Publ. Math. Debrecen
86/3-4 (2015), 363–376
DOI: 10.5486/PMD.2015.6096

Rings whose unit graphs are planar

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Abstract. The unit graph of a ring R is the simple graph $G(R)$ with vertex set R, where two distinct vertices x and y are adjacent if and only if $x + y$ is a unit of R. In this paper, we completely characterize the rings whose unit graphs are planar.

1. The result

Throughout, rings are associative with $1 \neq 0$. The group of units of a ring R is denoted by $U(R)$. This paper concerns the unit graph associated with a ring. Recall that the unit graph of a ring R, denoted $G(R)$, is the simple graph with vertex set R, where two distinct vertices x and y are adjacent if and only if $x + y \in U(R)$. The unit graph was first investigated in 1990 by Grimaldi in [5] for $\mathbb{Z}_n$, the ring of integers modulo n. In 2010, Ashrafi, *et al.* [2] generalized the unit graph $G(\mathbb{Z}_n)$ to $G(R)$ for an arbitrary ring R. The unit graph is also the topic of several other publications (see [1], [3] [6], [7], [8], [9], [10]).

The concentration is on the planarity of the unit graph of a ring. A graph is said to be planar if it can be drawn on the plane in such a way that its edges intersect only at their endpoints. The planarity is an important invariant in graph theory. This work is motivated by the following result of Ashrafi, *et al.* [2] who completely determined the finite commutative rings whose unit graphs are planar. We write $\mathbb{F}_p$ for the field of p elements and $R[t]$ for the polynomial ring over a ring R in the indeterminate t.

Mathematics Subject Classification: Primary: 05C25, 13A99, 16U60, 16L99.

Key words and phrases: unit graph, planar graph, ring, characteristic of a ring, unit.

Theorem 1.1 ([2])**.** *Let R be a finite commutative ring. Then $G(R)$ is planar if and only if R is isomorphic to one of the following rings:*

$$\mathbb{Z}_3, \mathbb{F}_4, \mathbb{Z}_5, \mathbb{Z}_3 \times \mathbb{Z}_3, B, \mathbb{Z}_3 \times B, \mathbb{F}_4 \times B, \mathbb{Z}_4, \frac{\mathbb{Z}_2[t]}{(t^2)}, \mathbb{Z}_4 \times B, \frac{\mathbb{Z}_2[t]}{(t^2)} \times B,$$

where B is a finite Boolean ring.

A natural question is to characterize the rings whose unit graphs are planar. This question is settled in this paper. We denote by $\mathrm{char}(R)$ the characteristic of a ring R and by $|X|$ the cardinal of a set X. Let $\mathbf{c} = |\mathbb{R}|$ be the cardinality of the continuum. Our main result is the following characterization of rings with planar unit graphs.

Theorem 1.2. *Let R be a ring. Then $G(R)$ is planar if and only if one of the following holds:*

(1) $|U(R)| \leq 3$ *and* $|R| \leq \mathbf{c}$.

(2) $|U(R)| = 4$, $\mathrm{char}(R) = 0$ *and* $|R| \leq \mathbf{c}$.

(3) $R \cong \mathbb{Z}_5$.

(4) $R \cong \mathbb{Z}_3 \times \mathbb{Z}_3$.

2. The proof

We proceed with a series of lemmas. The first one is a quick consequence of Theorem 1.1.

Lemma 2.1. *Let R be a finite commutative ring. If $G(R)$ is planar, then* $2 \leq \mathrm{char}(R) \leq 6$. *Furthermore,*

(1) *If* $\mathrm{char}(R) = 2$, *then* $|U(R)| \leq 3$.

(2) *If* $\mathrm{char}(R) = 3$, *then* $|U(R)| \leq 4$.

(3) *If* $\mathrm{char}(R) = 4$, *then* $|U(R)| \leq 2$.

(4) *If* $\mathrm{char}(R) = 5$, *then* $|U(R)| \leq 4$.

(5) *If* $\mathrm{char}(R) = 6$, *then* $|U(R)| \leq 2$.

Let G be a simple graph. For a vertex v in G, the degree of v is the number of edges of G incident with v. For an integer $k > 0$, the graph G is called k-regular if the degree of each vertex of G is equal to k. The next lemma was proved in [2, Proposition 2.4] for a finite ring R and it can be shown by the same argument there.

Lemma 2.2. *Let R be a ring with $|U(R)| = k < \infty$. If $2 \notin U(R)$, then $G(R)$ is k-regular.*

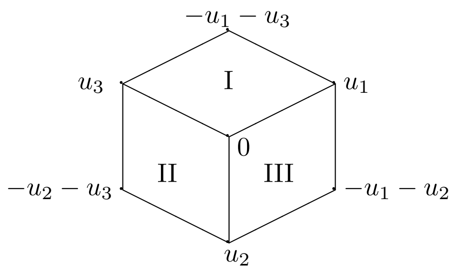
Let $K_{m,n}$ and K_n denote the complete bipartite graph with partitions of size m and n, and the complete graph of n vertices, respectively. A classical result of Kuratowski says that a graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$ (see [11, Theorem 6.2.2]), where a subdivision of a graph G is a graph obtained from G by subdividing some of the edges, that is, by replacing the edges by paths having at most their endvertices in common. A quick consequence of Kuratowski's Theorem is that if the maximal degree of a graph is less than 3, then this graph must be planar. If a planar graph is finite, then the minimal degree of vertex is at most five. For an infinite graph, however, the situation is quite different. In fact, there exists a k-regular planar infinite graph for any positive integer k (see [4]). Of course, any subgraph of a planar graph is clearly planar.

