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A common structure of n_k's for which $n_k\alpha$ mod $1 \to x$

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Abstract. Let α be an irrational number and $\varepsilon_k \le 1$, $k = 1, 2, \dots$, be an arbitrary decreasing sequence of real numbers such that $\varepsilon_k \to 0$. In this paper we show a construction of sequences n_k, $k = 1, 2, \dots$, for which the fractional parts $\{n_k\alpha\} \to x$, where $x \in [0, 1]$ is fixed but arbitrary and $k/n_k \ge \varepsilon_k$ for $k = 1, 2, \dots$. Here $\{n_k\alpha\} \in I_j$ for $k_{j-1} < k \le k_j$ and the length $|I_j| = \{h_j\alpha\}$, where h_j is a positive integer for $j = 1, 2, \dots$. The increasing sequence k_j is independent of x. Moreover, the differences $n_{k+1} - n_k$ satisfy the three gaps property with parameters a_j, b_j and $a_j + b_j$ not depending on x for every $k_{j-1} < k < k_j$ and $j = 2, 3, \dots$.

1. Introduction

In what follows α denotes an irrational number and $\{n\alpha\}$ denotes the fractional part of $n\alpha$. The H. Weyl's classical result [17, Satz 2] that the sequence $\{n\alpha\}$, $n = 1, 2, \dots$, is uniformly distributed in the unit interval $[0, 1]$ implies that to every $x \in [0, 1]$ there exists an increasing sequence $n_k = n_k(x)$, $k = 1, 2, \dots$, of positive integers such that $\{n_k\alpha\} \to x$ as $k \to \infty$. In a previous paper [16, Satz 6] he proved that given an increasing sequence n_k, $k = 1, 2, \dots$, the sequence $\{n_k\alpha\}$, $k = 1, 2, \dots$, is uniformly distributed in $[0, 1]$ for almost all real numbers α.

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P. Erdős asked whether there exists a real number $x \in [a,b]$ such that the sequence $n_k x$ is not everywhere dense mod 1. He and S. J.Taylor [3] proved that if n_k is a lacunary sequence of real positive numbers, that is if $n_{k+1}/n_k \geq \lambda$ for some $\lambda > 1$ and every k, then the set of the real numbers α belonging to any interval $[a,b]$, $a < b$, such that the sequence $n_k\alpha$ is not u.d. mod 1 has Hausdorff dimension 1.

A. Dubickas found an α and $n_1 < n_2 < n_3 < \dots$ such that $\{n_k\alpha\}$ converges to zero while k/n_k tends to 0 arbitrarily slowly. More precisely, he proved [2, Theorem 1]

Theorem 1.1. *Let α be a real quadratic algebraic number, and let*[1] $1 \geq \varepsilon_1 \geq \varepsilon_2 \geq \varepsilon_3 \geq \dots$ *be a sequence of real numbers such that $\varepsilon_n \to 0$ as $n \to \infty$. Then there exists an increasing sequence of positive integers $n_1 < n_2 < \dots$ satisfying $\varepsilon_k \leq \frac{k}{n_k}$ for each $k \geq 1$ such that $\lim_{k\to\infty}\{n_k\alpha\} = 0$.*

The condition A. Dubickas $\lim_{k\to\infty} \varepsilon_k = 0$ cannot be weakened as there follows from Theorem 2 of [2] saying:

Theorem 1.2. *Let n_k be an increasing sequence of positive integers with positive upper asymptotic density,*[2] *and let α be an irrational real number. Then the set of limit points of the sequence $\{n_k\alpha\}$, $k = 1, 2 \dots,$ is infinite.*

Another proof of Dubickas' condition also follows from the following reasoning: The asymptotic distribution function $g(x)$ of $\{n\alpha\}$ is $g(x) = x$, and thus it is a continuous function over $[0,1]$. Consequently there follows from [7, Example 3.1] that the sequence $n_k = n_k(x)$ such that $\{n_k\alpha\} \to x$, has zero asymptotic density for every $x \in [0,1]$, i.e.

$$\lim_{k\to\infty} \frac{k}{n_k} = 0, \tag{1}$$

for every $x \in [0,1]$.

