

Units in FD_{2p}

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Abstract. In this paper, we present the structure of the group of $*$ -unitary units in the group algebra FD_{2p} , where F is a finite field of characteristic $p > 2$, D_{2p} is the dihedral group of order $2p$, and $*$ is the canonical involution of the group algebra FD_{2p} . We also provide the structure of the maximal p -subgroup of the unit group $\mathcal{U}(FD_{2p})$ and compute a basis of its center.

1. Introduction

Let FG be the group algebra of a group G over a field F . For a normal subgroup H in G , the natural homomorphism $G \rightarrow G/H$ can be extended to an algebra homomorphism from FG to $F[G/H]$, defined by $\sum_{g \in G} a_g g \mapsto \sum_{g \in G} a_g gH$. The kernel of this homomorphism, denoted by $\Gamma(H)$, is the ideal generated by $\{h - 1 \mid h \in H\}$. Therefore, $FG/\Gamma(H) \cong F[G/H]$. In particular, for $H = G$, $\Gamma(G)$ is known as the augmentation ideal of the group algebra FG . Since $FG/\Gamma(G) \cong F$, it follows that the Jacobson radical $J(FG)$ is contained in $\Gamma(G)$. The equality occurs if G is a finite p -group and F is a field of characteristic p and hence $1 + \Gamma(G)$ is same as the normalized unit group $V(FG)$ of the group algebra FG . Therefore, $\mathcal{U}(FG) = V(FG) \times F^*$, where F^* is the cyclic group of all nonzero elements of F .

If $x = \sum_{g \in G} x_g g$ is an element of FG , then the element $x^* = \sum_{g \in G} x_g g^{-1}$ is called the conjugate of x . The map $x \mapsto x^*$ is an anti-automorphism of FG of order 2, which is known as the canonical involution of the group algebra FG . An element $x \in \mathcal{U}(FG)$ is called unitary if $x^* = x^{-1}$. The unitary units of the

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unit group $\mathcal{U}(FG)$ form a subgroup $\mathcal{U}_*(FG)$, and is called unitary subgroup of $\mathcal{U}(FG)$.

SANDLING in [6] described the structure of $V(FG)$ for an arbitrary finite abelian p -group G and a finite field F of p elements. In [7] Sandling provided generators and relators of $V(FG)$ for each 2-group G of order dividing 16 over a finite field F with 2 elements. CREEDON and GILDEA in [10] described the structure of $V(FD_8)$ for the dihedral group D_8 of order 8 and a finite field F of characteristic 2. For dihedral groups of orders 6 and 10, the structure of unit group is presented in [2] and [1]. In [13] KAUR and KHAN described the structure of the unit group $\mathcal{U}(FD_{2p})$ over a finite field F with two elements. GILDEA in [8] studied some properties of the center of the maximal p -subgroup of the unit group $\mathcal{U}(FD_{2p})$ over a finite field F of characteristic p . However, basis of its center is not known.

The set of all unitary units in the normalized unit group $V(FG)$ forms a subgroup of $\mathcal{U}_*(FG)$. We denote it by $V_*(FG)$. The unitary subgroup $\mathcal{U}_*(FG)$ coincides with $V_*(FG)$ if F is a finite field of characteristic 2. Otherwise, it coincides with $V_*(FG) \times \langle -1 \rangle$. BOVDI and SAKACS in [14] described the structure of $V_*(FG)$, where G is a finite abelian group and F is a finite field of characteristic p . BOVDI and ERDEI in [4] provided the structure of the unitary subgroup $V_*(F_2G)$, where G is a nonabelian group of order 8 and 16. V. BOVDI and ROSA in [3] computed the order of the unitary subgroup of the group of units, when G is either an extraspecial 2-group or the central product of such a group with a cyclic group of order 4 and F is a finite field of characteristic 2. They also computed the order of the unitary subgroup $V_*(FG)$, where G is a 2-group with a finite abelian subgroup A of index 2 and an element b such that b inverts every element in A and the order of b is 2 or 4. V. BOVDI and ROZGONYI in [5] described the structure of $V_*(F_2G)$ if order of b is 4. If G is a nonabelian group of order 8 and F is a finite field of characteristic 2, then the structure of $V_*(FG)$ is described in [12] and [11].

