

Geometry of space-time and generalised Lagrange gauge theory

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Dedicated to Professor Lajos Tamásy on his 70th birthday

Abstract. In §1 and §2 the authors present the Einstein and Maxwell equations for the generalised Lagrange space

$$GL^n = (M, g_{ij}(x, y) = e^{2\sigma(x, y)} \gamma_{ij}(x)),$$

and characterize the case of vanishing mixed curvature tensor field of the canonical linear d -connection. The Lagrangian gauge theory — in G.S. ASANOV's sense [1] is developed in §3 for the tangent bundle endowed with (h, v) -metrics, obtaining the generalised Einstein - Yang Mills equations with respect to the metric gauge tensor fields and to the gauge field $\sigma(x, y)$ for three remarkable cases in which the metrics are derived from the fundamental tensor field $g_{ij}(x, y)$. Proofs are, in most cases, mechanical but rather tedious calculations. They are omitted.

Introduction

In a previous paper [6] it was shown that the EPS conditions can be satisfied in a more general frame than that of Finsler spaces, namely for convenient generalised Lagrange spaces. More precise, was examined the space $GL^n = (M, g_{ij}(x, y))$, where M is a n -dimensional differentiable manifold and

$$(A1) \quad g_{ij}(x, y) = e^{2\sigma(x, y)} \gamma_{ij}(x),$$

$\gamma_{ij}(x)$ being a metric tensor field on M , and $\sigma : TM \rightarrow \mathbb{R}$ a function of class C^∞ on $\widetilde{TM} = TM \setminus \{0\}$, continuous on the null section of the tangent

bundle. The space was endowed with the non-linear connection

$$(A2) \quad N^i_j(x, y) = \left\{ \begin{matrix} i \\ kj \end{matrix} \right\} y^k,$$

where $\left\{ \begin{matrix} i \\ kj \end{matrix} \right\}$ are the Christoffel symbols for $\gamma_{ij}(x)$. Under these assumptions, GL^n represents a convenient relativistic model, since it has the same conformal and projective properties as the Riemannian space $V^n = (M, \gamma_{ij}(x))$.

§1. Einstein equations for GL^n

Developing the formalism presented in [3,4], we remind that the canonical h - and v -symmetrical d -connection of GL^n has the coefficients

$$(1.1) \quad \begin{aligned} L_{jk}^i &= \left\{ \begin{matrix} i \\ jk \end{matrix} \right\} + \Lambda_{jk}^i \\ C_{bc}^a &= \delta_b^a \dot{\sigma}_c + \delta_c^a \dot{\sigma}_b - \gamma_{bc} \dot{\sigma}^a \end{aligned}$$

where we used the notations

$$(1.1') \quad \begin{aligned} \Lambda_{jk}^i &= \delta_j^i \sigma_k + \delta_k^i \sigma_j - \gamma_{jk} \sigma^i \\ \sigma_k &= \delta_k \sigma, \quad \dot{\sigma}_a = \dot{\partial}_a \sigma, \quad \sigma^k = \gamma^{ks} \sigma_s, \quad \dot{\sigma}^a = \gamma^{ab} \dot{\sigma}_b \end{aligned}$$

and

$$(1.1'') \quad \delta_k = \partial_k - N_k^a(x, y) \dot{\partial}_a, \quad \dot{\partial}_b = \frac{\partial}{\partial y^b}; \quad \partial_k \equiv \frac{\partial}{\partial x^k}.$$

Then the torsion d -tensor fields of the canonical linear d -connection $CT(N)$ of GL^n have the coefficients given by

$$(1.2) \quad \begin{cases} T_{jk}^i = 0, & S_{bc}^a = 0, & C_{ja}^i, \\ R_{kl}^a = r_d^a{}_{kl} y^d, & P_{kb}^a = -\Lambda_{bk}^a. \end{cases}$$

and the curvature d -tensor fields have the expressions

$$(1.3) \quad \begin{cases} R_{jkl}^i = r_{jkl}^i + \delta_{(k}^i \sigma_{j)l} - \gamma^{is} \gamma_{j(k} \sigma_{s)l} + \gamma_{js} \dot{\sigma}^{(s} R_{kl}^{i)} \\ P_j^i{}_{kc} = \delta_k^i \dot{\sigma}_{jc} - \delta_c^i \dot{\sigma}_{jk} - \gamma^{is} \cdot \left[\gamma_{jk} \dot{\sigma}_{sc} - \gamma_{jc} \dot{\sigma}_{sk} \right] + \gamma^{is} \gamma_{ck} \sigma_{(s} \dot{\sigma}_{j)} \\ S_{bcd}^a = \delta_{(c}^a \dot{\sigma}_{b)d} - \gamma^{as} \gamma_{b(c} \dot{\sigma}_{s)d} \end{cases}$$

