z)\}] \\
 & = 2r[2C(X)A(Z) - 2C(Z)A(X) + 4D(X)A(Z) - 4D(Z)A(X)].
 \end{aligned}$$

This is the required generalization.

We can therefore state the following theorem:

Theorem 4. *In a conformally flat $G(PS)_n$ the associated and the auxiliary associated 1-forms satisfy the relation (4.9).*

If, in particular, $B = C = D = A$, then the $G(PS)_n$ reduces to a $(PS)_n$ and the relation (4.9) takes the form

$$\bar{A}(X)A(Z) - \bar{A}(Z)A(X) = 0,$$

a result already proved by the author elsewhere [1].

5. $G(PS)_n$ admitting a parallel vector field

In this section we suppose that a $G(PS)_n$ admits a parallel vector field V ([3] p. 124, [5] p. 322]).

Then

$$(5.1) \quad (\nabla_X V) = 0 \quad \forall X \in \chi(G(PS)_n).$$

Applying Ricci identity to (5.1) we get

$$(5.2) \quad R(X, Y, V) = 0.$$

From (5.2) it follows that

$$(5.3) \quad R(X, Y, Z, V) = 0.$$

In virtue of (5.3) we get

$$(5.4) \quad S(X, V) = 0.$$

Now, by (5.1) and (5.4)

$$(5.5) \quad (\nabla_X S)(Y, V) = \nabla_X S(Y, V) - S(\nabla_X Y, V) - S(Y, \nabla_X V) = 0$$

Again from (1.3) we get by (5.3) and (5.4)

$$(5.6) \quad \begin{aligned} (\nabla_X S)(Y, V) &= 2A(X)S(Y, V) + B(Y)S(X, V) \\ &+ R(Y, X, \mu, V) + R(Y, \nu, X, Y) + A(V)S(Y, X) = A(V)S(Y, X). \end{aligned}$$

From (5.5) and (5.6) we obtain

$$(5.7) \quad A(V)S(Y, X) = 0,$$

If $A(V) \neq 0$, i.e., if $g(P, V) \neq 0$, then from (5.7) we get $S(Y, X) = 0$. Hence

$$\tilde{C}(X, Y, Z) = R(X, Y, Z),$$

where $\tilde{C}$ is Weyl's conformal curvature tensor. Therefore $\tilde{C}(X, Y, Z) \neq 0$, for otherwise $R(X, Y, Z)$ will be zero implying that the manifold is flat which is inadmissible. Hence we can state the following theorem.

Theorem 5. *If a $G(PS)_n$ admits a parallel vector field which is not orthogonal to the basis vector field P , then the manifold cannot be conformally flat.*

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