Publ. Math. Debrecen
86/3-4 (2015), 465–471
DOI: 10.5486/PMD.2015.7101

A consequence of the ternary Goldbach theorem

By IMRE KÁTAI (Budapest) and BUI MINH PHONG (Budapest)

Dedicated to Professor János Aczél on his 90th anniversary

Abstract. Let

$$\mathcal{M}_k = \{p_1 + p_2 + \cdots + p_k \mid p_1, p_2, \ldots, p_k \in \mathcal{P}\},$$

where $\mathcal{P}$ is the set of primes. We proved that if an integer $k \geq 3$ and arithmetical functions f, g satisfy the functional equation

$$f(p_1 + p_2 + \cdots + p_k) = g(p_1) + g(p_2) + \cdots + g(p_k)$$

for all $p_1, p_2, \ldots, p_k \in \mathcal{P}$, then there are two constants A and B such that $f(n) = An + kB$ for all $n \in \mathcal{M}_k$ and $g(p) = Ap + B$ for all $p \in \mathcal{P}$.

1. In the following, let $\mathbb{N}, \mathbb{C}$ and $\mathcal{P}$ be the set of positive integers, complex and prime numbers, respectively. For each $k \in \mathbb{N}$, we denote by $\mathcal{M}_k$ the set of those $n \in \mathbb{N}$, which can be written as $n = p_1 + p_2 + \cdots + p_k$, $(p_1, p_2, \ldots, p_k \in \mathcal{P})$.

The Goldbach conjecture is that every even integer $n \geq 4$ can be written as the sum of two primes, and the ternary Goldbach conjecture is that every odd integer larger than 5 is the sum of three prime numbers. In 1742, Goldbach posed the problem in a letter to Euler. Attempts to solve it were not fruitful until 1923, Hardy and Littlewood used the newly formulated circle method

Mathematics Subject Classification: 11A25, 11A67, 11K65, 30D05.
Key words and phrases: arithmetical functions, functional equation, Goldbach conjecture.
This work was completed with the support of the Hungarian and Vietnamese TET (grant agreement no. TET 10-1-2011-0645).

discovered by Hardy and Ramanujan to show that the ternary Goldbach problem has a solution if we assume the generalized Riemann hypothesis for Dirichlet's L-functions. In a remarkable development, I. M. Vinogradov introduced a new method of trigonometric sums to show in 1937 that the ternary Goldbach conjecture is true in a weak form, namely that every large odd number is the sum of three prime numbers. His theorem was ineffective and gave no lower bound for what "sufficiently large" meant. In 2002, Liu and Wang [9] gave the bound $\exp(3100)$ and finally H. A. Helfgott proved recently [4]–[6] that the ternary Goldbach conjecture is true, i.e. every odd integer $n \geq 7$ belongs to $\mathcal{M}_3$. He proved somewhat more, namely that ([6], p. 71) *if $n \geq 9$, n is odd, then n is a sum of three odd primes.*

An almost immediate consequence of his result is

Theorem 1. *Let $f : \mathcal{M}_3 \to \mathbb{C}$, $g : \mathcal{P} \to \mathbb{C}$ be such functions for which*

$$f(p_1 + p_2 + p_3) = g(p_1) + g(p_2) + g(p_3) \tag{1.1}$$

holds for every $p_1, p_2, p_3 \in \mathcal{P}$. Then there exist suitable constants $A, B \in \mathbb{C}$ such that

$$f(n) = An + 3B \quad \text{for } n \in \mathcal{M}_3 \tag{1.2}$$

and

$$g(p) = Ap + B \quad \text{for } p \in \mathcal{P} \tag{1.3}$$

If the even Goldbach conjecture is true (that is $2n \in \mathcal{M}_2$ for every $n \in \mathbb{N}$, $n \geq 2$), then (1.2) *holds for every $n \geq 6$.*

Corollary 1. *Assume that complex valued functions f, g satisfy* (1.1) *in Theorem 1. If f is a multiplicative function and $f(n_0) \neq 0$ for some odd $n_0 > 1$, then either $f(n) = n$ and $g(p) = p$ or $f(n) = 1$ and $g(p) = \frac{1}{3}$ for all $n \in \mathbb{N}$, $p \in \mathcal{P}$.*

We note that this result for $f = g$ is proved by Fang [4].