Other well-known result on limit points of $\{n_k\alpha\}$ is Furstenberg's result [4, Theorem IV.1] implying that if the increasing sequence of positive integers $n_1 < n_2 < \dots$ forms a multiplicative semigroup which is not generated by powers of a single integer (e.g. $\{2^a3^b : a, b \text{ positive integers}\}$), then the sequence of fractional parts $n_k\alpha$, $k = 1, 2, \dots,$ is everywhere dense in $[0,1]$ for each real irrational α. P. Erdős and S. J. Taylor [3, Theorem 10] also proved that there exists a

[1]The decreasing property of the sequence ε_n is clearly not a limitation for one can redefine the sequence without loss of generality by taking $\varepsilon_k = \sup_{j\geq k} \varepsilon_j$.

[2]The asymptotic density of a sequence of positive integers n_k, $k = 1, 2, \dots,$ is defined as the limit $\lim_{n\to\infty} \#\{k; n_k \leq n\}/n$ if the limit exists and, if we write $0 < n_1 < n_2 < \dots$ then (cf. [15, 1.3]) $\lim_{n\to\infty} \#\{k; n_k \leq n\}/n = \lim_{k\to\infty} k/n_k$, providing one of the limits exists.

constant C, and an increasing sequence n_k such that $n_{k+1} - n_k < C$, $k = 1, 2, \ldots$, and the set of x such that $\{n_k x\}$ is not uniformly distributed is not enumerable.

Dubickas' proof of Theorem 1.1 is quite complex and actually the constructed sequence n_k, $k = 1, 2, \ldots$, depends on α. Almost immediately, Y. Bugeaud [1] answered Dubickas' question whether given a real algebraic number of degree at least 3 or a real transcendental number α there exists a slowly increasing sequence of positive $n_1 < n_2 < \ldots$ such that $\lim_{n\to\infty}\{n_k\alpha\} = 0$. Y. Bugeaud [1] using tools from the theory of continued fractions proved:

Theorem 1.3. *Let α be an irrational number and S a finite subset of $[0,1]$. Let $1 \geq \varepsilon_k \geq 0$, $k = 1, 2, \ldots$, be a given decreasing sequence of real numbers such that $\lim_{k\to\infty} \varepsilon_k = 0$. Then there exists an increasing sequence of positive integers $n_1 < n_2 < \ldots$ satisfying $\varepsilon_k \leq \frac{k}{n_k}$ for each $k \geq 1$ such that the set of limit points of $\{n_k\alpha\}$ coincides with S.*

Recently L. Mišík [8] proved the following extension in direction of the set of limit points

Theorem 1.4. *Let $X \subset [0,1]$ be a closed set, α be an irrational number, and $\varepsilon_n \leq 1$ be an arbitrary decreasing sequence such that $\lim_{k\to\infty} \varepsilon_k = 0$. Then there exists a sequence $n_1 < n_2 < \ldots$ of positive integers such that the set of limit points of $\{n_k\alpha\}$ coincides with X and $\varepsilon_k + |X| \leq \frac{k}{n_k}$ for each $k \geq 1$, where $|X|$ denotes the Lebesgue's measure of X.*

In this paper we prove that for every irrational α and every $x \in [0,1]$ there exists an increasing sequence of integers $n_1(x) < n_2(x) < \ldots$ such that on one hand $\{n_k\alpha\} \to x$ and the Dubickas' condition $k/n_k(x) \geq \varepsilon_k$ holds true for every k while the sequence $n_k(x)$ can be endowed with an additional structure. One of this structural properties extends properties of the set $A = \{n \in \mathbb{N} : \{n\alpha\} \in I\}$ investigated in [12].

2. The result

Theorem 2.1. *Let α be an irrational number and $\varepsilon_j \leq 1$, $j = 1, 2, \ldots$, be a decreasing sequence of positive numbers tending to 0. Then for every $x \in [0,1]$ there exist*

- *an increasing sequence $n_k(x)$, $k = 1, 2, \ldots$, of positive integers,*
- *an increasing sequence k_j, $j = 1, 2, \ldots$, of positive integers independent of x,*
- *a sequence of pairs a_j, b_j, $j = 1, 2, \ldots$, also independent of x, such that*

(I) $\{n_k(x)\alpha\} \to x$,

(II) $\varepsilon_k \leq \frac{k}{n_k(x)}$ *for* $k = 1, 2, \dots$, *and*

(III) *for every* k, $k_{j-1} < k \leq k_j$, $j = 2, 3, \dots$, *we have*

$$n_{k+1}(x) - n_k(x) = \begin{cases} a_j, & \text{or} \\ b_j, & \text{or} \\ a_j + b_j, & \end{cases}$$

where a_j *and* b_j *are coprime for every* $j = 1, 2, \dots$.