where $r_j^i{}_{kl}$ is the curvature tensor field of $\gamma_{ij}(x)$, $t_{(ij)} = t_{ij} - t_{ji}$, and we considered the following d -tensor fields

$$(1.3') \quad \begin{aligned} \sigma_{sl} &= \sigma_s|_l + \sigma_s \sigma_l - \frac{1}{2} \gamma_{sl}^H \sigma = \sigma_{s\uparrow l} - \sigma_s \sigma_l + \frac{1}{2} \gamma_{sl}^H \sigma \\ \overset{1}{\sigma}_{sl} &= \sigma_s|_l + \sigma_s \dot{\sigma}_l - \frac{1}{2} \gamma_{sl}^M \sigma = \dot{\sigma}_l \sigma_s - \dot{\sigma}_s \sigma_l + \frac{1}{2} \gamma_{sl}^M \sigma \\ \overset{2}{\sigma}_{sl} &= \dot{\sigma}_s|_l + \dot{\sigma}_s \sigma_l - \frac{1}{2} \gamma_{sl}^M \sigma = \dot{\sigma}_{s\uparrow l} - \sigma_s \dot{\sigma}_l + \frac{1}{2} \gamma_{sl}^M \sigma \\ \dot{\sigma}_{sl} &= \dot{\sigma}_s|_l + \dot{\sigma}_s \dot{\sigma}_l - \frac{1}{2} \gamma_{sl}^V \sigma = \dot{\sigma}_l \dot{\sigma}_s - \dot{\sigma}_s \dot{\sigma}_l + \frac{1}{2} \gamma_{sl}^V \sigma \end{aligned}$$

with $\overset{H}{\sigma} = \sigma_s \sigma^s$, $\overset{M}{\sigma} = \sigma_s \dot{\sigma}^s$, $\overset{V}{\sigma} = \dot{\sigma}_s \dot{\sigma}^s$.

We also used the covariant derivatives

$$(1.3'') \quad \sigma_s|_l = \delta_l \sigma_s - L_{sl}^t \sigma_t, \quad \sigma_s|_l = \dot{\sigma}_l \sigma_s - C_{sl}^t \sigma_t, \quad \sigma_{s\uparrow l} = \partial_l \sigma_s - \left\{ \begin{matrix} t \\ sl \end{matrix} \right\} \sigma_t.$$

Then, by contracting the indices, one can derive the four Ricci tensor fields, expressed by

$$(1.4) \quad \begin{aligned} R_{ij} &= r_{ij} - \gamma_{ij} \dot{\sigma} + (2 - n) \cdot \sigma_{ij} + \gamma_{is} \dot{\sigma}^{(s} r_t^{a)}{}_{ja} \\ \overset{1}{P}_{bk} &= (1 - n) \overset{2}{\sigma}_{bk} + \overset{1}{\sigma}_{bk} - \gamma_{bk} \overset{1}{\sigma} + \sigma_{(k} \dot{\sigma}_{b)} \\ \overset{2}{P}_{bc} &= (1 - n) \overset{1}{\sigma}_{bc} - \overset{2}{\sigma}_{bc} - \gamma_{bc} \overset{2}{\sigma} + \sigma_{(c} \dot{\sigma}_{b)} \\ S_{bc} &= (2 - n) \dot{\sigma}_{bc} - \gamma_{bc} \dot{\sigma} \end{aligned}$$

and then the curvature scalar fields

$$(1.5) \quad \begin{cases} R = e^{-2\sigma} [r + 2(1 - n) \overset{0}{\sigma} + 2r_{sc} y^s \dot{\sigma}^c] \\ \overset{1}{P} = -\overset{2}{P} = e^{-2\sigma} \cdot 2(1 - n) \overset{1}{\sigma}, \quad S = e^{-2\sigma} \cdot 2(1 - n) \dot{\sigma}, \end{cases}$$

where

$$\overset{0}{\sigma} = \gamma^{ij} \sigma_{ij}, \quad \overset{1}{\sigma} = \gamma^{ij} \overset{1}{\sigma}_{ij}, \quad \overset{2}{\sigma} = \gamma^{ij} \overset{2}{\sigma}_{ij}, \quad \dot{\sigma} = \gamma^{ab} \dot{\sigma}_{ab}, \quad r_{bk} = r_{bks}^s, \quad r = \gamma^{bk} r_{bk}.$$

