

Sublattices of verbal subgroups

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Abstract. The problem of classification of group varieties is still open. We consider four classes of verbal subgroups of a free group F of rank 2: $\{VN\text{-verbal}\} \subseteq \{P\text{-verbal}\} \subseteq \{R\text{-verbal}\} \subseteq \{M\text{-verbal}\}$. The subgroups in each class define specific properties in corresponding varieties, namely, VN -varieties have their 2-generator groups virtually nilpotent; P -varieties satisfy positive laws; R -varieties are restrained; and M -varieties contain no $\mathfrak{A}_p\mathfrak{A}$ as a subvariety. It is shown that each of these classes of verbal subgroups forms a sublattice of the lattice of subgroups in F . Three questions are posed.

1. Introduction

The problem of classification of group varieties attracted attention of many authors. We make a step in this direction by distinguishing four sublattices of group varieties according to their properties defined by 2-variable laws they satisfy.

Let $F = \langle x, y \rangle$ be a free group of rank 2 and $\mathcal{F}$ a free semigroup on the set $\{x, y\}$. We denote $[x, y] = x^{-1}y^{-1}xy$ and $x^y = y^{-1}xy$. The normal closure of $\langle x \rangle$ in F may appear denoted as one of the following

$$\langle x^F \rangle = \langle x^{\langle y \rangle} \rangle = \langle x^{y^i} : i \in \mathbb{Z} \rangle.$$

By V we denote any verbal subgroup in F and by $\hat{F}^n$ – the verbal subgroup defining the variety of locally finite groups of exponent dividing n . The fact that the class of these groups is actually a variety is a consequence of Zelmanov's solution of the restricted Burnside problem. Writing $\gamma_1(F) = F$, $\gamma_c = [\gamma_{c-1}(F), F]$

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for $c > 1$, we have that F/V is virtually nilpotent if and only if $V \supseteq \gamma_c(\hat{F}^k)$ for some $c, k \in \mathbb{N}$. We call such a verbal subgroup V a VN -verbal subgroup. The set of the VN -verbal subgroups in F , denoted briefly by $\{VN\text{-verbal}\}$, forms a sublattice of the lattice of all subgroups because the inclusions $V_1 \supseteq \gamma_c(\hat{F}^k)$ and $V_2 \supseteq \gamma_d(\hat{F}^\ell)$ imply $V_1 \cap V_2 \supseteq \gamma_m(\hat{F}^n)$ for $m = \max(c, d)$, $n = lcm(k, \ell)$.

We show that each VN -verbal subgroup V has the following properties:

- P -property: $V \cap \mathcal{FF}^{-1} \neq 1$,
- R -property: $F'V/V$ is finitely generated,
- M -property: $V \not\subseteq F''(F')^p$.

To each of these properties there is associated the set of verbal subgroups satisfying it. We call these respectively

P -verbal, R -verbal, and M -verbal subgroups.

We denote corresponding sets of verbal subgroups respectively:

$$\{P\text{-verbal}\}, \{R\text{-verbal}\}, \{M\text{-verbal}\}.$$

They also determine three types of varieties $\text{var}(F/V)$:

- a P -variety: satisfies a positive law,
- an R -variety: G' is finitely generated for each two-generator group G in it,
- an M -variety: has no subvariety of the form $\mathfrak{A}_p\mathfrak{A}$ for any prime p .

The above property of the R -varieties is much stronger since every its finitely generated group G has G' finitely generated [8, Proposition 9]. This fact follows also from [1, Lemma 1] as

Proposition 1. *Let $\mathfrak{V}$ be an R -variety. Then every finitely generated group in $\mathfrak{V}$ has finitely generated commutator subgroup.*

We show that each of these properties is defined by a binary law. The following inclusions hold for the respective sets of verbal subgroups and corresponding types of varieties:

$$\begin{aligned} \{VN\text{-verbal}\} &\subseteq \{P\text{-verbal}\} \subseteq \{R\text{-verbal}\} \subseteq \{M\text{-verbal}\}, \\ \{VN\text{-varieties}\} &\subseteq \{P\text{-varieties}\} \subseteq \{R\text{-varieties}\} \subseteq \{M\text{-varieties}\}. \end{aligned}$$

Moreover every subset shown here is a sublattice of the lattice of all subgroups in F or the lattice of all varieties, as the case may be. We now discuss various inclusion problems.