Here, we obtain generators of the unitary subgroup $\mathcal{U}_*(FD_{2p})$ for dihedral group D_{2p} of order $2p$ over a finite field F of characteristic p . We also obtain a basis of the center of the maximal p -subgroup of the unit group $\mathcal{U}(FD_{2p})$. Finally, we establish that the maximal p -subgroup of $\mathcal{U}(FG)$ is a general product of the unitary subgroup with a metabelian group.

Let $D_{2p} = \langle a, b \mid a^p = 1 = b^2, b^{-1}ab = a^{-1} \rangle$. The distinct conjugacy classes of D_{2p} are $C_0 = \{1\}$, $C_i = \{a^i, a^{-i}\}$, for $1 \leq i \leq l$, where $l = \frac{p-1}{2}$ and $C = \{b, ab, a^2b, \dots, a^{p-1}b\}$. For a set H , if $\hat{H}$ denotes the sum of all the elements

of H , then $\{\hat{C}_0, \hat{C}_1, \hat{C}_2, \dots, \hat{C}_l, \hat{C}\}$ forms a F -basis for the center $Z(FD_{2p})$ of the group algebra FD_{2p} .

2. Unit group of FD_{2p}

If A denotes the normal subgroup of D_{2p} generated by element a in D_{2p} , then $FD_{2p}/\Gamma(A) \cong FC_2$. Since $\Gamma(A)$ is a nilpotent ideal of FD_{2p} , we have $\mathcal{U}(FD_{2p})/(1 + \Gamma(A)) \cong \mathcal{U}(FC_2)$, which is isomorphic to $F^* \times F^*$. Let $\theta : \mathcal{U}(FD_{2p}) \rightarrow \mathcal{U}(FC_2)$ be a group epimorphism defined by

$$\sum_{i=0}^{p-1} \alpha_i a_i + \sum_{j=0}^{p-1} \beta_j a^j b \mapsto \sum_{i=0}^{p-1} \alpha_i + \sum_{j=0}^{p-1} \beta_j x,$$

where $C_2 = \langle x \rangle$. We can define a group homomorphism $\psi : \mathcal{U}(FC_2) \rightarrow \mathcal{U}(FD_{2p})$ by $a_0 + a_1 x \mapsto a_0 + a_1 b$. Since $\theta \circ \psi = 1$, we have $\mathcal{U}(FD_{2p}) \cong (1 + \Gamma(A)) \rtimes F^* \times F^*$. If order of F is p^n , then $1 + \Gamma(A)$ is a nilpotant group of order p^{4nl} with exponent p , where $l = \frac{p-1}{2}$.

We first establish the structure of the unitary subgroup $\mathcal{U}_*(FD_{2p})$.

3. Structure of the unitary subgroup $\mathcal{U}_*(FD_{2p})$

Theorem 1. *If A is the normal subgroup of D_{2p} , then the unitary subgroup $\mathcal{U}_*(FD_{2p})$ of the group algebra FD_{2p} is the semidirect product of the normal subgroup $V_*(FA)$ with an elementary abelian 2-group.*

We need following lemmas:

Lemma 2. *The group of unitary units in $1 + \Gamma(A)$ is $V_*(FA)$.*

PROOF. Assume that $v = 1 + x_1 + x_2 b$, where $x_i \in \omega(FA)$ is an arbitrary element of $1 + \Gamma(A)$. Since $\omega(FA)$ is a nilpotent ideal, $1 + x_1$ is an invertible element and thus v can be written as $v = u(1 + xb)$, where $u \in V(FA)$ and $x \in \omega(FA)$. In particular, if v is a unitary unit, then from the equation $v^*v = 1$, we obtain $u^*u + bx^*u^*uxb = 1$ and $bx^*u^*u + u^*uxb = 0$. Moreover, since $by = y^*b$ for any $y \in FA$, we have $u \in V_*(FA)$. Further, note that $u^{-1}v = 1 + xb$ is a symmetric unit as well as a unitary unit. Since the exponent of $1 + \Gamma(A)$ is p , it implies that $1 + xb = 1$ and hence $v = u$. $\square$