Regarding the mixed curvature tensor field, the following result obtained by direct calculation holds true:

Theorem. *The following assertions are valid:*

- a) $P_{bkc}^a - P_{bck}^a = 0$ iff $\overset{1}{\sigma}_{bc} + \overset{1}{\sigma}_{cb} = 0$
- b) $P_{bkc}^a + P_{bck}^a = 0$ iff $\overset{1}{\sigma}_{bc} - \overset{1}{\sigma}_{cb} = \frac{2}{n} \sigma_{(b} \dot{\sigma}_{c)} = 0$.
- c) $P_b^a{}_{kc} = 0$ implies $\overset{HV}{\sigma} \sigma = \overset{M}{\sigma}^2$ and $\overset{1}{\sigma} = \overset{2}{\sigma} = 0$

$$\begin{aligned} d) \quad & \overset{1}{P}_{bk} = \overset{2}{P}_{bk} = 0 \quad \text{iff} \quad \overset{1}{\sigma}_{bk} = \frac{1}{n} \sigma_{(b} \dot{\sigma}_{k)} \\ e) \quad & \overset{1}{\sigma}_{jk} = \overset{2}{\sigma}_{kj}. \end{aligned}$$

Remark. Using a), b), and e) it becomes obvious that $P_b^a{}_{kc} = 0$ implies $\overset{1}{\sigma}_{bk} = \overset{2}{\sigma}_{bk} = 0$.

The Einstein equations of the space GL^n have the generic form

$$(1.6) \quad \begin{cases} R_{ij} - \frac{1}{2} R g_{ij} = \kappa \overset{H}{T}_{ij} \\ S_{ab} - \frac{1}{2} S g_{ab} = \kappa \overset{V}{T}_{ab} \end{cases}$$

where $\overset{H}{T}_{ij}$ and $\overset{V}{T}_{ab}$ are called the h - and the v -components of the energy-momentum tensor field of the space GL^n respectively, $g_{ab} = \delta_a^i \delta_b^j \cdot g_{ij}$ and κ is the gravific constant. Using the expressions of the Ricci d -tensor fields (1.4) and of the curvature scalars (1.5) we obtain that the Einstein equations of the space GL^n admit the equivalent form

$$(1.6') \quad \begin{cases} r_{ij} - \frac{1}{2} r \gamma_{ij} + t_{ij} = \kappa \overset{H}{T}_{ij} \\ (2-n)(\dot{\sigma}_{ab} - \dot{\sigma} \gamma_{ab}) = \kappa \overset{V}{T}_{ab} \end{cases}$$

where

$$t_{ij} = (n-2)(\gamma_{ij} \overset{0}{\sigma} - \sigma_{ij}) + \gamma_{ij} r_{st} y^s \dot{\sigma}^t + \gamma_{is} \dot{\sigma}^{(s} r_{tj}^{a)} y^t$$

We remark that in the first equation of the system (1.6') the term t_{ij} is complementary to the classical Einstein equations of the Riemannian space $V^n = (M, \gamma_{ij}(x))$.

§2. Maxwell equations for GL^n

Let the h - and v -deflection tensor fields be given respectively by

$$D^i{}_j = y^i|_j, \quad d^i{}_j = y^i|_j$$

and the corresponding ones having the indices lowered

$$D_{ij} = g_{is} D^s{}_j, \quad d_{ij} = g_{is} d^s{}_j.$$

Then we define the h - and v -electromagnetic tensor fields by

$$(2.1) \quad F_{ij} = \frac{1}{2}(D_{ij} - D_{ji}), \quad f_{ij} = \frac{1}{2}(d_{ij} - d_{ji})$$

respectively, and infer the Maxwell equations

$$(2.2) \quad \begin{cases} \sigma_{ijk} F_{ij|k} = \sigma_{ijk} e^{2\sigma} r_0^s r_{kj} y_i \dot{\sigma}_s \\ \sigma_{ijk} F_{ij|k} + \sigma_{ijk} f_{ij|k} = 0 \\ \sigma_{ijk} f_{ij|k} = 0 \end{cases}$$