Lemma 2.3. *Let R be a ring. If $G(R)$ is planar, then $|U(R)| < \infty$.*

Proof. Assume on the contrary that $|U(R)| = \infty$. Take $u_1 \in U(R)$ and $u_2 \in U(R) \backslash \{u_1, -u_1\}$. We show next that there is a contradiction.

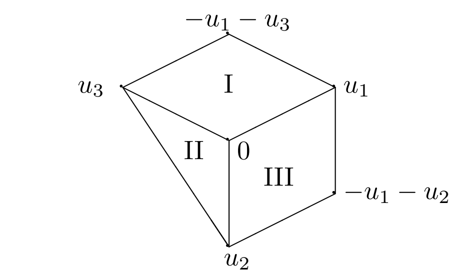
Case 1: $u_1 \neq -u_1 - u_2 \neq u_2$. In this case, we take $u_3 \in U(R) \backslash \{u_1, u_2, -u_1, -u_2, -u_1 - u_2\}$.

Subcase 1.1: $u_1 \neq -u_1 - u_3 \neq u_3$ and $u_2 \neq -u_2 - u_3 \neq u_3$. Then the following graph is a subgraph of $G(R)$:



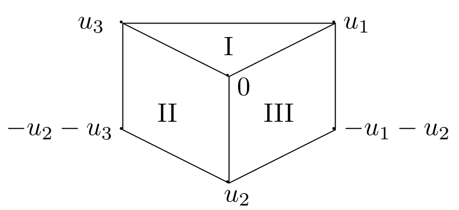
Now, take $v \in U(R) \backslash S$, where $S = \{u_1, u_2, u_3, -u_1, -u_2, -u_3, -u_1 - u_2, -u_1 - u_3, -u_2 - u_3, u_1 + u_2 - u_3, u_1 + u_3 - u_2, u_2 + u_3 - u_1\}$. Since $G(R)$ is planar and v is adjacent to 0, v must be in one of the regions (I), (II) and (III). Without loss of generality, put v into region (I). Note that $-v - u_2$ is adjacent to both v and u_2. As $G(R)$ is planar, $-v - u_2$ must be one of the vertices 0, u_1, u_3, $-u_1 - u_3$. But this contradicts the choice of v.

Subcase 1.2: $u_1 \neq -u_1 - u_3 \neq u_3$ and $-u_2 - u_3 = u_2$ or u_3. Then the following graph is a subgraph of $G(R)$:



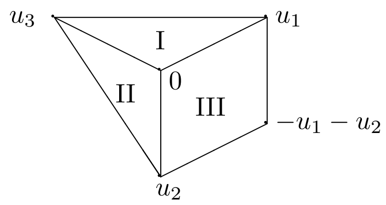
Now, take $v \in U(R)\backslash S$, where $S = \{u_1, u_2, u_3, -u_1, -u_2, -u_3, -u_1 - u_2, -u_1 - u_3, u_1 + u_2 - u_3, u_1 + u_3 - u_2\}$. Since $G(R)$ is planar and v is adjacent to 0, v must be in one of the regions (I), (II) and (III). Without loss of generality, put v into region (I). Note that $-v - u_2$ is adjacent to both v and u_2. As $G(R)$ is planar, $-v - u_2$ must be one of the vertices 0, u_1, u_3, $-u_1 - u_3$. But this contradicts the choice of v.

Subcase 1.3: $-u_1 - u_3 = u_1$ or u_3, and $u_2 \neq -u_2 - u_3 \neq u_3$. Then the following graph is a subgraph of $G(R)$:



Now, take $v \in U(R)\backslash S$, where $S = \{u_1, u_2, u_3, -u_1, -u_2, -u_3, -u_1 - u_2, -u_2 - u_3, u_1 + u_2 - u_3, u_2 + u_3 - u_1\}$. Since $G(R)$ is planar and v is adjacent to 0, v must be in one of the regions (I), (II) and (III). Without loss of generality, put v into region (I). Note that $-v - u_2$ is adjacent to both v and u_2. As $G(R)$ is planar, $-v - u_2$ must be one of the vertices 0, u_1, u_3. But this contradicts the choice of v.