Since A is a normal subgroup of D_{2p} , $FD_{2p}/\Gamma(A)$ is an algebra over F . Also note that $\Gamma(A)$ is $*$ -stable nil ideal, and therefore the set of all unitary units in $FD_{2p}/\Gamma(A)$ form a subgroup of $\mathcal{U}(FD_{2p}/\Gamma(A))$ and is denoted by $\mathcal{U}_*(FD_{2p}/\Gamma(A))$.

Lemma 3. *The unitary subgroup $\mathcal{U}_*(FD_{2p}/\Gamma(A))$ of $\mathcal{U}(FD_{2p}/\Gamma(A))$ is the group generated by $\{-1 + \Gamma(A), b + \Gamma(A)\}$.*

PROOF. If $u + \Gamma(A)$ is a unitary unit of $\mathcal{U}(FD_{2p}/\Gamma(A))$, then $(u + \Gamma(A))^*(u + \Gamma(A)) = 1 + \Gamma(A)$ and hence u^*u is a symmetric unit in $1 + \Gamma(A)$ because $\Gamma(A)$ is a $*$ -stable nil ideal. Further, if $u = x_0 + x_1b$, where $x_0, x_1 \in FA$, then $uu^* = x_0x_0^* + x_1x_1^* + 2x_0x_1b$. For $x = \sum_{i=0}^{p-1}(\alpha_i a^i + \beta_i a^i b)$, we can define $\chi(x) = \sum_{i=0}^{p-1}(\alpha_i + \beta_i)$. Since uu^* is an element of $1 + \Gamma(A)$, it follows that $\chi(x_0x_0^* + x_1x_1^*) = 1$ and $\chi(x_0x_1) = 0$. In particular, if $x_0 = \sum_{i=0}^{p-1} \alpha_i a^i$ and $x_1 = \sum_{i=0}^{p-1} \beta_i a^i$, then we obtain

$$\sum_{i=0}^{p-1} \alpha_i^2 + \sum_{i=0}^{p-1} \beta_i^2 + 2 \sum_{\substack{i,j=0 \\ i < j}}^{p-1} \alpha_i \alpha_j + 2 \sum_{\substack{i,j=0 \\ i < j}}^{p-1} \beta_i \beta_j = 1 \quad (1)$$

$$\left(\sum_{i=0}^{p-1} \alpha_i \right) \left(\sum_{j=0}^{p-1} \beta_j \right) = 0. \quad (2)$$

Now if $(\sum_{i=0}^{p-1} \alpha_i) = 0$, then from equation (1) we obtain $(\sum_{i=0}^{p-1} \beta_i)^2 = 1$ and hence $\sum_{i=0}^{p-1} \beta_i = \pm 1$. Thus, either x_1 or $-x_1$ is an element of $V(FA)$ and so the unitary units in $FD_{2p}/\Gamma(A)$ are $\pm b + \Gamma(A)$. Further, if $(\sum_{i=0}^{p-1} \beta_i) = 0$, then, in a similar way, one can show that the unitary units in $FD_{2p}/\Gamma(A)$ are $\pm 1 + \Gamma(A)$. $\square$

PROOF OF THE THEOREM. Note that $V_*(FA)$ is a normal subgroup of $\mathcal{U}_*(FG)$ and $\langle V_*(FA), b, -1 \rangle \subseteq \mathcal{U}_*(FG)$. Now suppose that $u \in \mathcal{U}(FG) \setminus V_*(FA)$ is a unitary unit, then $u + \Gamma(A)$ is a unitary unit in $\mathcal{U}(FG/\Gamma(A))$. In particular, if $u + \Gamma(A) = b + \Gamma(A)$, then $u = bx$ for some $x \in V_*(FA)$. Hence, $\mathcal{U}_*(FG) = V_*(FA) \rtimes (\langle b \rangle \times \langle -1 \rangle)$. $\square$