2. P -verbal subgroups

Definition 1. A verbal subgroup V is called P -verbal if

$$V \cap \mathcal{F}\mathcal{F}^{-1} \neq 1.$$

Thus a verbal subgroup V is P -verbal if and only if the group F/V satisfies a positive law $u(x, y) \equiv v(x, y)$ for some words u and v in $\mathcal{F}$.

By result of A. I. MAL'TSEV [9], nilpotent groups and hence nilpotent-by-(finite exponent) groups satisfy positive laws, so we have the set inclusion

$$\{VN\text{-verbal}\} \subset \{P\text{-verbal}\}.$$

The inclusion is proper since the verbal subgroups defining infinite Burnside groups are P -verbal but not VN -verbal. Other examples of P -verbal but not VN -verbal subgroups are given by A. YU. OL'SHANSKII and A. STOROZHEV [12].

Theorem 1. *The set of P -verbal subgroups forms a sublattice in the lattice of all subgroups in F .*

PROOF. Let V_1 and V_2 be P -verbal subgroups defining in F/V_i for $i = 1, 2$ the following positive laws $a(x, y) \equiv b(x, y)$ and $u(x, y) \equiv v(x, y)$ respectively. Since every positive law implies a balanced positive law, we shall assume that the laws $a(x, y) \equiv b(x, y)$ and $u(x, y) \equiv v(x, y)$ are balanced, that is the exponent sum of x (of y) in $a(x, y)$ and in $b(x, y)$ (resp. in $u(x, y)$ and in $v(x, y)$) is the same.

The join $V_1 V_2$ provides each of these laws, so it suffices to show only that the intersection $V_1 \cap V_2$ yields a positive law. We consider the law

$$a(u(x, y), v(x, y)) \equiv b(u(x, y), v(x, y)). \quad (1)$$

This law is positive and by assumption on V_1 it is satisfied in F/V_1 . In the group F/V_2 the law (1) has a form $a(u, u) \equiv b(u, u)$ and hence $u^k \equiv u^k$ for some integer k since the law $a(x, y) \equiv b(x, y)$ is assumed to be balanced. Thus the law (1) is satisfied in F/V_2 and hence is satisfied modulo $V_1 \cap V_2$, which finishes the proof. $\square$

3. R -verbal subgroups

Definition 2. A verbal subgroup V is called R -verbal (R for restrained) if the commutator subgroup F' is finitely generated modulo V .

It follows from Proposition 1 that if V is R -verbal, then every finitely generated group in $\text{var}(F/V)$ has finitely generated commutator subgroup.

Corollary 1. *Every verbal subgroup $V \subseteq F$ such that $V \not\subseteq F'$ is the R -verbal subgroup.*

PROOF. It is known that F' is generated by commutators $[x^i, y^j]$, $i, j \in \mathbb{Z}$. If $V \not\subseteq F'$ then F/V has finite exponent, which implies that F' is finitely generated modulo V . $\square$

It is shown in [7, Corollary 6.4] that each positive law define R -verbal subgroup, so we have the inclusions

$$\{VN\text{-verbal}\} \subset \{P\text{-verbal}\} \subseteq \{R\text{-verbal}\}.$$

By the example below, n -Engel laws define R -verbal subgroups. These subgroups are P -verbal for $n < 5$ [15], so the question whether the second inclusion is strict is related to the Problem 2.82 in [5], asking whether each variety of groups satisfying n -Engel law $[x, {}_n y] = 1$ is defined by positive laws.

To prove that R -verbal subgroups form a lattice, we need to find an appropriate criterion for V to be R -verbal.

For $n \in \mathbb{N}$, we introduce an important subgroup E_n in $F = \langle x, y \rangle$, setting $E_0 = \langle x \rangle$, and for $n > 0$,

$$E_n := \langle x, x^y, x^{y^2}, \dots, x^{y^n} \rangle = \langle x, [x, y], [x, y^2], \dots, [x, y^n] \rangle \quad (2)$$

Lemma 1. *A verbal subgroup V is R -verbal if and only if the subgroup $\langle x^F \rangle$ is finitely generated modulo V .*

PROOF. By definition, V is R -verbal if F' is finitely generated modulo V . We prove that the latter holds if and only if $\langle x^F \rangle$ is finitely generated modulo V . Indeed, if F' is finitely generated modulo V then, since $\langle x \rangle F' = \langle x^F \rangle$, the ‘only if’ part follows.

Assume now that $\langle x^F \rangle$ is finitely generated modulo V . Since $\langle x^F \rangle$ is normal, the conjugation by suitable y^i implies that there is $n \in \mathbb{N}$ such that $\langle x^F \rangle$ coincides modulo V with E_{n-1} , which we write as $\langle x^F \rangle \stackrel{V}{\equiv} E_{n-1}$.