Subcase 1.4: $-u_1 - u_3 = u_1$ or u_3, and $-u_2 - u_3 = u_2$ or u_3 (of course, it can't occur that $-u_1 - u_3 = u_3$ and $-u_2 - u_3 = u_3$). Then the following graph is a subgraph of $G(R)$:



Now, take $v \in U(R)\backslash S$, where $S = \{u_1, u_2, u_3, -u_1, -u_2, -u_3, -u_1 - u_2, u_1 + u_2 - u_3\}$. Since $G(R)$ is planar and v is adjacent to 0, v must be in one of the regions (I), (II) and (III). Without loss of generality, put v into region (I). Note that $-v - u_2$ is adjacent to both v and u_2. As $G(R)$ is planar, $-v - u_2$ must be one of the vertices 0, u_1, u_3. But this contradicts the choice of v.

Case 2: $-u_1 - u_2 = u_1$ or u_2. Take $u_3 \in U(R)\backslash\{u_1, u_2, -u_1, -u_2\}$. A similar argument as in Case 1 yields a contradiction. □

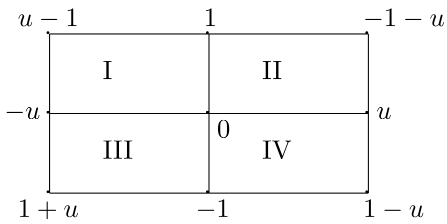
Lemma 2.4 is a self-strengthening of Lemma 2.3.

Lemma 2.4. *Let R be a ring. If $G(R)$ is planar, then $|U(R)| \leq 4$.*

Proof. Assume on the contrary that $|U(R)| \geq 5$. To get a contradiction, we proceed with two cases.

Case 1: $\mathrm{char}(R) = 0$. Then R contains $\mathbb{Z}$ as a subring. Since $|U(R)| < \infty$ by Lemma 2.3, $n \notin U(R)$ for all $\pm 1 \neq n \in \mathbb{Z}$. Take $\pm 1 \neq u \in U(R)$.

Subcase 1.1: $2u \neq -2$ and $2u \neq 2$. That is, $-1 - u \neq 1 + u$ and $u - 1 \neq 1 - u$. In this case, the following graph is a subgraph of $G(R)$:



Now, take $v \in U(R)\backslash\{1, -1, u, -u\}$. Since $G(R)$ is planar and v is adjacent to 0, either v is in one of the regions (I), (II), (III) and (IV), or v is one of the vertices $u - 1$, $-1 - u$, $1 - u$ and $1 + u$.

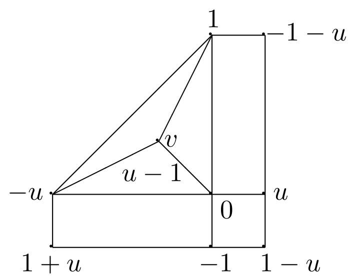
If v is in region (I), consider the vertices $1 - v$ and $-u - v$. As $1 - v$ is adjacent to both v and -1, we have $1-v=-u$ or $1-v=u-1$. As $-u-v$ is adjacent to both v and u, we have $-u-v=1$ or $-u-v=u-1$. Thus, we must have a contradiction: If $1-v=-u$ and $-u-v=1$, then $2v=0$, i.e. $2=0$; If $1-v=u-1$ and $-u-v=1$, then $3=0$; If $1-v=-u$ and $-u-v=u-1$, then $3u=0$, i.e. $3=0$; If $1-v=u-1$ and $-u-v=u-1$, then $u=-1$.

If v is in region (II), consider the vertices $1-v$ and $u-v$. Arguing as above, we have $1-v=-1-u$ or $1-v=u$, and $u-v=1$ or $u-v=-1-u$. This clearly leads to a contradiction.

If v is in region (III), consider the vertices $-1-v$ and $-u-v$. Then we have $-1-v=-u$ or $-1-v=1+u$, and $-u-v=1+u$ or $-u-v=-1$. This also leads to a contradiction.

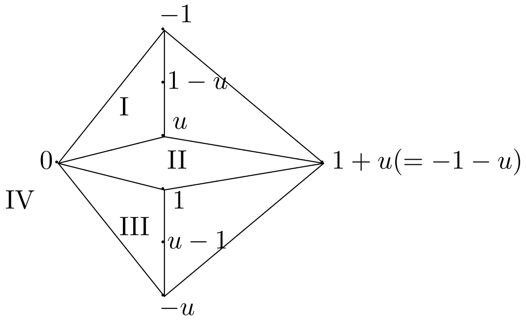
If v is region in (IV), consider the vertices $-1-v$ and $u-v$. Then we have $-1-v=u$ or $-1-v=1-u$, and $u-v=-1$ or $u-v=1-u$, and this also leads to a contradiction.

If v is one of the vertices $u-1$, $-1-u$, $1-u$ and $1+u$, we can assume that $v=u-1$ (the other cases are similar). Note that 1 is adjacent to $-u$. So we have the following subgraph of $G(R)$:



As $-u-v$ is adjacent to both v and u, we must have $-u-v=1$. As $1-v$ is adjacent to both v and -1, we must have $1-v=-u$. Thus, $2v=0$, i.e. $2=0$, a contradiction.

Subcase 1.2: $2u = -2$, i.e. $-1 - u = 1 + u$. In this case, $u - 1 \neq 1 - u$, so the following graph is a subgraph of $G(R)$:



Take $v \in U(R)\backslash\{1, -1, u, -u\}$. Then either v is in one of the regions (I), (II), (III) and (IV) or $v \in \{1-u, u-1, 1+u\}$.

