

Curvature properties of certain pseudosymmetric manifolds

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1. Introduction

Let (M, g) be a connected n -dimensional, $n \geq 3$, semi-Riemannian manifold of class C^∞ with the metric g and the Levi-Civita connection ∇ . Moreover, let S and κ be the Ricci tensor and the scalar curvature of (M, g) , respectively. A semi-Riemannian manifold (M, g) , $n \geq 3$, is said to be Einstein manifold ([1], p. 3) if

$$(1) \quad S = \frac{\kappa}{n}g$$

holds on M . Certain Einstein manifolds can be realized as hypersurfaces immersed isometrically in a semi-Riemannian space of constant curvature $M^{n+1}(c)$ ([18], [20], [21], [22]). Such hypersurfaces have also a curvature property of pseudosymmetry type. Namely, in [15] (Proposition 3.2) it was proved that every Einstein hypersurface immersed isometrically in a semi-Riemannian space of constant curvature $M^{n+1}(c)$, $n \geq 4$, is a pseudosymmetric manifold. A semi-Riemannian manifold (M, g) , $n \geq 3$, is said to be pseudosymmetric [12] if at every point of M the following condition is satisfied:

(*) the tensors $R \cdot R$ and $Q(g, R)$ are linearly dependent.

The manifold (M, g) is pseudosymmetric if and only if

$$(2) \quad R \cdot R = L_R Q(g, R)$$

holds on the set $U_R = \{x \in M \mid Z(R) \neq 0 \text{ at } x\}$, where L_R is some function on U_R . The definitions of the tensors $R \cdot R$, $Q(g, R)$ and $Z(R)$ will be given in the next section. Pseudosymmetric manifolds are a generalization of the notion of locally symmetric manifolds ($\nabla R = 0$) as well as of the notion of semisymmetric manifolds ($R \cdot R = 0$, [23]). We refer [10] as a review paper on pseudosymmetric manifolds.

Another class of semi-Riemannian manifolds are manifolds with pseudosymmetric Weyl tensor. A semi-Riemannian manifold (M, g) , $n \geq 4$, is

said to be manifold with pseudosymmetric Weyl tensor if at every point of M its Weyl conformal curvature tensor C satisfies the following condition ([16], [17]):

(**) the tensors $C \cdot C$ and $Q(g, C)$ are linearly dependent.

We note that the condition (**) is invariant under the conformal deformations of the metric tensor g . The manifold (M, g) is a manifold with pseudosymmetric Weyl tensor if and only if

$$(3) \quad C \cdot C = L_C Q(g, C)$$

holds on the set $U_C = \{x \in M \mid C \neq 0 \text{ at } x\}$, where L_C is some function on U_C . The definitions of the tensors $C \cdot C$ and $Q(g, C)$ will be given also in the next section. Note that $U_C \subset U_R$. If (M, g) is an Einstein manifold then $U_C = U_R$. The condition (**) arose during the study of 4-dimensional warped products ([9]). It was shown in [9] (Theorem 2) that every warped product $M_1 \times_F M_2$, $\dim M_1 = \dim M_2 = 2$, fulfils (**). Furthermore, it is easy to verify that if (M, g) , $n \geq 4$, is an Einstein manifold then the relations (2) and (3), with $L_C = L_R - \frac{\kappa}{n(n-1)}$, are equivalent on U_R (cf. [3], Theorem 3.1). Examples of not Einsteinian semi-Riemannian manifolds realizing (*) and (**) are described in [3] (Theorem 4.1) and [11] (Theorem 4.1). In addition, these manifolds satisfying a certain another condition, weaker than (1). Namely, it is easy to check that at every point of the manifolds described in the above theorems also the following condition is fulfilled:

(***) the tensors $S^2 - \frac{\text{tr}(S^2)}{n}g$ and $S - \frac{\kappa}{n}g$ are linearly dependent.