If denote $H := \langle [x, y], [x, y^2], \dots, [x, y^{n-1}] \rangle$, then in view of (2), $\langle x^F \rangle \stackrel{V}{\equiv} \langle x, H \rangle$ and then

$$\langle x^F \rangle \stackrel{V}{\equiv} \langle x \rangle H^{\langle x \rangle},$$

Now, since $H^{\langle x \rangle} \subseteq F' \subseteq \langle x^F \rangle$, we have by Dedekind’s law

$$\begin{aligned} F' \stackrel{V}{\equiv} F' \cap \langle x \rangle H^{\langle x \rangle} &= (F' \cap \langle x \rangle) H^{\langle x \rangle} = H^{\langle x \rangle} \\ &= \langle [x, y]^{\langle x \rangle}, [x, y^2]^{\langle x \rangle}, [x, y^3]^{\langle x \rangle}, \dots, [x, y^n]^{\langle x \rangle} \rangle. \end{aligned}$$

Since $\langle x^{\langle y \rangle} \rangle = \langle x^F \rangle$, the assumption that the subgroup $\langle x^F \rangle$ is finitely generated modulo V implies that $\langle x^{\langle y \rangle} \rangle$ is finitely generated and hence each subgroup of the form $\langle [x, y^i]^{\langle x \rangle} \rangle$ also is finitely generated modulo V . It follows that F' is finitely generated modulo V , which proves the ‘if’ part. $\square$

Lemma 2. *The subgroup $\langle x^F \rangle$ is finitely generated modulo V if and only if there exists $n \in \mathbb{N}$ such that*

$$[x, {}_n y] \in E_{n-1}V. \quad (3)$$

PROOF. Let $\langle x^F \rangle$ be finitely generated modulo V . Then, as above, the conjugation by suitable y^i implies that for some $n \in \mathbb{N}$, $\langle x^F \rangle \stackrel{V}{\equiv} E_{n-1}$. Hence $[x, {}_n y] \in F' \subseteq \langle x^F \rangle \subseteq E_{n-1}V$, which gives (3).

Conversely, let (3) hold. To show that $\langle x^F \rangle$ is finitely generated modulo V , it suffices to prove that

$$\langle x^F \rangle \stackrel{V}{\equiv} E_{n-1} = \langle x, x^y, \dots, x^{y^{n-1}} \rangle.$$

It is shown in [7, Corollary 5.4] that $E_n = \langle x, [x, y], \dots, [x, {}_n y] \rangle$. So if $[x, {}_n y] \in E_{n-1}V$ then by (2) $[x, y^n] \in E_{n-1}V$ and hence

$$x^{y^n} \in E_{n-1}V. \quad (4)$$

All inclusions below are meant modulo V , so we write (4) modulo V as:

$$x^{y^n} \in \langle x, x^y, x^{y^2}, \dots, x^{y^{n-1}} \rangle. \quad (5)$$

Substitution $y \rightarrow y^{-1}$ gives $x^{y^{-n}} \in \langle x, x^{y^{-1}}, x^{y^{-2}}, \dots, x^{y^{-(n-1)}} \rangle$. Now, conjugation by y^{n-1} gives $x^{y^{-1}} \in \langle x, x^y, x^{y^2}, \dots, x^{y^{n-1}} \rangle$ and by induction $x^{y^{-i}} \in \langle x, x^y, x^{y^2}, \dots, x^{y^{n-1}} \rangle \stackrel{(5)}{\subseteq} E_{n-1}$ for all $i > 0$.

Similarly, by conjugating (5) by y we obtain

$$x^{y^{n+1}} \in \langle x^y, x^{y^2}, \dots, x^{y^n} \rangle \stackrel{(5)}{\subseteq} E_{n-1}.$$

Repeated conjugation gives by induction $x^{y^i} \in E_{n-1}$ for all $i \geq 0$ and implies that $\langle x^F \rangle \stackrel{V}{\equiv} E_{n-1}$ is finitely generated, which finishes the proof. $\square$

By Lemmas 1, 2, the condition (3) allows us to formulate the following

Criterion. A verbal subgroup V is R -verbal if and only if

$$\exists n \in \mathbb{N}, \quad [x, {}_n y] \stackrel{V}{=} u(x, y), \quad u(x, y) \in E_{n-1}. \quad (6)$$

Example. If F/V satisfies the n -Engel law $[x, {}_n y] \equiv 1$ then V is R -verbal.