where σ_{ijk} denotes cyclic summation in the indices i, j, k and

$$r_0^s r_{kj} = r_t^s r_{kj} y^t, \quad y_i = \gamma_{is} y^s.$$

Indeed, (2.2) takes place since D_{ij} and d_{ij} satisfy the relations

$$\begin{cases} D_{ij|k} - D_{ik|j} = R_{0ijk} - d_{is} R_{jk}^s \\ D_{ij|k} - d_{ik|j} = P_{0ijk} - D_{is} C_{jk}^s - d_{is} P_{jk}^s \\ d_{ij|k} - d_{ik|j} = S_{0ijk} \end{cases}$$

and the Bianchi identities lead to (2.2). Also, the same equations come up clearly if we consider the expressions

$$F_{ij} = e^{2\sigma} (y_i \sigma_j - y_j \sigma_i), \quad f_{ij} = e^{2\sigma} (y_i \cdot \sigma_j - \gamma_j \sigma_i)$$

using direct computation.

In the following we examine the Maxwell equations for some remarkable particular examples of spaces GL^n :

1. If $\sigma(x, y) = \frac{1}{2} \gamma_{rs} y^r y^s$, then the Maxwell equations are trivially satisfied, since as $\sigma_k = 0$ and $\dot{\sigma}_k = y_k$, we infer that

$$F_{ij} = f_{ij} = 0.$$

2. If GL^n is locally Minkowskian, i.e. at any point $(x, y) \in TM$ there exists a domain of a local map in which

$$g_{ij}(x, y) = e^{2\sigma(y)} \gamma_{ij} \quad \text{with} \quad \partial_k \gamma_{ij} = 0,$$

then $F_{ij} = 0$ and $f_{ij} = e^{2\sigma(y)} y_i \dot{\sigma}_j$, and the Maxwell equations

$$\sigma_{ijk} f_{ij|k} = 0$$

take place.

3. For a dispersive medium [7] $(M, V^i(x), n(x, V(x)))$ where $V^i(x)$ is the speed of the particle and $n(x, V(x))$ is the refraction index, we can consider

$$\sigma(x, y) = \alpha \cdot \left(1 - \frac{1}{n^2(x, y)} \right), \quad \alpha \in \mathbb{R}, \alpha > 0.$$

Remark that for $y^i = V^i(x)$ the Maxwell equations have to be computed directly, being not implied by (2.2).

§3. Generalized Einstein - Yang Mills equations

Let TM be endowed with a (h, v) -metric G_* [3] given by

$$(3.1) \quad G_* = g_{ij} dx^i \otimes dx^j + h_{ab} \delta y^a \otimes \delta y^b$$

where $\{dx^i, \delta y^a \mid i, a = \overline{1, n1, n}\}$ is the adapted basis for the cotangent bundle, dual to (1.1''), and $g_{ij}(x, y)$, $h_{ab}(x, y)$ are symmetric d -tensor fields of rank n .

Let the coordinate transform on TM be given by

$$(3.2) \quad \begin{cases} x^i = x^i(x^j), & \det(\partial x^i / \partial \bar{x}^j) \neq 0, \\ y^a = \frac{\partial x^a}{\partial \bar{x}^b} \bar{y}^b, & i, j, a, b = \overline{1, n}. \end{cases}$$

A generalized gauge transformation on TM is a diffeomorphism of TM compatible with the tangent bundle structure, given locally by

$$(3.3) \quad \begin{cases} x^i = X^i(\tilde{x}^j), & \det(\partial X^i / \partial \tilde{x}^j) \neq 0, \\ y^a = Y_b^a(\tilde{x}) \tilde{y}^b, & \det(Y_b^a(\tilde{x})) \neq 0. \end{cases}$$

A gauge d -tensor field is a d -tensor field which obeys tensor-type transformation rules with respect to (3.3); e.g. the gauge d -tensor field $\{w_{jb}^i(x, y)\}$ transforms relative to (3.2) and (3.3) according to

$$(3.4) \quad \begin{cases} \bar{w}_{jb}^i(\bar{x}, \bar{y}) \frac{\partial x^k}{\partial \bar{x}^i} \frac{\partial x^c}{\partial \bar{x}^a} = w_{ld}^{kc}(x, y) \frac{\partial x^l}{\partial \bar{x}^j} \frac{\partial x^d}{\partial \bar{x}^b}, \\ \tilde{w}_{jb}^{ia}(\tilde{x}, \tilde{y}) \frac{\partial X^k}{\partial \tilde{x}^i} \frac{\partial Y^c}{\partial \tilde{y}^a} = w_{ld}^{kc}(x, y) \frac{\partial X^l}{\partial \tilde{x}^j} \frac{\partial Y^d}{\partial \tilde{y}^b}. \end{cases}$$