The tensor S^2 is defined by $S^2(X, Y) = S(\tilde{S}X, Y)$, where the tensor $\tilde{S}$, defined by $S(X, Y) = g(\tilde{S}X, Y)$, is the Ricci operator of (M, g) and X, Y are vector fields on M . However, the classes of manifolds satisfying (*) or (**) do not coincide. A suitable example is given in [3] (Example 5.2). Warped products realizing (**) are considered in [16]. Hypersurfaces in Euclidean spaces realizing (**) are studied in [5].

In this paper we will investigate curvature properties of pseudosymmetric manifolds having pseudosymmetric Weyl tensor. Certain results concerned with this subject are present in [19]. Namely, by Theorems 1 and 2 and Proposition 4 of [19] (cf. [2], Theorem 4.2) we obtain some curvature properties of pseudosymmetric manifolds (M, g) satisfying the condition:

$$(4) \quad \omega(X)\tilde{C}(Y, Z) + \omega(Y)\tilde{C}(Z, X) + \omega(Z)\tilde{C}(X, Y) = 0,$$

where $\tilde{C}$ is the Weyl curvature operator of (M, g) , ω is an 1-form on M and X, Y, Z are vector fields on M . We note that the tensor $C \cdot C$ vanishes at all point of a semi-Riemannian manifold fulfilling (4) at which the 1-form ω is nonzero ([13], Corollary 1).

Our main result (Theorem 3.1) states: If (M, g) , $\dim M \geq 4$, is a semi-Riemannian manifold realizing $(*)$ and $(**)$ then at every point of M the condition $(***)$ is satisfied.

Furthermore, we will find also another curvature relations of manifolds realizing these both conditions. In the last section we shall investigate curvature properties of pseudosymmetry type of a certain four-dimensional Riemannian manifold described in [6]. This manifold is an example of a non Ricci-pseudosymmetric manifold with pseudosymmetric Weyl tensor.

2. Preliminaries

Let (M, g) be a connected n -dimensional, $n \geq 3$, semi-Riemannian manifold. We define on M the endomorphisms $\tilde{R}(X, Y)$, $X \wedge Y$ and $\tilde{C}(X, Y)$ by

$$\tilde{R}(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X, Y]}Z,$$

$$(X \wedge Y)Z = g(Y, Z)X - g(X, Z)Y,$$

$$\tilde{C}(X, Y) = \tilde{R}(X, Y) - \frac{1}{n-2} \left((X \wedge \tilde{S}Y + \tilde{S}X \wedge Y) - \frac{\kappa}{n-1} X \wedge Y \right),$$

respectively, where $X, Y, Z \in \Xi(M)$, $\Xi(M)$ being the Lie algebra of vector fields on M . Furthermore, we define the Riemann-Christoffel curvature tensor R , the Weyl conformal curvature tensor C , the tensor G and the concircular tensor $Z(R)$ of (M, g) by

$$R(X_1, X_2, X_3, X_4) = g(\tilde{R}(X_1, X_2)X_3, X_4),$$

$$C(X_1, X_2, X_3, X_4) = g(\tilde{C}(X_1, X_2)X_3, X_4),$$

$$G(X_1, X_2, X_3, X_4) = g((X_1 \wedge X_2)X_3, X_4)$$

and $Z(R) = R - \frac{\kappa}{n(n-1)}G$, respectively. For a $(0, k)$ -tensor T on M , $k \geq 1$, we define the $(0, k+2)$ -tensors $R \cdot T$ and $Q(g, T)$ by

$$\begin{aligned} R \cdot T(X_1, \dots, X_k; X, Y) &= -T(\tilde{R}(X, Y)X_1, X_2, \dots, X_k) \\ &\quad - \dots - T(X_1, \dots, X_{k-1}, \tilde{R}(X, Y)X_k), \end{aligned}$$

$$\begin{aligned} Q(g, T)(X_1, \dots, X_k; X, Y) &= T((X \wedge Y)X_1, X_2, \dots, X_k) \\ &\quad + \dots + T(X_1, \dots, X_{k-1}, (X \wedge Y)X_k), \end{aligned}$$

respectively. Similarly, we can define the $(0, k+2)$ -tensor $C \cdot T$. Let now A and T be a symmetric $(0, 2)$ -tensor and a $(0, k)$ -tensor on a semi-Riemannian manifold (M, g) , $n \geq 3$, respectively. We define on M the