Using the above criterion we prove the following

Theorem 2. *The set of R -verbal subgroups forms a sublattice of the lattice of all subgroups of F .*

PROOF. Let U and V be the R -verbal subgroups of F . Then by (6) there exist $k, m \in \mathbb{N}$, and words $u(x, y) \in E_{k-1}$, $v(x, y) \in E_{m-1}$ such that

$$(i) \quad [x, {}_k y] \stackrel{U}{=} u(x, y), \quad (ii) \quad [x, {}_m y] \stackrel{V}{=} v(x, y). \quad (7)$$

It is clear that the join UV provides both of these laws, hence by (6), UV is R -verbal. We prove now that the intersection $U \cap V$ yields a law of the form (6), namely:

$$[x, {}_{k+m} y] \stackrel{U \cap V}{=} w(x, y), \quad \text{for some } w(x, y) \in E_{k+m-1}.$$

Construction of the law

In (7)(i) we put $[x, {}_m y]$ for x , and also in (7)(i) we put v for x to get the following two laws satisfied in F/U :

$$[x, {}_{k+m} y] \stackrel{U}{=} u([x, {}_m y], y) \quad \text{and} \quad [v, {}_k y] \stackrel{U}{=} u(v, y). \quad (8)$$

The laws (8) imply in F/U a law of the form $[x, {}_{k+m} y] \stackrel{U}{=} w(x, y)$:

$$[x, {}_{k+m} y] \stackrel{U}{=} u([x, {}_m y], y) \cdot \overbrace{(u(v, y))^{-1}}^{\in U} [v, {}_k y]. \quad (9)$$

The law (7)(ii) $[x, {}_m y] \stackrel{V}{=} v$ also implies two laws. For the first we take k -repeated commutator on both sides with y , and for the second we put each side of (7)(ii) for x into $u(x, y)$. So we get:

$$[x, {}_{m+k} y] \stackrel{V}{=} [v, {}_k y], \quad u([x, {}_m y], y) \stackrel{V}{=} u(v, y).$$

These two laws imply in F/V the law

$$[x, {}_{m+k} y] \stackrel{V}{=} \underbrace{u([x, {}_m y], y) \cdot (u(v, y))^{-1}}_{\in V} [v, {}_k y],$$

which coincides with (9), hence is satisfied modulo $U \cap V$, and has a form $[x, {}_{k+m}y] \equiv w(x, y)$. So to finish the proof we have to check that

$$u([x, {}_m y], y)(u(v, y))^{-1}[v, {}_k y] \in E_{k+m-1}.$$

We shall consider the factors in the order: $[v, {}_k y]$, $u(v, y)$, $u([x, {}_m y], y)$.

Since $v(x, y) \in E_{m-1} = \langle x, [x, y], [x, {}_2 y], \dots, [x, {}_{m-1} y] \rangle$, by means of the commutator identity $[ab, y] = b^{-1}[a, y]b[b, y]$, we conclude that

$$[v, {}_k y] \in E_{(m-1)+k}. \quad (10)$$

Since $u(x, y) \in E_{k-1}$, we get $u(v, y) \in \langle v, [v, y], [v, {}_2 y], \dots, [v, {}_{k-1} y] \rangle$. So by (10),

$$u(v, y) \in E_{(m-1)+(k-1)} \subseteq E_{m+k-1}.$$

For the third factor we have $u(x, y) \in E_{k-1}$ and hence $u([x, {}_m y], y) \in$

$$\langle [x, {}_m y], [[x, {}_m y], y], [[x, {}_m y], {}_2 y], \dots, [[x, {}_m y], {}_{k-1} y] \rangle \subseteq E_{m+k-1}.$$

Thus the law (9) of the form $[x, {}_{k+m}y] \stackrel{U \cap V}{\equiv} w$, has $w \in E_{k+m-1}$, and by (6) defines the R -verbal subgroup $U \cap V$, which finishes the proof. $\square$

4. M -verbal subgroups

Definition 3. A verbal subgroup $V \subseteq F$, for $F = \langle x, y \rangle$ is called M -verbal if for all primes p

$$V \not\subseteq F''(F')^p, \quad \text{i.e. } \text{var}(F/V) \not\preceq \mathfrak{A}_p \mathfrak{A}.$$

The name M -verbal is chosen because F/V satisfies so called Milnor identity defined by F. POINT [13], that is, a law not holding in any of the varieties $\mathfrak{A}_p \mathfrak{A}$ [14, Proposition 1.1].