Proof Of Theorem 1. Let $g(2), g(3)$ be arbitrary complex numbers. Let A, B be defined by $A = g(3) - g(2)$, $B = -2g(3) + 3g(2)$. Then (1.3) is true for $p = 2, 3$. Since $5+2+2 = 3+3+3$, $7+2+2 = 5+3+3$, $11+2+2 = 7+5+3$ and $13 + 2 + 2 = 11 + 3 + 3$, we deduce from (1.1) that (1.3) holds for $p \leq 13, p \in \mathcal{P}$.

Let $P \in \mathcal{P}$, $P \geq 17$. Then $P + 2 + 2 = P + 4$ can be written as the sum of odd primes p_1, p_2, p_3:

$$P + 2 + 2 = p_1 + p_2 + p_3. \tag{1.4}$$

We have $p_j \leq P + 4 - 6 < P$. We can use induction. If (1.3) holds for all the primes $p < P$, then from (1.4):

$$g(P) = g(p_1) + g(p_1) + g(p_1) - 2g(2) = A(p_1 + p_2 + p_3) + 3B - 2(2A + B)$$

$$= A(P+4) - 4A + B = AP + B.$$

Consequently, (1.3) holds for every $p \in \mathcal{P}$, and so (1.2) is true as well. □

Assume that $2n \in \mathcal{M}_2$. Then $2n+2 \in \mathcal{M}_3$. The last assertion is true as well. Theorem 1 is proved.

Let us prove the corollary.

If there exists an odd $n_0 > 1$ for which $0 \neq f(n_0) = An_0 + 3B$, then $A = B = 0$ cannot be occur.

Assume first that $A = 0$. Then $f(n) = 3B$ for every $n \in \mathcal{M}_3$, and so $3B = f(15) = f(3)f(5) = (3B)^2$. This implies that $B = \frac{1}{3}$, consequently

$$g(p) = \frac{1}{3} \quad \text{for } p \in \mathcal{P}$$

and

$$f(n) = 1 \quad \text{for } n \in \mathbb{N},\ (n,2) = 1.$$

Since the density of those even integers which cannot be written as the sum of two primes is 0 (see [3]), therefore for every k there exists an odd m for which $2^k m - 2$ is the sum of two primes. Thus $2^k m = 2 + \pi_1 + \pi_2$ $(\pi_1, \pi_2 \in \mathcal{P})$, and so

$$f(2^k) = f(2^k)f(m) = f(2^k m) = f(2 + \pi_1 + \pi_2) = g(2) + g(\pi_1) + g(\pi_2) = 1.$$

Thus $f(n) = 1$ for all $n \in \mathbb{N}$.

Assume that $A \neq 0$. Let $n \in \mathbb{N}$, $(n,7) = 1$. Then

$$7An+3B = f(7n) = f(7)f(n) = (7A+3B)(An+3B) = A(7A+3B)n+3B(7A+3B),$$

which with $A \neq 0$ implies that $7A + 3B = 7$ and $3B = 3B(7A + 3B) = 21B$. Therefore, we have $A = 1$, $B = 0$ and $f(n) = n$ for every odd $n \in \mathbb{N}$, $g(p) = p$ for every $p \in \mathcal{P}$.

We can prove that $f(2^k) = 2^k$ for all $k \in \mathbb{N}$. Indeed, there is an odd m for which $2^k m - 2$ is the sum of $\pi_1, \pi_2 \in \mathcal{P}$, that is $2^k m = 2 + \pi_1 + \pi_2$. Consequently

$$\begin{aligned} f(2^k)m &= f(2^k)f(m) = f(2^k m) = f(2 + \pi_1 + \pi_2) \\ &= g(2) + g(\pi_1) + g(\pi_2) = 2 + \pi_1 + \pi_2 = 2^k m, \end{aligned}$$

and so $f(2^k) = 2^k$.

The corollary is proved.

2. Let $k \in \mathbb{N}$ and $k \geq 4$. Then by the theorem of Helfgott [6], we can observe that

$$\mathcal{M}_k = \{2k, 2k+1, \dots\}.$$

We can prove

Theorem 2. *Let $k \geq 4, k$ fix, $f : \mathcal{M}_k \to \mathbb{C}$, $g : \mathcal{P} \to \mathbb{C}$. Assume that*

$$f(p_1 + p_2 + \cdots + p_k) = g(p_1) + g(p_2) + \cdots + g(p_k) \tag{2.1}$$

holds for every $p_1, p_2, \ldots, p_k \in \mathcal{P}$. Then there exist suitable constants $A, B \in \mathbb{C}$ such that