If v is in region (I), consider the vertices $-1 - v$ and $u - v$. As $-1 - v$ is adjacent to both v and 1, we have $-1 - v = u$. As $u - v$ is adjacent to both v and $-u$, we have $u - v = -1$. It follows that $-2v = 0$, i.e. $2 = 0$, a contradiction.

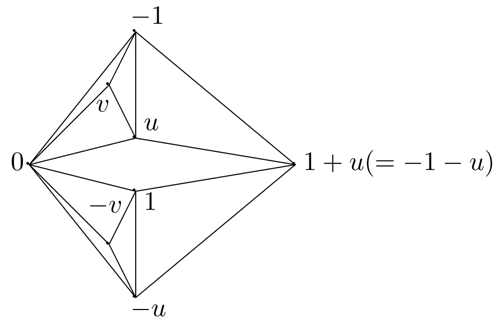
If v is in region (II), consider the vertices $1 - v$ and $u - v$. Arguing as above, we have $1 - v = u$ and $u - v = 1$, which gives $-2v = 0$, i.e. $v = 0$, a contradiction.

If v is in region (III), consider the vertices $1 - v$ and $-u - v$ and we have $1 - v = -u$ and $-u - v = 1$, giving $-2v = 0$, i.e. $v = 0$, a contradiction.

If v is in region (IV), consider the vertices $-1 - v$ and $-u - v$ and we have $-1 - v = -u$ and $-u - v = -1$, giving $-2v = 0$, i.e. $v = 0$, a contradiction.

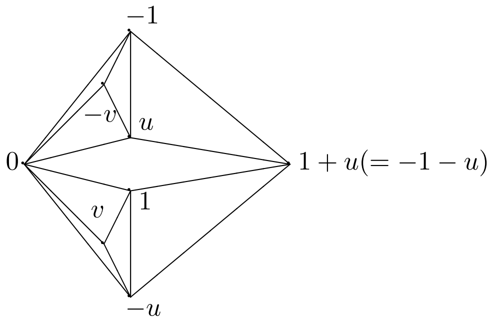
Now assume $v \in \{1-u, u-1, 1+u\}$. If $v = 1+u$, then 0 is adjacent to $1+u$ and 1 is adjacent to u. This is impossible.

If $v = 1 - u$, then $G(R)$ has the following subgraph:



In this case, we consider the vertices $-1 - v$ and $u - v$. As $-1 - v$ is adjacent to both 1 and v, we have $-1-v=u$; as $u-v$ is adjacent to both $-u$ and v, we have $u-v=-1$. So $-2v=0$, i.e. $2=0$, a contradiction.

If $v=u-1$, $G(R)$ has the following subgraph:



In this case, we consider the vertices $1-v$ and $-u-v$. As $1-v$ is adjacent to both -1 and v, we have $1-v=-u$; as $-u-v$ is adjacent to both u and v, we have $-u-v=1$. So $-2v=0$, i.e. $2=0$, a contradiction.

Subcase 1.3: $2u=2$, i.e. $u-1=1-u$. In this case, $-1-u\neq 1+u$. By a similar process as Subcase 1.2, we also can get a contradiction.

Case 2: $\mathrm{char}(R)=n\geq 2$. Then R contains $\mathbb{Z}_n$ as a subring. Since $G(\mathbb{Z}_n)$ is planar, we have $n\leq 6$ by Lemma 2.1. We need two notations. For any $a\in R$, let $\mathbb{Z}_n[a]$ be the subring of R generated by $\mathbb{Z}_n\cup\{a\}$. Note that $G(\mathbb{Z}_n[a])$ is also planar. For $u\in U(R)$, let $o(u)$ be the order of u in the multiplicative group $U(R)$. Then $o(u)<\infty$ for all $u\in U(R)$ by Lemma 2.3.

Subcase 2.1: $n=6$. Take $\pm 1\neq u\in U(R)$. As $o(u)<\infty$, $\mathbb{Z}_6[u]$ is a finite commutative ring. So, by Lemma 2.1(5), $|U(\mathbb{Z}_6[u])|\leq 2$. But $\mathbb{Z}_6[u]$ has at least three units, a contradiction.

Subcase 2.2: $n=5$. Take $u\in U(R)\backslash U(\mathbb{Z}_5)$. Then $\mathbb{Z}_5[u]$ is a finite commutative subring of R. So, by Lemma 2.1(4), $|U(\mathbb{Z}_5[u])|\leq 4$. But $\mathbb{Z}_5[u]$ has at least five units, a contradiction.

Subcase 2.3: $n=4$. Take $\pm 1\neq u\in U(R)$. Then $\mathbb{Z}_4[u]$ is a finite commutative subring of R. So, by Lemma 2.1(3), $|U(\mathbb{Z}_4[u])|\leq 2$. But $\mathbb{Z}_4[u]$ has at least three units, a contradiction.