Proof. The roadmap of the proof is as follows:

1° Given an arbitrary but fixed $x \in [0, 1]$ we construct a sequence $0 = k_0 < k_1 < k_2 < k_3 < \dots$ of positive integers independent of the limit point x, and an increasing sequence $n_k(x)$ for k's with $k_{j-1} < k \leq k_j$ and $j = 2, 3, \dots$. The construction of k_j's for $j = 1$ and n_k's for $k_0 < k \leq k_1$ will be postponed to step 3°. The reason for this partly unusual placing of this step on the third position is to stress the main idea of construction of k's and n_k's presented in the previous two steps. The construction of the initial segment of n_k's for $k \leq k_1$ may cause that the values of k_j for $j > 1$ should be increased, nevertheless the crucial estimates used in the first and third steps remain true also for the increased values of k's and the corresponding blowing-up of values of n_k's for $k > k_1$.

2° In the previous step the constructions of k_j's and n_k's depended on an arbitrary but fixed $x \in [0, 1]$. In this step we modify the sequences k_j's and n_k's in such a way that the obtained estimations remain true for arbitrary $x' \in [0, 1]$.[3]

3° In this step we complete the construction from 1° for the case $j = 1$.

4° We prove property (III).

The details of the proof:

1° Given an $x \in [0, 1]$, select a sequence $I_2(x), I_3(x), \dots$ of subintervals of $[0, 1]$ satisfying the following four conditions

- $x \in I_j(x)$, for $j = 2, 3, \dots$,
- $\frac{1}{2} > |I_2(x)| > |I_3(x)| > \dots$, [4]
- $|I_j(x)| \to 0$ as $j \to \infty$,

[3] More precisely, k_j's for x' must satisfy new inequalities (7), (8), (11) and (13).

[4] In the proof of (III) we use Slater's original theorem holding for intervals $I \subset [0, 1]$ of length $\leq 1/2$. Actually, from a generalization of Slater's theorem proved in [12] and holding also for intervals of length $> 1/2$, an analogous but complicated result can be proved for k's satisfying $k_{j-1} < k \leq k_j$ with $j = 1$ and $k_0 = 0$.

- $|I_j(x)| = \{h_j\alpha\}$, where h_j, $j = 2, 3, \dots$, are positive integers such that the above three conditions are fulfilled.

Now, select a preliminary increasing sequence of positive integers (not depending on x)

$$k_0 = 0 < k_1 < k_2 < k_3 < \dots < k_j < k_{j+1} < \dots. \tag{2}$$

The sequence will be subject of a series of modifications in the process of construction of the next sequence $n_k(x)$, if necessary. Since $n\alpha$ is dense in $[0, 1]$ construct inductively an increasing sequence of positive integers

$$0 < n_1(x) < n_2(x) < n_3(x) < \dots \tag{3}$$

in such a way that for every k such that $k_j < k \le k_{j+1}$, $j = 1, 2, \dots$, take for $n_k(x)$, in increasing order, a corresponding $n = n_k$ satisfying

$$n_{k_j}(x) < n \le n_{k_{j+1}}(x) \quad \text{and} \quad \{n\alpha\} \in I_{j+1}(x). \tag{4}$$

in such a way that we exhaust all possible n's such that $\{n\alpha\} \in I_{j+1}(x)$ while if necessary we increase k_{j+1}.

Note that we need that the members of sequence $n_{k_{j+1}}(x)$ are sufficiently large and satisfy inequalities (7), (8), (11) and (13) in the next steps, so we also increase adequately k_{j+1} in these steps. In this process the values of k_j's and n_{k_j}'s are stepwise modified in such a way that both initial segments of these sequences remain unchanged for indices less than the actual j.

The crucial tool in the subsequent parts of the proof will be an old result by E. Hecke [5], A. Ostrowski [9] and H. Kesten [6] which says: If the interval $I \subset [0, 1]$ is of the form $|I| = \{h\alpha\}$ where h is a positive integer, then for every $M < N$ we have

$$(N - M).|I| - 2h \le \#\{M < n \le N;\ \{n\alpha\} \in I\} \le (N - M)|I| + 2h. \tag{5}$$

Write for the sake of simplicity $|I_{j+1}(x)| = |I_{j+1}|$ and $n_k(x) = n_k$. Using this abbreviated notation, (5) can be rewritten in the form

$$k - k_j = (n_k - n_{k_j})|I_{j+1}| + O(h_{j+1}) \tag{6}$$

for every $k_j < k \le k_{j+1}$, where the O-constant is ≤ 2.