$(0, k+2)$ -tensor $Q(A, T)$ by

$$Q(A, T)(X_1, \dots, X_k; X, Y) = T((X \wedge_A Y)X_1, X_2, \dots, X_k) \\ + \dots + T(X_1, \dots, X_{k-1}, (X \wedge_A Y)X_k),$$

where $X \wedge_A Y$ is the endomorphism defined by

$$(X \wedge_A Y)Z = A(Y, Z)X - A(X, Z)Y.$$

In particular, we have $X \wedge_g Y = X \wedge Y$. Let (M, g) be a semi-Riemannian manifold covered by a system of charts $\{U; x^k\}$. We denote by

$$g_{ij}, R_{hijk}, S_{ij}, S_i^j = g^{jk}S_{ij}, G_{hijk} = g_{hk}g_{ij} - g_{hj}g_{ik}$$

and

$$C_{hijk} = R_{hijk} + \frac{\kappa}{(n-1)(n-2)}G_{hijk} \\ - \frac{1}{n-2}(g_{hk}S_{ij} - g_{hj}S_{ik} + g_{ij}S_{hk} - g_{ik}S_{hj}),$$

the local components of the metric tensor g , the Riemann-Christoffel curvature tensor R , the Ricci tensor S , the Ricci operator $\tilde{S}$, the tensor G and the Weyl tensor C , respectively. By $R \cdot T_{i_1 \dots i_k \ell m}$ we will denote the local components of the tensor $R \cdot T$.

The semi-Riemannian manifold (M, g) is said to be Ricci-pseudosymmetric ([14], [7]) if at every point of M the following condition is satisfied:

(***) the tensors $R \cdot S$ and $Q(g, S)$ are linearly dependent.

The manifold (M, g) is Ricci-pseudosymmetric if and only if

$$R \cdot S = L_S Q(g, S)$$

holds on the set $U_S = \{x \in M \mid S - \frac{\kappa}{n}g \neq 0 \text{ at } x\}$, where L_S is some function on U_S . It is clear that if at a point x of a manifold (M, g) (*) is satisfied then also (***) holds at x . The converse statement is not true ([14], [7]). For instance, the Cartan hypersurfaces of dimensions > 3 are not pseudosymmetric, Ricci-pseudosymmetric manifolds ([17], Theorem 1). These hypersurfaces satisfy also the condition (***) .

At the end of this section we present the following two lemmas.

Lemma 2.1. *The Weyl curvature tensor of every Ricci-pseudosymmetric semi-Riemannian manifold (M, g) , $\dim M \geq 4$, fulfils the following equality*

$$(5) \quad C(\tilde{S}X, Y, Z, W) + C(\tilde{S}Z, Y, W, X) + C(\tilde{S}W, Y, X, Z) = 0,$$

where $X, Y, Z, W \in \Xi(M)$.

PROOF. Evidently, (5) is satisfied at every point of $M - U_S$. Let now x be a point of U_S . Thus, by our assumptions, the following relation

$$(6) \quad \begin{aligned} & -R(\tilde{S}X_1, X_2, X_3, X_4) - R(X_1, \tilde{S}X_2, X_3, X_4) = \\ & = L_S(g(X_4, X_1)S(X_3, X_2) - g(X_3, X_1)S(X_4, X_2) \\ & \quad + g(X_4, X_2)S(X_3, X_1) - g(X_3, X_2)S(X_4, X_1)) \end{aligned}$$

holds at x , where $X_1, \dots, X_4 \in T_x(M)$. Summing this cyclically in X_1, X_3, X_4 we get

$$R(\tilde{S}X_1, X_2, X_3, X_4) + R(\tilde{S}X_3, X_2, X_4, X_1) + R(\tilde{S}X_4, X_2, X_1, X_3) = 0.$$

But on the other hand, we can easily verify that the following identity

$$\begin{aligned} & C(\tilde{S}X, Y, Z, W) + C(\tilde{S}Z, Y, W, X) + C(\tilde{S}W, Y, X, Z) \\ & = R(\tilde{S}X, Y, Z, W) + R(\tilde{S}Z, Y, W, X) + R(\tilde{S}W, Y, X, Z) \end{aligned}$$

holds on any semi-Riemannian manifold. Using now the last two equations, we can conclude that (5) is fulfilled at x , which completes the proof.