Subcase 2.4: $n=3$. Take $\pm 1\neq u\in U(R)$. As above, $\mathbb{Z}_3[u]$ is a finite commutative subring of R. So, by Lemma 2.1(2), we have $|U(\mathbb{Z}_3[u])|\leq 4$. In particular, $o(u)\leq 4$. If $o(u)=4$ and $u^2=-1$, then $\mathbb{Z}_3[u]$ contains at least 8 units: 1, -1, u, $-u$, $1+u$, $1-u$, $-1+u$ and $-1-u$, a contradiction. If $o(u)=4$ and $u^2\neq -1$, then 1, 2, u, u^2, u^3 are five distinct units of $\mathbb{Z}_3[u]$, a contradiction. If $o(u)=3$, then 1, 2, u, $2u$, u^2, $2u^2$ are six distinct units of $\mathbb{Z}_3[u]$, a contradiction.

Hence $o(u) = 2$, and in this case, $U(\mathbb{Z}_3[u]) = \{1, 2, u, 2u\}$. Note that the argument above already shows that $v^2 = 1$ for all $v \in U(R)$. So the group $U(R)$ is abelian. As $|U(R)| \geq 5$, take $v \in U(R)\backslash U(\mathbb{Z}_3[u])$. Consider the subring $\mathbb{Z}_3[u, v]$ of R generated by $\mathbb{Z}_3[u] \cup \{v\}$. Then $\mathbb{Z}_3[u, v]$ is a finite commutative ring containing at least 5 units: 1, 2, u, $2u$, v. This contradicts Lemma 2.1(2).

Subcase 2.5: $n = 2$. Let $H = U(R)$. For $u \in H$, $\mathbb{Z}_2[u]$ is a finite commutative ring. So, by Lemma 2.1(1), we have $|U(\mathbb{Z}_2[u])| \leq 3$. In particular, $o(u) \leq 3$. Thus, we have proved that $o(u) \leq 3$ for all $u \in H$.

If $H \cong S_3$, the symmetric group of degree 3, then the subring $\mathbb{Z}_2[H]$ of R generated by $\mathbb{Z}_2 \cup H$ is a finite ring containing exactly six units such that 2 is not a unit of $\mathbb{Z}_2[H]$. Hence, by [2, Proposition 2.4], $G(\mathbb{Z}_2[H])$ is 6-regular. In particular, $G(\mathbb{Z}_2[H])$ is not planar, and so $G(R)$ is not planar. This contradiction shows that H is not isomorphic to S_3. To finish the proof, we need the following claim. □

Claim. There exist $u, v \in H\backslash\{1\}$ such that $uv = vu$ and $\langle u\rangle \cap \langle v\rangle = \{1\}$.

Proof of Claim. As above, we have $|H| = 2^k 3^l$, where $k, l \geq 0$. Note that $|H| \geq 5$ by hypothesis. If $k = 0$ or $l = 0$, there is nothing to prove because any finite p-group has nontrivial center. If $k > 1$, consider a Sylow 2-subgroup P of H. Being a finite p-group, P contains a non-trivial central element, say u. As $|\langle u\rangle| \leq 3$ and $|P| \geq 2^k \geq 4$, we can take $v \in P\backslash\langle u\rangle$. Then $uv = vu$ and $\langle u\rangle \cap \langle v\rangle = \{1\}$. If $l > 1$, we can consider a Sylow 3-subgroup and a similar argument also shows the existence of such elements u and v. If $k = l = 1$, then $|H| = 6$. As $H \ncong S_3$, H is a cyclic group of order 6. But this is impossible, as every element of H has order less than or equal to 3. The completes the proof of Claim.

Now by the Claim, take $u, v \in H\backslash\{1\}$ such that $uv = vu$ and $\langle u\rangle \cap \langle v\rangle = \{1\}$. Then the subring $\mathbb{Z}_2[u, v]$ of R generated by $\mathbb{Z}_2 \cup \{u, v\}$ is a finite commutative ring, containing at least four distinct units 1, u, v, uv. This contradicts Lemma 2.1(1).

The proof is now complete. □

Our last lemma is about the genus of a simple graph. A surface is said to be of genus g if it is topologically homeomorphic to a sphere with g handles. A graph G that can be drawn without crossing on a compact surface of genus g, but not on one of genus $g - 1$, is called a graph of genus g. The genus of a graph G is denoted by $\gamma(G)$. Not that a graph is planar if and only if it has genus zero.

Lemma 2.5 ([12, Corollaries 6.14, 6.15])**.** *Suppose that a simple graph G is connected with $p \geq 3$ vertices and q edges. Then $\gamma(G) \geq \frac{q}{6} - \frac{p}{2} + 1$. Furthermore, if G has no triangles, then $\gamma(G) \geq \frac{q}{4} - \frac{p}{2} + 1$.* □

Now we are ready to prove our main result.