Using (6) we obtain

$$\frac{k}{n_k} = \frac{k_j}{n_k} + |I_{j+1}| - \frac{n_{k_j}}{n_k}|I_{j+1}| + O\left(\frac{h_{j+1}}{n_k}\right)$$

$$= |I_{j+1}| + \frac{n_{k_j}}{n_k}\left(\frac{k_j}{n_{k_j}} - |I_{j+1}|\right) + O\left(\frac{h_{j+1}}{n_k}\right) \geq |I_{j+1}| - 2\,\frac{h_{j+1}}{n_{k_j}} \tag{7}$$

under the condition

$$\frac{k_j}{n_{k_j}} - |I_{j+1}| \geq 0. \tag{8}$$

Note that our construction implies $k_j/n_{k_j} \to |I_j|$ for $k_j \to \infty$ (note that in this moment j is fixed),[5] and therefore (8) can be achieved by increasing adequately k_j and n_{k_j}.[6]

If we increase the values of both k_j and n_{k_j} in such a way that (8) holds and moreover we also have

$$|I_{j+1}| - 2\frac{h_{j+1}}{n_{k_j}} \geq \varepsilon_{k_j},$$

then using (7) we obtain $k/n_k \geq \varepsilon_k$ for $k_j < k \leq k_{j+1}$ and consequently

$$\frac{k}{n_k} \geq \varepsilon_k \quad \text{for } k_{j+1} \geq k > k_1.$$

An additional increase of k_1 and the mentioned subsequent 'chain' increment of already found k's and n_k's, does not influence the validity of the estimations above.

2° As mentioned, the just constructed sequence n_k depended on a fixed $x \in [0,1]$. Now we shall adapt it to fit the required estimates for an arbitrary $x' \in [0,1]$. Assume therefore that $x' \in [0,1]$ is arbitrary.

Let $I_j(x')$, $j = 2, 3, \ldots$, be a sequence of subintervals of $[0,1]$ now satisfying the previous conditions in the following form

$$x' \in I_j(x'), \qquad |I_j(x')| = |I_j(x)| = \{h_j\alpha\}.$$

In step 1° we constructed conjugated sequences $n_1 < n_2 < \ldots$ and $k_1 < k_2 < \ldots$. Based on the just constructed sequence of k's we shall construct a new sequence

$$n_1(x') < n_2(x') < n_3(x') < \ldots \tag{9}$$

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[5]To see this note that the definition of u.d. says that $\frac{\#\{0<n\leq n_{k_j}; n\alpha\in I_j\}}{n_{k_j}} \to |I_j|$ provided $n_{k_j} \to \infty$. Then divide the fraction into two parts

1) $\frac{\#\{0<n\leq n_{k_{j-1}}; n\alpha\in I_j\}}{n_{k_j}} \to 0$ for sufficiently large n_{k_j}, and

2) $\frac{\#\{n_{k_{j-1}}<n\leq n_{k_j}; n\alpha\in I_j\}}{n_{k_j}} = \frac{k_j - k_{j-1}}{n_{k_j}} \to \frac{k_j}{n_{k_j}}$ is n_{k_j} is sufficiently large.

[6]According to condition $k_j < k \leq k_{j+1}$ in (4) we have $n_{k_j+1} < n_{k_j+2} < n_{k_j+3} < \cdots < n_{k_j+1}$. Thus $n_{k_{j+1}} - n_{k_j} \geq k_{j+1} - k_j$ and consequently both $n_{k_{j+1}}$ and k_{j+1} strictly increase.

such that for every k with $k_j < k \le k_{j+1}$ the increasing sequence $n_k(x')$ contains all possible n's in the interval $n_{k_j}(x') < n \le n_{k_{j+1}}(x')$ for which $\{n\alpha\} \in I_{j+1}(x')$.

Abbreviate again $n_k(x') = n'_k$.