4. The structure of center $Z(1 + \Gamma(A))$

To find a basis of center $Z(1 + \Gamma(A))$ of the maximal p -subgroup $1 + \Gamma(A)$ of the unit group $\mathcal{U}(FD_{2p})$, we need following lemma:

Lemma 4. *If $\omega_i = (a^i - a^{-i})(1 + b)$ and $\omega'_i = (a^i - a^{-i})(1 - b)$ for $1 \leq i \leq l$, then the set $\{\omega_i, \omega'_i, \omega_i \omega'_i, \omega'_i \omega_i \mid 1 \leq i \leq l\}$ is a free F -basis of $\Gamma(A)$ as a free F -module.*

PROOF. It is known that the set $\{(a^i - 1), (a^i - 1)b \mid 1 \leq i \leq 2l\}$ is a free F -basis of $\Gamma(A)$ as a free F -module. Observe that $\omega_i \omega_j = 0, \omega'_i \omega'_j = 0$ for $1 \leq i, j \leq l$. Also note that $\omega_i \omega'_i = 2(a^{2i} + a^{-2i} - 2)(1 - b)$ and $\omega'_i \omega_i = 2(a^{2i} + a^{-2i} - 2)(1 + b)$. Thus, if $t = 2i$, then

$$\begin{aligned} (a^t - 1) &= \frac{1}{4}(\omega_{2i} + \omega'_{2i}) + \frac{1}{8}(\omega_i \omega'_i + \omega'_i \omega_i) \\ (a^t - 1)b &= \frac{1}{4}(\omega_{2i} - \omega'_{2i}) - \frac{1}{8}(\omega_i \omega'_i - \omega'_i \omega_i) \end{aligned} \left\} 0 < 2i \leq l$$

$$\begin{aligned} (a^t - 1) &= -\frac{1}{4}(\omega_{p-2i} + \omega'_{p-2i}) + \frac{1}{8}(\omega_i \omega'_i + \omega'_i \omega_i) \\ (a^t - 1)b &= -\frac{1}{4}(\omega_{p-2i} - \omega'_{p-2i}) - \frac{1}{8}(\omega_i \omega'_i - \omega'_i \omega_i) \end{aligned} \left\} l < 2i \leq (p-1)$$

If $t = p - 2i$, then

$$\begin{aligned} (a^t - 1) &= -\frac{1}{4}(\omega_{2i} + \omega'_{2i}) + \frac{1}{8}(\omega_i \omega'_i + \omega'_i \omega_i) \\ (a^t - 1)b &= -\frac{1}{4}(\omega_{2i} - \omega'_{2i}) - \frac{1}{8}(\omega_i \omega'_i - \omega'_i \omega_i) \end{aligned} \left\} 0 < 2i \leq l$$

$$\begin{aligned} (a^t - 1) &= \frac{1}{4}(\omega_{p-2i} + \omega'_{p-2i}) + \frac{1}{8}(\omega_i \omega'_i + \omega'_i \omega_i) \\ (a^t - 1)b &= \frac{1}{4}(\omega_{p-2i} - \omega'_{p-2i}) - \frac{1}{8}(\omega_i \omega'_i - \omega'_i \omega_i) \end{aligned} \left\} l < 2i \leq (p-1)$$

Therefore, $\Gamma(A) = \text{span}\{\omega_i, \omega'_i, \omega_i \omega'_i, \omega'_i \omega_i \mid 1 \leq i \leq l\}$. Since the dimension of $\Gamma(A)$ over F is $4l$, the result follows. $\square$

Assume that F is a finite field of order p^n . Let $f(x)$ be a monic irreducible polynomial of degree n over F_p such that $F \cong F_p[x]/\langle f(x) \rangle$ and α denote the residue class of x modulo $\langle f(x) \rangle$.