$$f(n) = An + kB \quad \text{for } n \in \mathbb{N},\ n \geq 2k \tag{2.2}$$

and

$$g(p) = Ap + B \quad \text{for } p \in \mathcal{P}. \tag{2.3}$$

Corollary 2. *Assume that complex valued functions f, g satisfy* (2.1) *in Theorem 2. If f is multiplicative and defined on $\mathbb{N}$, then either*

$$A = B = 0,\ (f(n), g(p)) = (0, 0) \quad \text{for } n \in \mathbb{N},\ p \in \mathcal{P},$$

or

$$A = 0, B = \frac{1}{k},\ (f(n), g(p)) = (1, \frac{1}{k}) \quad \text{for } n \in \mathbb{N},\ p \in \mathcal{P},$$

or

$$A = 1, B = 0,\ (f(n), g(p)) = (n, p) \quad \text{for } n \in \mathbb{N},\ p \in \mathcal{P}.$$

Proof Of Theorem 2. This is a direct consequence of Theorem 1. Let $p_4, \ldots, p_k$ be arbitrary fix primes, p_1, p_2, p_3 run over all primes. Let

$$\varphi(p_1 + p_2 + p_3) = f(p_1 + p_2 + p_3 + p_4 + \cdots + p_k) - (g(p_4) + \cdots + g(p_k)).$$

Then we have

$$\varphi(p_1 + p_2 + p_3) = g(p_1) + g(p_2) + g(p_3) \quad \text{for } p_1, p_2, p_3 \in \mathcal{P}.$$

From Theorem 1 we obtain that $g(p) = Ap + B$ $(p \in \mathcal{P})$, consequently $f(n) = An + kB$ holds for all $n \in \mathcal{M}_k$. Since $n \in \mathcal{M}_k$ if $k \geq 4$, $n \geq 2k$, the proof of Theorem 2 is completed. □

Corollary 2 can be proved similarly as Corollary 1. We omit it.

3. Conjectures

C. Spiro [13] proved in 1992 that if f is a multiplicative function satisfying

$$f(p_1 + p_2) = f(p_1) + f(p_2) \quad \text{for } p_1, p_2 \in \mathcal{P}$$

and there exists a prime p' such that $f(p') \neq 0$, then $f(n) = n$ for all $n \in \mathbb{N}$. For some generalization and further results of this theorem see the papers of [1], [2], [4], [8], [10]–[12].

We hope that the following conjectures are true:

Conjecture 1. *Let* $f : \mathcal{M}_2 \to \mathbb{C}$, $g : \mathcal{P} \to \mathbb{C}$ *and*

$$f(p_1 + p_2) = g(p_1) + g(p_2) \quad \text{for } p_1, p_2 \in \mathcal{P}.$$

Let $g(3)$, $g(5)$ *be arbitrary complex numbers. Let* A, B *be defined so that*

$$g(3) = 3A + B, g(5) = 5A + B.$$

Then

$$g(p) = Ap + B \text{ and } f(n) = An + 2B \quad \text{for } p \in \mathcal{P},\ n \in \mathcal{M}_2.$$

If f *is multiplicative, then either* $(f(n), g(p)) = (0, 0)$, *or* $(f(n), g(p)) = (1, \frac{1}{2})$, *or* $(f(n), g(p)) = (n, p)$ *for all* $p \in \mathcal{P}$ *and* $n \in \mathbb{N}$.

Conjecture 2. *If* $P \geq 7, P \in \mathcal{P}$, *then there exist* $p_1, p_2, p_3 \in \mathcal{P}$, p_1, p_2, $p_3 < P$ *for which*

$$P = p_1 + p_2 - p_3.$$

Theorem 3. *There exists an effective constant* C *such that Conjecture 2 holds for every prime* $P \geq C$.

Theorem 3 can be proved by the method of I. M. Vinogradov, which was used for the proof on the ternary Goldbach problem. The method of Helfgott seems to be applicable to prove it for $C \approx 10^{27}$. Thus a massive computer computation perhaps would help to prove Conjecture 2.

Theorem 4. *Conjecture 2 implies Conjecture 1.*

Theorem 4 can be proved by the method used in the proof of Theorem 1. We omit the details.

Let $\mathcal{P}_1 = \mathcal{P} \setminus \{2\}$, $\mathcal{B}$ = set of odd integers which are either primes or the product of two distinct primes. According to a nice theorem of J. Chen every large even number can be written as $p + Q$, where $p \in \mathcal{P}_1$, $Q \in \mathcal{B}$. Tomás Oliveira e Silva, Siegfried Herzog, and Silvio Pardi [14] proved that if an even number $4 \leq n \leq 4 \cdot 10^{18}$, then $n = p_1 + p_2$, $p_1, p_2 \in \mathcal{P}_1$.