Relation (6) implied for $k_j < k \le k_{j+1}$ that

$$\begin{aligned} k - k_j &= |I_{j+1}|(n_k - n_{k_j}) + O(h_{j+1}),\\ k - k_j &= |I_{j+1}|(n'_k - n'_{k_j}) + O(h_{j+1}), \end{aligned}$$

where the involved O-constants are smaller than 2. Consequently we get with $n_{k_0} = n'_{k_0} = 0$ that

$$\begin{aligned} &(n'_k - n'_{k_j}) = (n_k - n_{k_j}) + O\left(\frac{h_{j+1}}{|I_{j+1}|}\right)\\ &\frac{\frac{k}{n_k}}{\frac{k}{n'_k}} = \frac{n'_k}{n_k} = \frac{(n'_{k_1} - n'_{k_0}) + (n'_{k_2} - n'_{k_1}) \cdots + (n'_{k_j} - n'_{k_{j-1}}) + (n'_k - n'_{k_j})}{(n_{k_1} - n_{k_0}) + (n_{k_2} - n_{k_1}) \cdots + (n_{k_j} - n_{k_{j-1}}) + (n_k - n_{k_j})}\\ &\qquad\quad = \frac{(n_{k_1} - n_{k_0}) + O\big(\frac{h_1}{|I_1|}\big) + \cdots + (n_k - n_{k_j}) + O\big(\frac{h_{j+1}}{|I_{j+1}|}\big)}{(n_{k_1} - n_{k_0}) + (n_{k_2} - n_{k_1}) \cdots + (n_{k_j} - n_{k_{j-1}}) + (n_k - n_{k_j})}\\ &\qquad\quad = 1 + O\left(\sum_{i=1}^{j+1} \frac{h_i}{|I_i|}\right)\frac{1}{n_k} \end{aligned} \tag{10}$$

where the last O constant in (10) is ≤ 4. Relation (10) thus implies

$$\frac{k}{n_k} \le \frac{k}{n'_k} + 4\left(\sum_{i=1}^{j+1} \frac{h_i}{|I_i|}\right)\frac{1}{n_k} \cdot \frac{k}{n'_k}.$$

Since $n_k \ge n_{k_j}$ and $\frac{k}{n'_k} \le 1$, we have

$$\frac{k}{n_k} - 4\left(\sum_{i=1}^{j+1} \frac{h_i}{|I_i|}\right)\frac{1}{n_{k_j}} \le \frac{k}{n'_k}, \tag{11}$$

for an arbitrary sequence k_j in (2). Assuming that the k_j's satisfy the inequality (8), and adding (7) to (11) we obtain

$$|I_{j+1}| - 2\,\frac{h_{j+1}}{n_{k_j}} - 4\left(\sum_{i=1}^{j+1} \frac{h_i}{|I_i|}\right)\frac{1}{n_{k_j}} \le \frac{k}{n'_k}. \tag{12}$$

Now, assume that our sequence k_j from (2) and the sequence n_{k_j} from (3) satisfy not only (8) but also

$$\varepsilon_{k_j} \le |I_{j+1}| - 2\,\frac{h_{j+1}}{n_{k_j}} - 4\left(\sum_{i=1}^{j+1} \frac{h_i}{|I_i|}\right)\frac{1}{n_{k_j}} \tag{13}$$

for every $j = 1, 2, \ldots$. Then (12) implies that for every $j = 1, 2, \ldots$ we have

$\varepsilon_{k_j} \le \frac{k}{n'_k}$, for k, $k_j < k \le k_{j+1}$. Since ε_k is non-increasing, then $\varepsilon_k \le \frac{k}{n'_k}$, for $k_j < k \le k_{j+1}$ and thus

$$\varepsilon_k \le \frac{k}{n'_k} \quad \text{for every } k_1 < k. \tag{14}$$

That the property (13) can be fulfilled for a suitable n_{k_j} follows from the fact that $|I_{j+1}|$ is fixed and n_{k_j} can be shifted to a sufficiently large value.[7]

We recall that (8) can be fulfilled for suitable n_{k_j}'s since the sequence $\{n\alpha\}$ is uniformly distributed which implies

$$\frac{k_j}{n_{k_j}} \to |I_j|, \tag{15}$$

while $|I_j| > |I_{j+1}|$ for every $j = 1, 2, \dots$. Globally, k/n_k tends to zero, but due to the construction we always have $k/n_k \to |I_j|$ for $k_{j-1} < k \le k_j$.

Again the eventual necessary increments of values of k's and n_k's does not influence the used inequalities.