Theorem 3. A verbal subgroup V is M -verbal if and only if it satisfies

$$VF'' \cap \mathcal{FF}^{-1} \neq 1, \quad (11)$$

that is, if and only if it yields a positive law in metabelian groups.

PROOF. By result of BELYAEV and SESEKIN [2] the wreath product $C_p wr C$ contains a free semigroup. Since $C_p wr C$ generates the product variety $\mathfrak{A}_p \mathfrak{A}$ of the variety of all abelian groups of exponent p by the variety of all abelian groups (see e.g. [10, 17.6 and Corollary 22.44]), it follows that the equality $F''(F')^p \cap \mathcal{F}\mathcal{F}^{-1} = 1$ holds for every prime p . Hence (11) implies $V \not\subseteq F''(F')^p$.

Conversely, by result of J. Groves [4, Theorem C (ii)], the group $G := F/VF''$ is either nilpotent-by-finite or $\text{var } G$ contains a subvariety $\mathfrak{A}_p \mathfrak{A}$ for some prime p . So if $V \not\subseteq F''(F')^p$ for all prime p , then F/VF'' must be virtually nilpotent, hence it satisfies a positive law and then (11) follows. $\square$

Now, each R -verbal subgroup V is M -verbal. Indeed, if V is R -verbal then $(F/V)'$ is finitely generated. Since $F'/F''(F')^p$ is infinitely generated $V \not\subseteq F''(F')^p$ and hence $\text{var}(F/V) \not\subseteq \mathfrak{A}_p \mathfrak{A}$.

$$\{VN\text{-verbal}\} \subset \{P\text{-verbal}\} \subseteq \{R\text{-verbal}\} \subseteq \{M\text{-verbal}\}.$$

If V defines a pseudo-abelian variety $\mathfrak{V}$, i.e. nonabelian variety in which every metabelian group is abelian, then $\mathfrak{V}$ does not contain any of $\mathfrak{A}_p \mathfrak{A}$ hence, by definition, V is M -verbal but need not be P -verbal. For example, it is shown in [6], that the pseudo-abelian relatively free groups F/V by A. YU. OL'SHANSKII [11] contain free non-abelian semigroups which do not satisfy any positive law, so that V is M -verbal but not P -verbal. Thus we have the following strict inclusions:

$$\{VN\text{-verbal}\} \subset \{P\text{-verbal}\} \subset \{M\text{-verbal}\}.$$

Theorem 4. *The set of M -verbal subgroups forms a sublattice of the subgroup lattice of F .*

PROOF. The property $VF'' \cap \mathcal{F}\mathcal{F}^{-1} \neq 1$ means that F/VF'' satisfies a positive law. Let V_1 and V_2 yield respectively the following positive laws modulo F''

$$a(x, y) \equiv b(x, y)f_1'' \text{ and } c(x, y) \equiv d(x, y)f_2'', \quad a, b, c, d \in \mathcal{F}, \quad f'' \in F''.$$

The join $V_1 V_2$ provides each of these laws. To speak of $V_1 \cap V_2$ we can assume that laws $a(x, y) \equiv b(x, y)$ and $c(x, y) \equiv d(x, y)$ are balanced. Now we consider the following law

$$a(c, df_2'') \equiv b(c, df_2'') \cdot f_1''(c, df_2''). \quad (12)$$

This law is positive modulo F'' , and by assumption, is satisfied modulo V_1 . By assumption on V_2 , there is $v_2 \in V_2$ such that $df_2'' = cv_2$. Then modulo V_2 , (12) has a form $a(c, c) \equiv b(c, c) \cdot f_1''(c, c)$, and since $a \equiv b$ is balanced, it is trivial modulo V_2 . So (12) is satisfied modulo $V_1 \cap V_2$. Hence by Theorem 3, the subgroup $V_1 \cap V_2$ is M -verbal, which finishes the proof. $\square$

Question 1. Which R -verbal subgroups are not P -verbal?

Question 2. Which M -verbal subgroups are not R -verbal?

A group is *locally graded* if every non-trivial finitely generated subgroup has a proper subgroup of finite index. Considering the class of verbal subgroups V for which F/V is locally graded, we infer from [3, Theorem B] that the properties of being VN -, P -, and R -verbal coincide, since every R -verbal subgroup with a locally graded F/V , is VN -verbal.

By [3, Theorem A] this also holds for M -verbal subgroups if F/V belongs to the smaller class $\mathcal{S}$ defined in [3].

Question 3. Is an M -verbal subgroup VN -verbal if F/V is locally graded?

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