Lemma 2.2. *Let (M, g) , $\dim M \geq 4$, be a pseudosymmetric semi-Riemannian manifold with pseudosymmetric Weyl tensor. Then the relation*

$$(7) \quad \begin{aligned} & ((n-2)(L_R - L_C) + \frac{\kappa}{n-1})Q(g, C)(X_1, \dots, X_4; X, Y) \\ & = Q(S, C)(X_1, \dots, X_4; X, Y) \\ & + g(X_1, Y)C(\tilde{S}X, X_2, X_3, X_4) - g(X_1, X)C(\tilde{S}Y, X_2, X_3, X_4) \\ & - g(X_2, Y)C(\tilde{S}X, X_1, X_3, X_4) + g(X_2, X)C(\tilde{S}Y, X_1, X_3, X_4) \\ & + g(X_3, Y)C(\tilde{S}X, X_4, X_1, X_2) - g(X_3, X)C(\tilde{S}Y, X_4, X_1, X_2) \\ & - g(X_4, Y)C(\tilde{S}X, X_3, X_1, X_2) + g(X_4, X)C(\tilde{S}Y, X_3, X_1, X_2) \end{aligned}$$

holds on U_C , where $X, Y, X_1, \dots, X_4$ are vector fields on U_C .

PROOF. First of all we note that from (2) it follows that the relation

$$(8) \quad R \cdot C = L_R Q(g, C)$$

holds on U_C . Moreover, it is easy to verify that the following identity is satisfied on U_C

$$C \cdot C(X_1, \dots, X_4; X, Y) = R \cdot C(X_1, \dots, X_4; X, Y)$$

$$\begin{aligned}
& + \frac{\kappa}{(n-1)(n-2)} Q(g, C)(X_1, \dots, X_4; X, Y) \\
& - \frac{1}{n-2} Q(S, C)(X_1, \dots, X_4; X, Y) \\
& - \frac{1}{n-2} (g(X_1, Y)C(\tilde{S}X, X_2, X_3, X_4) - g(X_1, X)C(\tilde{S}Y, X_2, X_3, X_4) \\
& - g(X_2, Y)C(\tilde{S}X, X_1, X_3, X_4) + g(X_2, X)C(\tilde{S}Y, X_1, X_3, X_4) \\
& + g(X_3, Y)C(\tilde{S}X, X_4, X_1, X_2) - g(X_3, X)C(\tilde{S}Y, X_4, X_1, X_2) \\
& - g(X_4, Y)C(\tilde{S}X, X_3, X_1, X_2) + g(X_4, X)C(\tilde{S}Y, X_3, X_1, X_2)).
\end{aligned}$$

Now, applying in this (3) and (8) we get (7), completing the proof.

3. Manifolds satisfying (*) and (**)

Theorem 3.1. *Let (M, g) , $\dim M \geq 4$, be a pseudosymmetric manifold with pseudosymmetric Weyl tensor. Then the conditions (***) and*

$$(9) \quad Q\left(S - \left(\frac{\kappa}{n-1} - L_R + L_C\right)g, C - \frac{\mu}{n(n-2)}G\right) = 0, \quad \mu \in \mathbb{R},$$

hold at every point of U_C .