Let TM be endowed with the nonlinear connection $N = \{N_i^a(x, y)\}$. If δ_k given by (1.1) yields d -covector fields when acting on gauge scalar fields (i.e. functions $f \in F(TM)$ which obey $\bar{f}(\bar{x}, \bar{y}) = f(x, y)$, $\tilde{f}(\tilde{x}, \tilde{y}) = f(x, y)$), then N is called a generalized gauge non-linear connection. Further, if $|_k$ and $|_c$ are the h - and v -covariant derivations associated with a linear d -connection D on TM (i.e. a connection that preserves by parallelism the distributions N and VTM [3]), having the coefficients

$$D\Gamma(N) = \{L_{jk}^i(x, y), L_{bk}^a(x, y), C_{ja}^i(x, y), C_{bc}^a(x, y)\}$$

given by the relations

$$(3.5) \quad \begin{cases} D_{\delta_j} \delta_i = L_{ij}^k \delta_k, & D_{\dot{\partial}_a} \delta_j = C_{ja}^i \delta_i \\ D_{\delta_k} \dot{\partial}_b = L_{bk}^a \dot{\partial}_a, & D_{\dot{\partial}_c} \dot{\partial}_b = C_{bc}^a \dot{\partial}_a \end{cases}$$

then we say that D is a gauge linear d -connection iff $|_k$ and $|_c$ preserve the gauge tensorial character.

In the following we suppose that N is given by (A2), that g_{ij} and h_{ab} in (3.1) are gauge d -tensor fields, that γ_{ij} and σ in (A1) are gauge fields (tensor, respectively scalar), and we fix the coefficients

$$C_{ja}^i = 0, \quad L_{bk}^a = \dot{\partial}_b N_k^a = \left\{ \begin{matrix} a \\ bk \end{matrix} \right\}.$$

Then we can infer the following

Proposition. *If $g_{ij|k} = 0$ and $h_{ab|c} = 0$, $L_{jk}^i = L_{kj}^i$ and $C_{bc}^a = C_{cb}^a$, then*

$$(3.6) \quad \left\{ \begin{array}{l} L_{jk}^i = \frac{1}{2} g^{is} (\delta_j g_{sk} + \delta_k g_{sj} - \delta_s g_{jk}) \\ C_{bc}^a = \frac{1}{2} h^{as} (\dot{\partial}_b h_{sc} + \dot{\partial}_c h_{sb} - \dot{\partial}_s h_{bc}). \end{array} \right.$$

We remark that if g_{ij} is given by (A1) and $h_{ab} = \delta_a^i \delta_b^j \cdot g_{ij}$, then L_{jk}^i and C_{bc}^a are those given by (1.1).

It can be shown by direct calculation that the torsion and curvature d -tensor fields of $D\Gamma(N)$ are gauge d -tensor fields, and we have [3,4]

$$(3.7) \quad T_{jk}^i = 0, \quad S_{bc}^a = 0, \quad P_{kb}^a = 0, \quad C_{ja}^i = 0, \quad R_{kl}^a = r_{skl}^a y^s$$

and

$$(3.8) \quad \begin{aligned} R_{bkl}^a &= r_{bkl}^a + C_{bd}^a R_{kl}^d, \quad P_{jkc}^i = \dot{\partial}_c L_{jk}^i \\ P_{bkc}^a &= -C_{bc|k}^a, \quad S_{jbc}^i = 0, \end{aligned}$$

with R_{jkl}^i and S_{bcd}^a given by (1.3) for L_{jk}^i and C_{bc}^a given by (1.1). So that $D\Gamma(N)$ admits one torsion and five curvature non-trivial gauge d -tensor fields. It is obvious that the mixed Lagrangian