Theorem 5. *If $u_{i,k} = 1 + \alpha^i(a - 1)^k$, then the center $Z(1 + \Gamma(A))$ is an elementary abelian p group of order $p^{n(l+1)}$ with the set $\{u_{i,k}^* u_{i,k}, 1 + \alpha^i \hat{A}b \mid 0 \leq i \leq (n-1), 1 \leq k \leq (p-1) \text{ and } k \text{ is even}\}$ as a basis.*

PROOF. First we prove that $Z(1 + \Gamma(A)) = Z(FD_{2p}) \cap (1 + \Gamma(A))$. Let $x \in Z(FD_{2p}) \cap (1 + \Gamma(A))$. It is clear that x is of the form

$$x = 1 + \sum_{i=1}^l \alpha_i(\hat{C}_i - 2) + \beta \hat{A}b,$$

where $\alpha_i, \beta \in F$. Since $Z(FD_{2p}) \cap 1 + \Gamma(A) \subseteq Z(1 + \Gamma(A))$, it implies that

$|Z(1 + \Gamma(A))| \geq p^{n(l+1)}$. To show the equality, we compute the dimension of $Z(\Gamma(A))$ over F . Take $x \in Z(\Gamma(A))$ such that

$$x = \sum_{i=1}^l \alpha_i \omega_i + \sum_{i=1}^l \alpha'_i \omega'_i + \sum_{i=1}^l \alpha_{ii} \omega_i \omega'_i + \sum_{i=1}^l \alpha'_{ii} \omega'_i \omega_i.$$

First, note that if $i = 2k + 1$, where $1 \leq i \leq l$ then

$$\omega_1 \omega'_i = \omega_{k+1} \omega'_{k+1} - \omega_k \omega'_k \quad \text{and} \quad \omega'_i \omega_1 = \omega'_{k+1} \omega_{k+1} - \omega'_k \omega_k.$$

For $i = 2k$, where $1 \leq i \leq l$

$$\omega_1 \omega'_i = \omega_{l-k} \omega'_{l-k} - \omega_{l-(k-1)} \omega'_{l-(k-1)} \quad \text{and} \quad \omega'_i \omega_1 = \omega'_{l-k} \omega_{l-k} - \omega'_{l-(k-1)} \omega_{l-(k-1)}.$$

Next, observe that if l is odd, then

$$\begin{aligned} \omega_1 \omega'_j \omega_j &= 4\omega_{2j+1} - 4\omega_{2j-1} - 8\omega_1, \quad \text{for } 1 \leq j \leq \frac{l-1}{2} \\ \omega_1 \omega'_j \omega_j &= -4\omega_{2l-2j} + 4\omega_{2l-2j+2} - 8\omega_1, \quad \text{for } \frac{l+3}{2} \leq j \leq l-1 \\ \omega_1 \omega'_{\frac{l+1}{2}} \omega_{\frac{l+1}{2}} &= -4\omega_{l-1} - 4\omega_l - 8\omega_1 \\ \omega_1 \omega'_l \omega_l &= 4\omega_2 - 8\omega_1 \end{aligned}$$

and if l is even, then

$$\begin{aligned} \omega_1 \omega'_j \omega_j &= 4\omega_{2j+1} - 4\omega_{2j-1} - 8\omega_1, \quad \text{for } 1 \leq j \leq \frac{l-2}{2} \\ \omega_1 \omega'_j \omega_j &= -4\omega_{2l-2j} + 4\omega_{2l-2j+2} - 8\omega_1, \quad \text{for } \frac{l+2}{2} \leq j \leq l-1 \\ \omega_1 \omega'_l \omega_l &= -4\omega_l - 4\omega_{l-1} - 8\omega_1 \\ \omega_1 \omega'_l \omega_l &= 4\omega_2 - 8\omega_1 \end{aligned}$$

Now, after substituting these values in the equations $\omega_1 x = x \omega_1$ and $\omega'_1 x = x \omega'_1$, we obtain $\alpha_i = \alpha'_i = 0 \forall 1 \leq i \leq l$ and the following set of l equations in $2l$ variables.