PROOF. Let x be a point of U_C . We can present (7) in the following form

$$\begin{aligned}
(10) \quad & \left((n-2)(L_R - L_C) + \frac{\kappa}{n-1}\right) Q(g, C)_{hijklm} = Q(S, C)_{hijklm} \\
& + g_{hm}S_\ell^p C_{pijk} - g_{hl}S_m^p C_{pijk} - g_{im}S_\ell^p C_{phjk} + g_{il}S_m^p C_{phjk} \\
& + g_{jm}S_\ell^p C_{pkhi} - g_{jl}S_m^p C_{pkhi} - g_{km}S_\ell^p C_{pjhi} + g_{kl}S_m^p C_{pjhi}.
\end{aligned}$$

Contracting this with g^{hm} and using the identities

$$\begin{aligned}
& g^{hm}Q(g, C)_{hijklm} = (n-1)C_{lijk}, \\
& g^{hm}Q(S, C)_{hijklm} = \kappa C_{lijk} + S_i^p C_{pljk} + S_j^p C_{pil k} + S_k^p C_{pij l} - S_\ell^p C_{pijk},
\end{aligned}$$

we obtain

$$\begin{aligned}
(11) \quad & (n-1)(n-2)(L_R - L_C)C_{lijk} = S_i^p C_{pljk} + S_j^p C_{pil k} \\
& + S_k^p C_{pij l} + (n-2)S_\ell^p C_{pijk} + g_{j\ell}A_{ki} - g_{k\ell}A_{ij},
\end{aligned}$$

where

$$(12) \quad A_{ij} = S^{pq}C_{ipqj} = S^{pq}R_{ipqj} + \frac{2}{n-2}S_{ij}^2 - \frac{n\kappa}{(n-1)(n-2)}S_{ij} \\ + \frac{1}{n-2} \left(\frac{\kappa^2}{n-1} - S^{pq}S_{pq} \right) g_{ij}.$$

But (11), in view of Lemma 2.1, reduces to

$$(13) \quad (n-1)(n-2)(L_R - L_C)C_{\ell ijk} = S_i^p C_{p\ell jk} \\ + (n-1)S_\ell^p C_{pijk} + g_{j\ell}A_{ki} - g_{k\ell}A_{ij}.$$

Symmetrizing this in ℓ, i we find

$$(14) \quad S_i^p C_{p\ell jk} + S_\ell^p C_{pijk} = \frac{1}{n}(g_{k\ell}A_{ij} + g_{ki}A_{j\ell} - g_{j\ell}A_{ik} - g_{ij}A_{k\ell}).$$

Thus (13) turns into

$$(15) \quad (n-1)(n-2)(L_R - L_C)C_{\ell ijk} = (n-2)S_\ell^p C_{pijk} + g_{j\ell}A_{ki} \\ - g_{k\ell}A_{ij} + \frac{1}{n}(g_{k\ell}A_{ij} + g_{ki}A_{j\ell} - g_{j\ell}A_{ik} - g_{ij}A_{k\ell}).$$

It is easy to check that the identity

$$-S_i^p C_{pjkl} - S_j^p C_{pikl} = \frac{1}{n-2}(g_{jk}S_{il}^2 - g_{j\ell}S_{ik}^2 + g_{ik}S_{j\ell}^2 - g_{il}S_{jk}^2) \\ + R \cdot S_{ijkl} - \frac{\kappa}{(n-1)(n-2)}(g_{jk}S_{il} - g_{j\ell}S_{ik} + g_{ik}S_{j\ell} - g_{il}S_{jk}),$$

by making use of the relation $R \cdot S = L_R Q(g, S)$, which follows immediately from (2), turns into

$$(16) \quad -S_i^p C_{pjkl} - S_j^p C_{pikl} = g_{j\ell}B_{ik} - g_{jk}B_{il} + g_{il}B_{jk} - g_{ik}B_{j\ell},$$

$$(17) \quad B_{jk} = L_R S_{jk} + \frac{\kappa}{(n-1)(n-2)}S_{jk} - \frac{1}{n-2}S_{jk}^2.$$

Moreover, contracting the equality

$$-S_i^p R_{pj\ell m} - S_j^p R_{pilm} = L_R(g_{im}S_{j\ell} + g_{jm}S_{il} - g_{il}S_{jm} - g_{j\ell}S_{im})$$

with $g^{\ell m}$ we obtain

$$S^{pq}R_{jpq\ell} = S_{j\ell}^2 - nL_R \left(S_{j\ell} - \frac{\kappa}{n}g_{j\ell} \right).$$