Proof of Theorem 1.2. ($\Longrightarrow$). Suppose that $G(R)$ is planar. Then R embeds in $\mathbb{R}\times\mathbb{R}$ as sets, so $|R|\le \mathbf{c}$. By Lemma 2.4, $|U(R)|\le 4$. If $|U(R)|=3$, we are done. So we can assume that $|U(R)|=4$, and we can further assume $n:=\mathrm{char}(R)>0$. Then R contains $\mathbb{Z}_n$ as a subring. Being a subgraph of $G(R)$, $G(\mathbb{Z}_n)$ is planar, so $2\le n\le 6$ by Lemma 2.1. Take $\pm1\ne u\in U(R)$. Then $\mathbb{Z}_n[u]$ is a finite commutative subring of R, and hence $G(\mathbb{Z}_n[u])$ is planar. If $n=4$ or $n=6$, then $\mathbb{Z}_n[u]$ contains at least three units; this is impossible by Lemma 2.1(3,4). So $n\ne4$ and $n\ne6$. Next we prove that $n\ne2$. Assume that $n=2$. Then, for any $1\ne u\in U(R)$, $\mathbb{Z}_2[u]$ is a finite commutative subring of R, and hence $o(u)\le3$ by Lemma 2.1(1). If $o(u)=3$, take $v\in U(R)\backslash\{1,u,u^2\}$ and we see 1, u, u^2, v, uv are five distinct units of R, contradicting that $|U(R)|=4$. Hence $o(u)\le2$ for all $u\in U(R)$. So $U(R)$ is a commutative multiplicative group. Take $1\ne u\in U(R)$ and $v\in U(R)\backslash\{1,u\}$. Then $\mathbb{Z}_2[u,v]$ is a finite commutative subring of R containing four units 1, u, v, uv. But this is impossible by Lemma 2.1(1). Hence $n\ne2$. Thus, we have proved that $n=3$ or $n=5$.

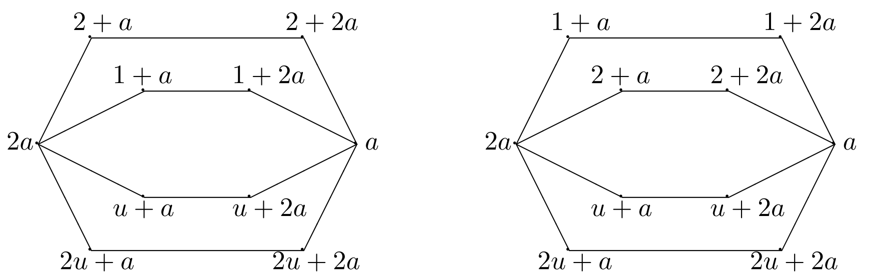
Suppose $n=3$. We prove that $R\cong\mathbb{Z}_3\times\mathbb{Z}_3$. Take $\pm1\ne u\in U(R)$. Then $\mathbb{Z}_3[u]$ is a finite commutative subring of R, and $U(\mathbb{Z}_3[u])=\{1,2,u,2u\}$ (as $|U(R)|=4$). If $R\ne\mathbb{Z}_3[u]$, take $a\in R\backslash\mathbb{Z}_3[u]$ and consider the subring $\mathbb{Z}_3[u,a]$ of R generated by $\mathbb{Z}_3\cup\{u,a\}$. Note that

$$a\longleftrightarrow 1+2a\longleftrightarrow 1+a\longleftrightarrow 2a\longleftrightarrow u+a\longleftrightarrow u+2a\longleftrightarrow a$$

and

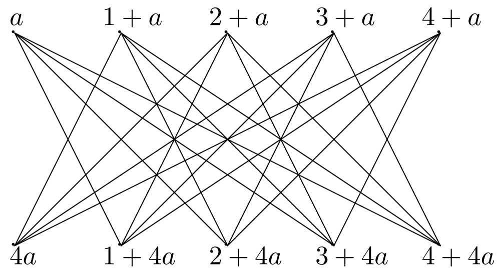
$$a\longleftrightarrow 2+2a\longleftrightarrow 2+a\longleftrightarrow 2a\longleftrightarrow 2u+a\longleftrightarrow 2u+2a\longleftrightarrow a$$

are two 6-cycles in $G(\mathbb{Z}_3[u,a])$. By symmetry, essentially there are two ways to draw the subgraph below:



For the subgraph on the left, as $u+2+2a$ is adjacent to both $1+a$ and $2u+a$, the planarity of $G(R)$ ensures that $u+2+2a=a$. On the other hand, as $u+2+a$ is adjacent to both $1+2a$ and $2u+2a$, the planarity of $G(R)$ ensures that $u+2+a=2a$. So, it follows that $a=-a$, i.e., $2a=0$ or $a=0$, a contradiction. For the subgraph on the right, as $u+1+2a$ is adjacent to both $2+a$ and $2u+a$, the planarity of $G(R)$ ensures that $u+1+2a=a$. On the other hand, as $u+1+a$ is adjacent to both $2+2a$ and $2u+2a$, the planarity of $G(R)$ ensures that $u+1+a=2a$. So, it follows that $a=-a$, i.e., $2a=0$ or $a=0$, a contradiction. Therefore, $R=\mathbb{Z}_3[u]$ with $\mathbb{Z}_3[u]\cong\mathbb{Z}_3\times\mathbb{Z}_3$.