$$(3.9) \quad L = l_1 R + l_5 S + \hat{L}$$

is a gauge scalar field. Here

$$(3.10) \quad \begin{aligned} \hat{L} &= l_0 L_0 + l_2 \hat{R} + l_3 \overset{h}{P} + l_4 \overset{v}{P}, \quad L_0 = R_{kl}^a R_a^{kl}, \quad \overset{v}{R} = R_{bkl}^a R_a^{bkl} \\ \overset{h}{P} &= P_{jkc}^i P_i^{jkc}, \quad \overset{v}{P} = P_{bkc}^a P_a^{bkc}, \quad l_0, l_1, l_2, l_3, l_4, l_5 \in \mathbb{R} \end{aligned}$$

and R, S are given by (1.5), the raising/lowering of the indices being performed using $g_{ij}(x, y)$ and $h_{ab}(x, y)$ for the corresponding index-types. We remark that the Lagrangian (3.9) depends functionally on the gauge fields $g_{ij}(x, y)$, $h_{ab}(x, y)$ and their derivatives, by means of (3.8) and (3.6). So that applying the gauge variational principle

$$\delta \int \mathcal{L} dx^n dy^n = 0$$

for the Lagrangian density

$$(3.11) \quad \mathcal{L} = LG, \quad G = |\det(g_{ij})|^{1/2} \cdot |\det(h_{ab})|^{1/2},$$

we infer the generalized Einstein-Yang Mills equations for $\mathcal{L}$, obtained by vanishing of the Euler-Lagrange derivatives

$$\frac{\delta \mathcal{L}}{\delta \phi} \equiv \partial_k \left(\frac{\partial \mathcal{L}}{\partial (\partial_k \phi)} \right) + \dot{\partial}_a \left(\frac{\partial \mathcal{L}}{\partial (\dot{\partial}_a \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0, \quad \phi \in \{g_{ij}(x, y), h_{ab}(x, y)\}.$$

By direct calculation, we obtain the following equivalent expressions for these equations:

$$(3.12) \quad R_{ij} - \frac{1}{2} R g_{ij} = \frac{1}{l_1} g_{il} g_{jm} \left\{ \frac{\partial \hat{L}}{\partial g_{lm}} + g^{st} \frac{\partial R_{st}}{\partial g_{lm}} - \frac{1}{G} \left[\partial_k \left(G \frac{\partial L}{\partial (\partial_k g_{lm})} \right) + \dot{\partial}_a \left(G \frac{\partial L}{\partial (\dot{\partial}_a g_{lm})} \right) \right] \right\} + \frac{1}{2l_1} g_{ij} (\hat{L} + l_5 S),$$

$$(3.13) \quad S_{ab} - \frac{1}{2} S h_{ab} = \frac{1}{l_5} h_{ae} h_{bf} \cdot \left\{ \frac{\partial \hat{L}}{\partial h_{ef}} + h^{uv} \frac{\partial S_{uv}}{\partial h_{ef}} - \frac{1}{G} \left[\partial_k \left(G \frac{\partial L}{\partial (\partial_k h_{ef})} \right) + \dot{\partial}_d \left(G \frac{\partial L}{\partial (\dot{\partial}_d h_{ef})} \right) \right] \right\} + \frac{1}{2l_5} h_{ab} (\hat{L} + l_1 R).$$

Remark that, for the case $h_{ab} = \delta_a^i \delta_b^j \cdot g_{ij}$, the equations (3.12, 3.13) contain explicitly the Einstein h - and v -gauge tensor fields considered in (1.6)

$$(3.14) \quad \begin{cases} E_{ij} = R_{ij} - \frac{1}{2} R g_{ij} \\ E_{ab} = S_{ab} - \frac{1}{2} S h_{ab} \end{cases}$$

and that the right-hand terms of these equations stand for the energy-momentum h - and v -tensor fields, respectively.

In order to obtain solutions $\{g_{ij}(x, y), h_{ab}(x, y)\}$ for (3.12) and (3.13) having the predefined form (A1), a necessary condition will be (as L will depend on $\sigma(x, y)$ and its derivatives) the vanishing of the corresponding Euler-Lagrange derivative

$$(3.15) \quad \frac{\delta L}{\delta \sigma} \equiv \frac{1}{G} \left\{ \partial_k \left[\frac{\partial \mathcal{L}}{\partial (\partial_k \sigma)} \right] + \dot{\partial}_a \left(\frac{\partial \mathcal{L}}{\partial \dot{\sigma}_a} \right) - \frac{\partial \mathcal{L}}{\partial \sigma} \right\} = 0.$$

We shall describe the generalised Einstein-Yang Mills equation (3.15) for the gauge Lagrangian L given in (3.9) for three special cases.