Substituting this in (12) and using (17) we get

$$(18) \quad A_{ij} = -nB_{ij} + \text{tr}(B)g_{ij}.$$

Next, transvecting (15) with S^{ij} we find

$$(19) \quad \left((n-2)(L_R - L_C) + \frac{\kappa}{n(n-1)} \right) A_{\ell k} = E_{\ell k} - \frac{1}{n} \text{tr}(E) g_{\ell k},$$

where $E_{\ell k} = S_\ell^p A_{pk}$. On the other hand, transvecting (13) with S_m^k and symmetrizing the resulting equality in j, m we obtain

$$(n-1)(n-2)(L_R - L_C)(S_m^p C_{pjil} + S_j^p R_{pmil}) = S_i^k (S_m^p C_{pjlk} + S_j^p C_{pmlk}) \\ + (n-1)S_\ell^k (S_m^p C_{pjik} + S_j^p C_{pmik}) + g_{\ell m} E_{ij} + g_{j\ell} E_{im} - S_{j\ell} A_{im} - S_{\ell m} A_{ij}.$$

This, by an application of (14), yields

$$\frac{(n-1)(n-2)}{n} (L_R - L_C) (g_{\ell m} A_{ij} + g_{j\ell} A_{im} - g_{ij} A_{\ell m} - g_{im} A_{j\ell}) \\ = -\frac{1}{n} (g_{\ell m} E_{ij} + g_{j\ell} E_{im} - S_{ij} A_{\ell m} - S_{im} A_{j\ell}) + \frac{n-1}{n} (S_{\ell m} A_{ij} \\ + S_{j\ell} A_{im} - g_{ij} E_{\ell m} - g_{im} E_{j\ell}) + g_{\ell m} E_{ij} + g_{j\ell} E_{im} - A_{ij} S_{\ell m} - A_{im} S_{j\ell}$$

which, after antisymmetrization in ℓ, i , turns into

$$(n-1)(n-2)(L_R - L_C)Q(g, A) = -Q(S, A) + (n-1)Q(g, E).$$

Now, in virtue of (19), we have

$$(20) \quad Q\left(S - \frac{\kappa}{n}g, A\right) = 0.$$

It is clear that if the tensor $S - \frac{\kappa}{n}g$ vanishes at x then also $(***)$ and

$$(21) \quad A = \mu\left(S - \frac{\kappa}{n}g\right), \quad \mu \in \mathbb{R},$$

are satisfied at x . Further, in view of Lemma 2.4(i) [15], we can conclude from (20) that if the tensor $S - \frac{\kappa}{n}g$ is nonzero at x then also (21) is fulfilled at x . Now (18), by (17) and (21), leads to $(***)$. Finally, we prove that (9) holds at x . In fact, the relation (15), in virtue of (21), turns into

$$S_\ell^p C_{pijk} = (n-1)(L_R - L_C)C_{\ell ijk} - \frac{1}{n^2}\mu\kappa G_{\ell ijk} + \frac{\mu}{n-2}(g_{\ell k}S_{ij} - g_{j\ell}S_{ik}) \\ (22) \quad - \frac{\mu}{n(n-2)}(g_{ik}S_{j\ell} + g_{\ell k}S_{ij} - g_{j\ell}S_{ik} - g_{ik}S_{\ell k}).$$

Applying this in (10) we get

$$Q\left(\left(\frac{\kappa}{n-1} - L_R + L_C\right)g, C\right) = Q\left(S, C - \frac{\mu}{n(n-2)}G\right),$$

which turns immediately into (9), completing the proof.

Combining Theorem 3.1 with Lemma 1.2 of [4] we get immediately the following corollary.