Suppose $n=5$. We prove that $R\cong\mathbb{Z}_5$. We see that R contains $\mathbb{Z}_5$ as a subring. Assume on the contrary that $R\neq\mathbb{Z}_5$. Take $a\in R\backslash\mathbb{Z}_5$. The following graph H is a subgraph of $G(\mathbb{Z}_5[a])$, and hence of $G(R)$:



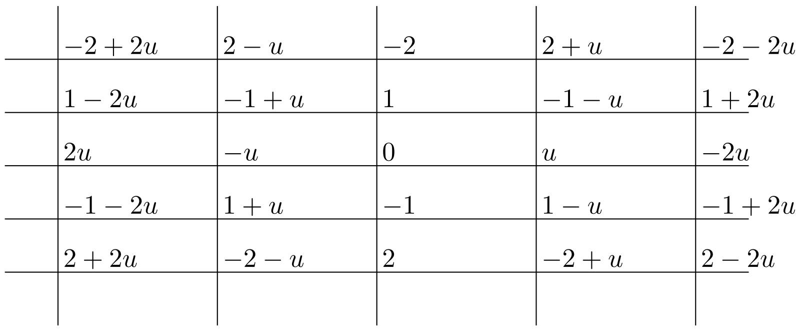
Note that H has 10 vertices and 20 edges with no triangles. So $\gamma(H)\geq 1$ by Lemma 2.5. This shows that H is not planar, giving the contradiction that $G(R)$ is not planar.

($\Longleftarrow$). We have $|R|\leq\mathbf{c}$. If $R\cong\mathbb{Z}_5$ or $R\cong\mathbb{Z}_3\times\mathbb{Z}_3$, then $G(R)$ is planar by Theorem 1.1. If $|U(R)|\leq 2$, then the maximal degree of $G(R)$ is at most two, so $G(R)$ must be planar.

Suppose that $|U(R)|=3$. Then we easily see that $2=0$ in R. So $G(R)$ is 3-regular by Lemma 2.2. Let $U(R)=\{u_1,u_2,u_3\}$. For a given $r\in R$, r is adjacent to u_i-r $(i=1,2,3)$. If u_1-r is adjacent to one of u_i-r $(i=2,3)$, say u_2-r, then $(u_1-r)+(u_2-r)=u_1+u_2$ is a unit of R, so it must be that $u_1+u_2=u_3$. Thus u_1-r is also adjacent to u_3-r and u_2-r is adjacent to u_3-r. Hence, the vertices r, u_1-r, u_2-r, u_3-r form a complete graph K_4. As $G(R)$ is 3-regular, $G(R)$ must be a disjoint union of copies of K_4, so $G(R)$ is planar. Therefore, we can let the neighborhoods of u_1-r be r, a, b, where $a,b\notin\{u_2-r,u_3-r\}$. We may assume $u_1-r+a=u_2$ and $u_1-r+b=u_3$. Then $u_2-r+a=u_1$ and $u_3-r+b=u_1$. This means that a is adjacent to u_2-r

and b is adjacent to $u_3 - r$. Let c be the third neighborhood of $u_2 - r$. Then $u_2 - r + c = u_3$, so $u_3 - r + c = u_2$. This means that c is also a neighborhood of $u_3 - r$. Now consider the vertex a. Let the neighborhoods of a be $u_1 - r$, $u_2 - r$, x. Then $a + x = u_3$. As $b + x = b + u_3 - a = r + u_1 - a = u_1 - r + a = u_2$, x is adjacent to b. Similarly, x is adjacent to c. So, the vertices r, $u_1 - r$, $u_2 - r$, $u_3 - r$, a, b, c and x form a cube, which is 3-regular. As $G(R)$ is 3-regular, $G(R)$ must be a disjoint union of copies of a cube. As a cube is a planar graph, $G(R)$ is planar.

Finally, suppose that $|U(R)| = 4$ and $\text{char}(R) = 0$. Then R contains $\mathbb{Z}$ as a subring. Take $\pm 1 \neq u \in U(R)$. As $|U(R)| = 4$, we have $U(R) = \{1, -1, u, -u\}$. By Lemma 2.2, both $G(\mathbb{Z}[u])$ and $G(R)$ are 4-regular. It follows that $G(R)$ is a disjoint union of $G(\mathbb{Z}[u])$. As shown below, $G(\mathbb{Z}[u])$ is planar, so $G(R)$ is planar.



Graph $G(\mathbb{Z}[u])$

$\square$

We end the paper by giving some examples of rings with planar unit graphs.

Example 2.6. *Let $\mathbb{T}_2(\mathbb{Z}_2)$ be the 2×2 upper triangular matrix ring over $\mathbb{Z}_2$ and let B be the zero ring or a finite Boolean ring. Then $R = \mathbb{T}_2(\mathbb{Z}_2) \times B$ has a planar unit graph.*

A ring R is semilocal if $R/J(R)$ is semisimple Artinian, where $J(R)$ is the Jacobson radical of R. The next example gives a countable non-semilocal ring whose unit graph is planar. Let D be a ring and C be a subring of D. With addition and multiplication defined componentwise, $\mathcal{R}[D, C] := \{(d_1, \cdots, d_n, c, c, \cdots) : d_i \in D, c \in C, n \geq 1\}$ becomes a ring. For a bimodule M over a ring R, the trivial extension of R by M is the ring $R \propto M := \{(a, x) : a \in R, x \in M\}$ with addition defined componentwise and with multiplication defined by $(a, x)(b, y) = (ab, ay + xb)$.