Conjecture 3. *Let* $Q_2 \in \mathcal{B}$, $Q_2 \geq 11$. *Then there exist* $p_1, p_2 \in \mathcal{P}_1$, $Q_1 \in \mathcal{B}$ *such that* $p_1, p_2, Q_1 < Q_2$, *and* $Q_2 = p_1 + Q_1 - p_2$.

To prove it seems to be easier than to prove Conjecture 2. An obvious consequence of it is

Theorem 5. *Let f defined on the set $\mathcal{P}_1+\mathcal{B}$ mapping into $\mathbb{C}$. Let $g:\mathcal{B}\to\mathbb{C}$. Assume that*

$$f(p+Q)=g(p)+g(Q).$$

Assume that Conjecture 3 is true. Let $g(3)$, $g(5)$ be arbitrary complex numbers. Let A, B be defined so that $g(3)=3A+B$, $g(5)=5A+B$. Then

$$g(Q)=AQ+B \quad \textit{and} \quad f(m)=Am+2B$$

for every $Q\in\mathcal{B}$ and for every $m\in\mathcal{P}_1+\mathcal{B}$.

Acknowledgment. The authors would like thank the referees for their important suggestions.

References

[1] K. K. Chen and Y. G. Chen, On $f(p)+f(q)=f(p+q)$ for all odd primes p and q, *Publ. Math. Debrecen* **76** (2010), 425–430.

[2] A. Dubickas and P. Šarka, On multiplicative functions which are additive on sums of primes, *Aequat. Math.* **86** (2013), 81–89.

[3] T. Estermann, On Goldbach's problem: Proof that almost all even positive integers are sums of two primes, *Proc. Lodon Math. Soc.* **S2-44** (1938), 307–314.

[4] J.-H. Fang, A characterisation of the identity function with equation $f(p+q+r)=f(p)+f(q)+f(r)$, *Combinatorica* **31** (2011), 697–701.

[5] H. A. Helfgott, Major arcs for Goldbach's problem, Preprint *arXiv: 1203.5712*.

[6] H. A. Helfgott, Minor arcs for Goldbach's problem, Preprint, *arXiv: 1205.5252*.

[7] H. A. Helfgott, The ternary Goldbach conjecture is true, Preprint, *arXiv: 1312.774, 1404.2224*.

[8] J.-M. De Koninck, I. Kátai and B. M. Phong, A new characteristic of the identity function, *J. Number Theory* **63** (1997), 325–338.

[9] M. Liu and T. Wang, On the Vinagradov bound in the three primes Goldbach conjecture, *Acta Arith.* **102** (2002), 261–293.

[10] B. M. Phong, On sets characterizing the identity function, *Ann. Univ. Sci. Budapest Sect. Comput.* **24** (2004), 295–306.

[11] B. M. Phong, A characterization of the identity function with functional equations, *Annales Univ. Sci. Budapest. Sect. Comp.* **32** (2010), 247–252.

[12] B. M. Phong, A characterization of the identity function with the equation of Hosszú type, *Publ. Math. Debrecen* **69** (2006), 219–226.

[13] C. Spiro, Additive uniqueness set for arithmetic functions, *J. Number Theory* **42** (1992), 232–246.

[14] T. Oliveira E Silva, S. Herzog and S. Pardi, Empirical verification of the even Goldbach conjecture and computation of prime gaps up to $4 \cdot 10^{18}$, *Math. Comp.* **83** (2014), 2033–2060.

IMRE KÁTAI
DEPARTMENT OF COMPUTER ALGEBRA
FACULTY OF INFORMATICS
EÖTVÖS LORÁND UNIVERSITY
H-1117 BUDAPEST
PÁZMÁNY PÉTER SÉTÁNY 1/C
HUNGARY

E-mail: katai@compalg.inf.elte.hu

BUI MINH PHONG
DEPARTMENT OF COMPUTER ALGEBRA
FACULTY OF INFORMATICS
EÖTVÖS LORÁND UNIVERSITY
H-1117 BUDAPEST
PÁZMÁNY PÉTER SÉTÁNY 1/C
HUNGARY

E-mail: bui@compalg.inf.elte.hu

(Received July 2, 2014; revised November 29, 2014)