Corollary 3.1. *Let (M, g) , $\dim M \geq 4$, be a pseudosymmetric manifold with pseudosymmetric Weyl tensor. Then the tensors $S^2 - \frac{\text{tr}(S^2)}{n}g$ and $S - \frac{\kappa}{n}g$ are linearly dependent at every point of M .*

Theorem 3.2. *Let (M, g) , $\dim M \geq 4$, be a pseudosymmetric manifold with pseudosymmetric Weyl tensor. Moreover, let V be a vector at $x \in U_C$ such that the scalar $\varrho = a(V)$ is nonzero, where a is a covector defined by*

$$a(X) = S(X, V) - \left(\frac{\kappa}{n-1} - L_R + L_C \right) g(X, V),$$

$X \in T_x(M)$ and let F be a $(0, 2)$ -tensor defined by

$$F(X, Y) = C(V, X, Y, V) - \frac{\mu}{n(n-2)} G(V, X, Y, V), \quad X, Y \in T_x M,$$

and μ is the real number given in (9).

(i) *If the tensor $S - (\frac{\kappa}{n-1} - L_R + L_C)g - \frac{1}{\varrho}a \otimes a$ vanishes then the relations:*

$$(23) \quad \begin{aligned} C(X, Y, Z, W) = \\ = \frac{1}{\varrho^2} (F((X \wedge_{a \otimes a} Y)Z, W) - F((X \wedge_{a \otimes a} Y)W, Z)), \end{aligned}$$

$$(24) \quad a(X)\tilde{C}(Y, Z) + a(Y)\tilde{C}(Z, X) + a(Z)\tilde{C}(X, Y) = 0,$$

$$(25) \quad C \cdot C = 0, \quad L_C = 0,$$

$$(26) \quad L_R = \frac{\kappa}{n(n-1)},$$

$$(27) \quad S = \frac{\kappa}{n}g + \frac{1}{\varrho}a \otimes a,$$

hold at x , where $X, Y, Z, W \in T_x(M)$.

(ii) *If the tensor $S - (\frac{\kappa}{n-1} - L_R + L_C)g - \frac{1}{\varrho}a \otimes a$ is nonzero then the equality*

$$\begin{aligned} \varrho C(X, Y, Z, W) = \\ \left(\frac{\mu \varrho}{(n-1)(n-2)} + \lambda \left(\frac{\kappa}{n-1} - L_R + L_C \right)^2 \right) G(X, Y, Z, W) \end{aligned}$$

$$\begin{aligned}
& +\lambda S((X \wedge_S Y)Z, W) - \lambda \left(\frac{\kappa}{n-1} - L_R + L_C \right) (g(X, W)S(Y, Z) \\
& + g(Y, Z)S(X, W) - g(X, Z)S(Y, W) - g(Y, W)S(X, Z))
\end{aligned}$$

holds at x , where $X, Y, Z, W \in T_x(M)$.

PROOF. We will use notations of Proposition 4.1 of [2].

(i) Applying Proposition 4.1(i) of [2] we get

$$(28) \quad \varrho^2 \left(C - \frac{\mu}{n(n-2)} G \right)_{\ell i j k} = a_\ell a_k F_{ij} + a_i a_j F_{\ell k} - a_\ell a_j F_{ik} - a_i a_k F_{\ell j}.$$

(i1) We assume that μ vanishes. Then (28) reduces to (23). Thus (24) is satisfied. Moreover, from (23), in view of Theorem 1 of [13], it follows that (25) is fulfilled. Next, applying Theorem 4.2 of [2] we get (26). Now we see that the Ricci tensor S takes the required form (27).

(i2) Suppose that μ is nonzero. Then (28), after contraction with g^{ij} , yields

$$(29) \quad a^p a_p F_{\ell k} = -\frac{n-1}{n(n-2)} \mu \varrho^2 g_{\ell k} + a_\ell b_k - a_k a_\ell,$$

where $b_k = a^p F_{\ell k}$. We note that if $a^p a_p$ vanishes then (29) reduces to

$$g_{\ell k} = \tau(a_\ell b_k + a_k b_\ell), \quad \tau \in \mathbb{R} - \{0\},$$

a contradiction. Thus $a^p a_p$ must be nonzero. Multiplying now (28) by $a^p a_p$ and using (29) we get

$$\begin{aligned}
& a^p a_p \left(C - \frac{\lambda}{n(n-2)} G \right)_{\ell i j k} = \\
& = -\frac{n-1}{n(n-2)} \mu (a_\ell a_k g_{ij} + a_i a_j g_{\ell k} - a_i a_k g_{\ell j} - a_\ell a_j g_{ik}).
\end{aligned}$$

Contracting this with $g^{\ell k}$ we get

$$g_{ij} = \tilde{\tau} a_i a_j, \quad \tilde{\tau} \in \mathbb{R} - \{0\},$$

a contradiction.