Example 2.7. *Let $S = R \propto R/I$ where $R = \mathcal{R}[\mathbb{Z}_2, \mathbb{Z}_2]$ and $I = \mathcal{R}[\mathbb{Z}_2, 0]$. Then S is not semilocal, but $G(S)$ is planar.*

Proof. We easily see that $J(S) = \{(0, x) : x \in R/I\}$, so $|J(S)| = |R/I| = 2$, and $S/J(S) \cong R$ is Boolean. Since $S/J(S)$ is an infinite Boolean ring, S is not semilocal. As $|U(S)| = 2$, $G(S)$ is planar by Theorem 1.2. □

Some other examples of rings with planar unit graphs can be constructed through polynomial rings. In [1], the authors determined the finite rings R with $G(R[t])$ planar. By Theorem 1.2, we now can characterize the rings R with $G(R[t])$ planar. Remark that, for a reduced ring R, $U(R[t]) = U(R)$ (we can't find a reference for this, but it can be easily proved).

Corollary 2.8. *Let R be a ring, and let $t_1, t_2, \ldots, t_n$ be commuting indeterminates over R. Then $G(R[t_1, t_2, \ldots, t_n])$ is planar if and only if R is reduced with $|R| \leq \mathbf{c}$ such that either $|U(R)| \leq 3$, or $|U(R)| = 4$ with $\mathrm{char}(R) = 0$.*

Proof. Without loss of generality, we can assume that $n = 1$.

($\Longleftarrow$). This is by Theorem 1.2 and the Remark above.

($\Longrightarrow$). As $G(R[t])$ is planar, R is reduced by [1, Proposition 6.1(ii)], and $|R[t]| \leq \mathbf{c}$. So $|R| \leq \mathbf{c}$. Moreover, by Theorem 1.2, either $|U(R[t])| \leq 3$, or $|U(R[t])| = 4$ with $\mathrm{char}(R) = 0$. Since R is reduced, $U(R[t]) = U(R)$. So the claim follows. □

Acknowledgments. We thank the anonymous referee for a very careful reading of the paper, and for his/her many valuable comments which improved the paper. This work was supported by a Discovery Grant from NSERC of Canada. The first two authors were also grateful for the support from the National Natural Science Foundation of China (11161006,11461010), the Guangxi Natural Sciences Foundation (2014GXNSFAA118005) and the Scientific Research Foundation of Guangxi Educational Committee(LX2014223).

References

[1] M. Afkhami and F. Khosh-Ahang, Unit graphs of rings of polynomials and power series, *Arabian J. Math.* **2** (2013), 233–246.

[2] N. Ashrafi, H. R. Maimani, M. R. Pournaki and S. Yassemi, Unit graphs associated with rings, *Comm. Algebra* **38** (2010), 2851–2871.

[3] A. K. Das, H. R. Maimani, M. R. Pournaki and S. Yassemi, Nonplanarity of unit graphs and classification of the toroidal ones, *Pacific J. Math.* **268** (2014), 371–387.

[4] A. Georgakopoulos, Infinite highly connected planar graphs of large girth, *Abh. Math. Sem. Univ. Hamburg* **76** (2006), 235–245.

[5] R. P. Grimaldi, Graphs from rings, Proceedings of the 20th Southeastern Conference on Combinatorics, Graph Theory, and Computing, (Boca Raton, FL, 1989), Congr. Numer. Vol. 71, pp. 95,-103, 1990.

[6] F. Heydari and M. J. Nikmehr, The unit graph of a left Artinian ring, *Acta Math. Hungar.* **139** (2013), 134–146.

[7] K. Khashyarmanesh and M. R. Khorsandi, A generalization of unit and unitary cayley graphs of a commutative ring, *Acta Math. Hungar.* **137** (2012), 242–253.

[8] H. R. Maimani, M. R. Pournaki and S. Yassemi, Necessary and sufficient conditions for unit graphs to be Hamiltonian, *Pacific J. Math.* **249** (2011), 419–429.

[9] H. Su, K. Noguchi and Y. Zhou, Finite commutative rings with higher genus unit graphs, *J. Algebra Appl.* **14** (2015), 1550002.

[10] H. Su and Y. Zhou, On the girth of the unit graph of a ring, *J. Algebra Appl.* **13** (2014), 1350082.

[11] D. B. West, Introduction to Graph Theory, second ed., *Prentice-Hall*, 2001.

[12] T. White, Graphs, Groups and Surfaces, 8, North-Holland Mathematics Studies, *North-Holland Publishing Co., Amsterdam*, 1984.

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(Received March 4, 2014; revised January 11, 2015)