1°. If $g_{ij}(x, y) = e^{2\sigma(x, y)} \gamma_{ij}(x)$ and $h_{ab} = \delta_a^i \delta_b^j g_{ij}$, then we obtain the case of the almost Hermitian model $H^{2n} = (TM, G_*, F)$ [3] given by the

N -lift of the generalized Lagrange metric (A1) to TM , and the almost complex structure F on TM given in the local adapted frame (1.1'') by

$$F(\delta_i) = -\dot{\partial}_i, \quad F(\dot{\partial}_i) = \delta_i.$$

In this case, we remark that L_{jk}^i and C_{bc}^a are given by (1.1), R and S are described in (1.5), and (3.15) becomes

$$(3.16) \quad \frac{\delta L}{\delta \sigma} = \frac{1}{G} \left[\partial_k \left(l_1 H \frac{\partial \sigma^0}{\partial \beta_k} - 2l_4 G \frac{\partial(C_{bc|l}^a)}{\partial \beta_k} P_a^{blc} \right) + \dot{\partial}_a \left(G \frac{\partial \bar{L}}{\partial \dot{\sigma}_a} \right) \right] + 2(1-n)\bar{L} + 2(\bar{L} - l_5 S) = 0$$

where $H = Ge^{-2\sigma}$, $\beta_k = \partial_k \sigma$, and

$$(3.17) \quad \bar{L} = l_2 \overset{v}{R} + l_3 \overset{h}{P} + l_4 \overset{v}{P} + l_5 S.$$

2°. For $g_{ij} = \gamma_{ij}(x)$, $h_{ab} = e^{2\sigma(x,y)} \gamma_{ab}(x)$, we have $L_{jk}^i = \{^i_{jk}\}$, $R = r$, C_{bc}^a given by (1.1-2), and

$$(3.18) \quad \frac{\delta L}{\delta \sigma} = \frac{1}{G} \left[-2l_4 \partial_k \left(G P_a^{blc} \frac{\partial(C_{bc|l}^a)}{\partial \beta_k} \right) + \dot{\partial}_a \left(G \frac{\partial \bar{L}}{\partial \dot{\sigma}_a} \right) \right] + 2(l_4 \overset{v}{P} + l_5 S - l_0 L_0) - nL = 0.$$

3°. For $g_{ij}(x, y) = e^{2\sigma(x,y)} \gamma_{ij}(x)$, $h_{ab} = \gamma_{ab}(x)$, we have L_{jk}^i given by (1.1-1), $P = S = 0$, and

$$(3.19) \quad \begin{aligned} \frac{\delta L}{\delta \sigma} = & \frac{1}{G} \left\{ l_1 \partial_k \left(H \frac{\partial \sigma^0}{\partial \beta_k} \right) + \right. \\ & + 2 \left[l_2 \frac{\partial}{\partial y^a} \left(G R_f^{bkl} R_{kl}^d (\delta_b^f \delta_d^a + \delta_d^f \delta_b^a - \gamma_{bd} \gamma^{af}) \right) \right. \\ & + l_3 \frac{\partial}{\partial y^a} \left(G P_i^{jkc} (\gamma_{kj} \gamma^{is} N_s^a - \delta_j^i N_k^a - \delta_k^i N_j^a) \right) \left. \right\} + \\ & + 2(2l_0 L_0 + l_1 R + 2l_2 \overset{v}{R} + l_3 \overset{h}{P}) - n\bar{L} = 0. \end{aligned}$$

An open problem is the one of determining valid solutions $\sigma(x, y)$ for the equation (3.15), for suitable constants $l_0, \dots, l_5$ in the Lagrangian field (3.9).

Conclusions. The space GL^n , as a model for the geometry of space-time [6] can be examined from the point of view of determining its Einstein

and Maxwell equations, using for the first ones two approaches: the theory of d -object fields for generalised Lagrange spaces and the generalized gauge theory for vector bundles endowed with (h, v) -metrics. Explicit forms for these equations are obtained, as a preliminary step for determining their solutions. The Einstein-Yang Mills equation with respect to the scalar gauge field $\sigma(x, y)$ is also derived for three cases of (h, v) -metrics related to the fundamental metric of GL^n .

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(Received September 15, 1992)