(ii) The second assertion is an immediate consequence of Proposition 4.1(ii) of [2]. Our theorem is thus proved.

4. Examples

In this section we present some examples of four-dimensional Riemannian manifolds with pseudosymmetric Weyl tensor.

Let (M, g) be the four-dimensional manifold defined in [6] (Lemme 1.1). As it was proved in [6] (see Lemme 1.1 and Remarqué 1.5), (M, g) is a non conformally flat and non semisymmetric, Weyl-semisymmetric manifold, i.e. the tensors C and $R \cdot R$ are nonzero and the condition $R \cdot C = 0$ holds on M . Furthermore, using formulas for the local components of the Ricci tensor and the Riemann-Christoffel tensor of the manifold (M, g) obtained in [6] (Lemme 1.1), we can easily verify that (M, g) is a non Ricci-pseudosymmetric manifold satisfying at every point the condition $(**)$.

Let V be a connected subset of the set $U = \{x \in M : u(x) \neq 0\}$, where u is the function defined in [6] (Lemme 1.1). By formula (10) of [6] we have $U = U_C$. Let on V be given the conformal deformation $g \rightarrow \bar{g}_k = \frac{1}{(u+k)^2}g$ of the metric g , where k is a constant such that $k \geq 0$ when $u > 0$ on V or $k \leq 0$ when $u < 0$ on V .

First we consider the deformation $g \rightarrow \bar{g}_0 = \frac{1}{u^2}g$. The manifold $(V, \bar{g}_0)$ is an Einstein manifold ([6], Lemme 1.1(viii)). Moreover, as it was stated in [8] (Example 3) the relation

$$\bar{R} \cdot \bar{R} = -\frac{1}{12}(u^3 - pq)Q(\bar{g}_0, \bar{R})$$

holds on $(V, \bar{g}_0)$, where $\bar{R}$ is the Riemann-Christoffel curvature tensor of $\bar{g}_0$ and p, q are some constants. Now, in view of Theorem 3.1 of [3], the equality

$$\bar{C} \cdot \bar{C} = -\frac{1}{12}(u^3 - pq + \bar{\kappa})Q(\bar{g}_0, \bar{C})$$

holds on V , where $\bar{C}$ is the Weyl conformal curvature tensor of $\bar{g}_0$ and the constant $\bar{\kappa}$ is the scalar curvature of $\bar{g}_0$. From this we get easily

$$C \cdot C = -\frac{u^3 - pq + \bar{\kappa}}{12u^2}Q(g, C).$$

Thus we see that the condition $(**)$ is fulfilled on (M, g) , i.e. (M, g) is a manifold with pseudosymmetric Weyl tensor.

Now we consider on V the conformal deformation $g \rightarrow \bar{g}_k = \frac{1}{(u+k)^2}g$, where k is a positive or negative constant. Since (V, g) is a manifold with pseudosymmetric Weyl tensor, then $(V, \bar{g}_k)$, $k \neq 0$, is also a manifold with pseudosymmetric Weyl tensor. Moreover, as it was stated in [8] (Example 2) this manifold is a non Ricci-pseudosymmetric manifold.

Finally, we state that (M, g) is a Riemannian manifold which cannot be realized as a hypersurface immersed isometrically in a 5-dimensional Euclidean space. This is an immediate consequence of Corollary 3.1 of [15] and the fact that the tensor $R \cdot R - Q(S, R)$ is a nonzero tensor on M . More generally, (M, g) cannot be realized as a hypersurface immersed isometrically in a 5-dimensional space of constant curvature. This statement follows immediately from Proposition 3.1 of [15] and the fact that

the tensors $R \cdot R - Q(S, R)$ and $Q(g, C)$ are not linearly dependent at every point of M .

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