3° The proof of (14) also for $k = 1, 2, \dots, k_1$ proceed as follows:

For $k \le k_1$ put $n_k = k$ and suppose that n_{k_1} is sufficiently large[8] to satisfy (13) i.e.

$$\varepsilon_{k_1} \le |I_2| - 2\,\frac{h_2}{n_{k_1}} - 4\left(\frac{h_1}{|I_1|} + \frac{h_2}{|I_2|}\right)\frac{1}{n_{k_1}}.$$

Then put

$$\{i_1.\alpha\} = \min(\{1.\alpha\}, \{2.\alpha\}, \dots, \{k_1.\alpha\}), \quad \{i_2.\alpha\} = \max(\{1.\alpha\}, \{2.\alpha\}, \dots, \{k_1.\alpha\}),$$

$$I_1 = [\{i_1.\alpha\}, \{i_2.\alpha\}], h_1 = |i_2 - i_1|.$$

Due to our construction we have automatically

$$\frac{k}{n_k} = 1 \ge \varepsilon_k \text{ for } k = 1, 2, \dots, k_1 \quad \text{and} \quad \frac{k_1}{n_{k_1}} - |I_2| \ge 0$$

and thus the inequality (8) also holds.

We did not assume that $x \in I_1$, but on the other hand the interval I_1 is fixed for every $x' \in [0, 1]$. Thus the proof of (II) is finished.

4° The proof of (III) follows from Slater's three gaps theorem [13] and [14] saying that if n and $n + s$ are immediately neighboring indices with the property

[7]Note again that in the course of the constructions of k_j's and n_{k_j}'s in the proof they are enlarged in such way that conditions (7), (8), (11) and (13) are satisfied. However after the constructions are closed the sequence k_j is fixed and independent of x.

[8]Again an eventual shift of n_{k_1} to a larger value does not influence the previous estimates, only forces an additional work with the reconstruction of n_k's.

that both $n\alpha$ and $(n+s)\alpha$ belong to interval I, then $s = a$ or $s = a+b$ or $s = b$ depending on the length $|I|$ of interval I. In accordance with (4) for $k_j < k \le k_{j+1}$, the numbers n_k and n_{k+1} are the closest ones with $n_k\alpha \in I_{j+1}$ and $n_{k+1}\alpha \in I_{j+1}$, and consequently $n_{k+1} - n_k = a_j$, or $a_j + b_j$, or b_j depending on the length $|I_{j+1}|$. The extension of Slater's theorem proved for instance in [12] says that the differences $n_{k+1} - n_k$ depend only on the length $|I|$ of the interval I but not on its position within the unit interval $[0,1]$.

For the sake of simplicity let $I = I_j(x)$, $a = a_j$, $b = b_j$ for every $j = 2, 3, \dots$. Since $|I| \le 1/2$, define a and b as the least positive integers such that $\{a\alpha\} \in (0, |I|)$ and $\{b\alpha\} \in (1-|I|, 1)$. Let $\{n\alpha\} \in I$ and let s be minimal with $\{(n+s)\alpha\} \in I$. Then

$$s = \begin{cases} a, & \text{if } 0 \le \{n\alpha\} < |I| - \{a\alpha\}, \\ a+b, & \text{if } |I| - \{a\alpha\} \le \{n\alpha\} < 1 - \{b\alpha\}, \\ b, & \text{if } 1 - \{b\alpha\} \le \{n\alpha\} < |I|. \end{cases} \tag{16}$$

In addition a and b are relatively prime. □

Concluding remark: The given proof of Theorem 1.1 also gives a general construction of sequence $n_1 < n_2 < \dots$ from Dubickas' Theorem 3.1 in the following sense: Let x'' is an arbitrary number from interval $(0,1)$. The intervals $I_1(x), I_2(x), I_3(x), \dots$ used in proof of Theorem 1.1 shift in such a way that they contain the given x''. Denote them as $I_1(x''), I_2(x''), I_3(x''), \dots$. Now select k_1 consecutive n's for which $\{n\alpha\} \in I_1(x'')$. Denote the last one as n_{k_1}. Then continue selecting $k_2 - k_1$ consecutive n's, $n > n_{k_1}$ for which again $\{n\alpha\} \in I_2(x'')$. The last one denote n_{k_2}. In the next step select $k_3 - k_2$ consecutive integers n such that $n > n_{k_2}$ and $\{n\alpha\} \in I_3(x'')$. The last one denote as n_{k_3}, etc. There follows from the construction that in this way selected sequence n_k has properties required by Dubickas' Theorem and $\{n_k\alpha\}$ converges to x'' and $k/n_k > \varepsilon_k$ for given ε_k, $k = 1, 2, \dots$.

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