$$\begin{aligned} \alpha'_{ii} - \alpha'_{i+1i+1} - \alpha_{ii} + \alpha_{i+1i+1} &= 0 \\ -3(\alpha'_{11} - \alpha_{11}) - 2 \sum_{i=2}^l (\alpha'_{ii} - \alpha_{ii}) &= 0 \end{aligned}$$

Further, observe that the last equation can be written as

$$-\sum_{k=1}^{l-1} (2k+1)(\alpha'_{kk} - \alpha'_{k+1k+1} - \alpha_{kk} + \alpha_{k+1k+1}) = 0$$

Therefore, the dimension of the solution space of the above system of linear equation is $l+1$ and hence $|Z(1+\Gamma(A))| \leq p^{n(l+1)}$. Thus, $|Z(1+\Gamma(A))| = p^{n(l+1)}$ and

$$Z(1+\Gamma(A)) = \left\{ 1 + \sum_{i=1}^l \alpha_i (\hat{C}_i - 2) + \beta \hat{A}b \mid \alpha_i, \beta \in F \right\} = S_*(FA) \times (1 + F\hat{A}b).$$

Further, note that

$$\mathcal{B} = \{u_{i,k}^* u_{i,k} \mid 0 \leq i \leq (n-1), 1 \leq k \leq (p-1) \text{ and } k \text{ is even}\}$$

forms a basis of $S_*(FA)$ and hence the result follows. $\square$

5. Structure of $1 + \Gamma(A)$

In this section, we obtain the structure of $1 + \Gamma(A)$. We shall use the following result:

Theorem 6 (PAVESIC [9]). *Let $(e_1, e_2, \dots, e_n)$ be an ordered n -tuple of orthogonal idempotents in a unital ring R such that $1 = e_1 + e_2 + \dots + e_n$ and each e_i strongly preserves a circle subgroup M of the circle group of R , i.e. $e_i M \subseteq M$ and $M e_i \subseteq M$, then $(M, \circ) = (L, \circ) \circ (D, \circ) \circ (U, \circ)$, where $L = \{m \in M \mid (\forall i) e_i m = e_i m \bar{e}_i\}$, $D = \{m \in M \mid (\forall i) e_i m = e_i m e_i\}$ and $U = \{m \in M \mid (\forall i) e_i m = e_i m \underline{e}_i\}$, where $\bar{e}_i = e_{i+1} + e_{i+2} + \dots + e_n$, $\underline{e}_i = e_1 + e_2 + \dots + e_{i-1}$, $\underline{e}_1 = 0$ and $\bar{e}_n = 0$.*

Theorem 7. *If A is the normal subgroup of D_{2p} , then $1 + \Gamma(A)$ is the general product of unitary subgroup with a metabelian group.*

PROOF. Note that $\Gamma(A)$ is a circle group and $\{e_1 = \frac{1+b}{2}, e_2 = \frac{1-b}{2}\}$ is a complete set of orthogonal idempotents that strongly preserve $\Gamma(A)$. Therefore, $(\Gamma(A), o) = (L, o) \circ (D, o) \circ (U, o)$, where $(L, o) = \bigoplus_{i=1}^l F\omega'_i$, $(U, o) = \bigoplus_{i=1}^l F\omega_i$ and $(D, o) = \bigoplus_{i=1}^l F\omega_i \omega'_i \oplus_{i=1}^l F\omega'_i \omega_i$. Hence, $1 + \Gamma(A)$ is a general product of elementary abelian p -groups $(1 + L)$, $(1 + D)$, and $1 + U$. Moreover, $1 + D$ normalize $1 + L$; thus the group $W = (1 + L) \rtimes (1 + D)$ is a metabelian group.

If $C_{1+\Gamma(A)}(a)$ is a centralizer of a in $1 + \Gamma(A)$, then it is clear that it is an elementary abelian p -group of order p^{np} . Since $V_*(FA)$ and $Z(1 + \Gamma(A))$ are in $C_{1+\Gamma(A)}(a)$ and their intersection is identity, it follows that $C_{1+\Gamma(A)}(a) = V_*(FA) \times Z(1 + \Gamma(A))$. Also note that $W \cap C_{1+\Gamma(A)}(a) = Z(1 + \Gamma(A))$. Thus we obtain that $1 + \Gamma(A) = WC_{1+\Gamma(A)}(a)$ is a general product of W and $V_*(FA)$. $\square$

If α is the residue class of $x \pmod{\langle f(x) \rangle}$, then the set $\{1 + \alpha^i \omega'_j \mid 0 \leq i \leq n-1, 1 \leq j \leq l\}$ generates $(1 + L)$ and $\{1 + \alpha^i \omega_j \mid 0 \leq i \leq n-1, 1 \leq j \leq l\}$ generates $(1 + U)$. The following lemma provides the structure of $1 + D$.

Lemma 8. *The set $\{1 + \alpha^i(a - a^{-1})^{2k}(1 - b), 1 + \alpha^i(a - a^{-1})^{2k}(1 + b) \mid 1 \leq k \leq l \text{ and } 0 \leq i \leq (n-1)\}$ forms a basis of $1 + D$.*

PROOF. First, we show that $\sum_{i=1}^l F\omega_i\omega'_i = \text{span}\{(a - a^{-1})^{2k}(1 - b) \mid 1 \leq k \leq l\}$. For this, we prove $\omega_k\omega'_k \in \text{span}\{(a - a^{-1})^{2k}(1 - b) \mid 1 \leq k \leq l\}$ by induction over k . The result is trivial for $k = 1$, as $\omega_1\omega'_1 = 2(a - a^{-1})^2(1 - b)$. Assume the result for $k - 1$. Consider $\omega_k\omega'_k$. Notice that

$$\omega_k\omega'_k = 2(a^k - a^{-k})^2(1 - b) = 2(a^{2k} + a^{-2k} - 2)(1 - b).$$

Now,

$$\begin{aligned} (a - a^{-1})^{2k} &= \sum_{j=0}^{k-1} (-1)^j \binom{2k}{j} (a^{2k-2j} + a^{-(2k-2j)}) + (-1)^k \binom{2k}{k} \\ &= \sum_{j=0}^{k-1} (-1)^j \binom{2k}{j} (a^{2k-2j} + a^{-(2k-2j)} - 2) \\ (a - a^{-1})^{2k}(1 - b) &= \sum_{j=0}^{k-1} (-1)^j \frac{1}{2} \binom{2k}{j} \omega_{k-j}\omega'_{k-j} \end{aligned}$$

Therefore, $\frac{1}{2}\omega_k\omega'_k = (a - a^{-1})^{2k}(1 - b) - (\sum_{j=1}^{k-1} (-1)^j \frac{1}{2} \binom{2k}{j} \omega_{k-j}\omega'_{k-j})$. Hence, by the induction hypothesis, we obtain

$$\omega_k\omega'_k \in \text{span}\{(a - a^{-1})^{2k}(1 - b) \mid 1 \leq k \leq l\}.$$

Therefore, $\sum_{i=1}^l F\omega_i\omega'_i = \sum_{k=1}^l F(a - a^{-1})^{2k}(1 - b)$. It implies that

$$1 + \sum_{i=1}^l F\omega_i\omega'_i = 1 + \sum_{k=1}^l F(a - a^{-1})^{2k}(1 - b) = \prod_{k=1}^l \prod_{i=0}^{n-1} (1 + \alpha^i(a - a^{-1})^{2k})(1 - b)$$

Similarly, we can show that

$$1 + \sum_{i=1}^l F\omega'_i\omega_i = \prod_{k=1}^l \prod_{i=0}^{n-1} (1 + \alpha^i(a - a^{-1})^{2k})(1 + b)$$

Since $1 + D = (1 + \sum_{i=1}^l F\omega_i\omega'_i) \times (1 + \sum_{i=1}^l F\omega'_i\omega_i)$, the result follows. $\square$

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