

Real hypersurfaces of non-flat complex space forms in terms of the Jacobi structure operator

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Abstract. Real hypersurfaces satisfying the condition $\phi l = l\phi$, ($l = R(., \xi)\xi$), have been studied by many authors, under at least one more condition, since the class of these hypersurfaces is too large. Moreover the operator l has been studied satisfying other conditions, including $\nabla_\xi l = 0$ and $lA = Al$. Even more, not much work has been done on the last equation. In the present paper we study condition $\phi l = l\phi$, combined with either $\nabla_\xi l = 0$ or $lA = Al$. All conditions are restricted in subspaces of the tangent space, in order to produce larger classes.

0. Introduction

An n -dimensional Kaehlerian manifold of constant holomorphic sectional curvature c is called complex space form, which is denoted by $M_n(c)$. A complete simply connected complex space form is a complex projective space $\mathbb{C}P^n$ if $c > 0$, a complex hyperbolic space $\mathbb{C}H^n$ if $c < 0$, or a complex Euclidean space $\mathbb{C}^n$ if $c = 0$. The induced almost contact metric structure of a real hypersurface M of $M_n(c)$ will be denoted by (ϕ, ξ, η, g) .

Homogeneous real hypersurfaces in $\mathbb{C}P^n$, were classified by R. TAKAGI [16]. J. BERNDT [1] classified real hypersurfaces with principal structure vector fields in $\mathbb{C}H^n$.

Another class of real hypersurfaces were studied by OKUMURA [14], and MONTIEL and ROMERO [12]. They classified real hypersurfaces satisfying $\phi A =$

Mathematics Subject Classification: 53C40, 53D15.

Key words and phrases: real hypersurfaces, almost contact manifold, Jacobi structure operator.

$A\phi$, in $\mathbb{C}P^n$ and $\mathbb{C}H^n$ respectively. In both cases, real hypersurfaces were categorized as *type A*, described in Section 1. For more details and examples on real hypersurfaces of type *A*, we refer to [13].

A Jacobi field along geodesics of a given Riemannian manifold (M, g) plays an important role in the study of differential geometry. It satisfies a well known differential equation which inspires Jacobi operators. For any vector field X , the Jacobi operator is defined by $R_X : R_X(Y) = R(Y, X)X$, where R denotes the curvature tensor and Y is a vector field on M . R_X is a self-adjoint endomorphism in the tangent space of M , and is related to the Jacobi differential equation, which is given by $\nabla_{\dot{\gamma}}(\nabla_{\dot{\gamma}}Y) + R(Y, \dot{\gamma})\dot{\gamma} = 0$ along a geodesic γ on M , where $\dot{\gamma}$ denotes the velocity vector along γ on M .

In a real hypersurface M of a complex space form $M_n(c)$, $c \neq 0$, the Jacobi operator on M with respect to the structure vector field ξ , is called the Jacobi structure operator and is denoted by $lX = R_\xi(X) = R(X, \xi)\xi$.

Many authors have studied real hypersurfaces from many points of view. Certain authors have studied real hypersurfaces under the condition $\phi l = l\phi$, equipped with one or two additional conditions. U-H. KI *et al.* [9] classified real hypersurfaces in complex space forms satisfying i) $\phi l = l\phi$ and $A^2\xi = \theta A\xi + \tau\xi$ (θ is a function, τ is constant) ii) $\phi l = l\phi$ and $Q\xi = \sigma\xi$ (where Q is the Ricci operator, σ is constant). U-HANG KI [7] classified real hypersurfaces in complex hyperbolic space satisfying $\phi l = l\phi$ and $lQ = Ql$. U-H. KI *et al.* [8], classified real hypersurfaces in complex space forms satisfying $\phi l = l\phi$, $lQ = Ql$, and additional conditions on the mean curvature. U-H. KI *et al.* [10] studied real hypersurfaces in complex space forms satisfying $\phi l = l\phi$ and $lQ = lQ$ ([7], [8], [9], [10]).

Other authors have studied real hypersurfaces under the conditions $\nabla_X l = 0$ ($X \in TM$) or $\nabla_\xi l = 0$ [5], [11], [15].

In the present paper, we consider $\phi l = l\phi$ (commuting structure Jacobi operator) and $\nabla_\xi l = 0$ (Reeb parallel structure Jacobi operator). Both conditions are restricted on the distribution on $M : \ker(\eta), (\ker(\eta))^\perp = \text{span}\{\xi\}$. Namely we prove:

Theorem 0.1. *Let M be a real hypersurface of a complex space form $M_n(c)$, ($n > 2$) ($c \neq 0$), satisfying $\phi l = l\phi$ on $\ker(\eta)$. If $\nabla_\xi l = 0$ holds on $\ker(\eta)$ or on $\text{span}\{\xi\}$, then M is a Hopf hypersurface. Furthermore, if $\eta(A\xi) \neq 0$, then M is locally congruent to a model space of type *A*.*

J. T. CHO and U-H. KI in [4] classified real hypersurfaces M of a projective space satisfying $\phi l = l\phi$ and $lA = Al$ on M . In the present paper we generalize

this result, studying the real hypersurfaces of any complex space form satisfying $\phi l = l\phi$ on $\ker(\eta)$ and $lA = Al$ on $\ker(\eta)$ or on $\text{span}\{\xi\}$. We prove:

Theorem 0.2. *Let M be a real hypersurface of a complex space form $M_n(c)$, ($n > 2$) ($c \neq 0$), satisfying $\phi l = l\phi$ on $\ker(\eta)$. If $lA = Al$ holds on $\ker(\eta)$ or on $\text{span}\{\xi\}$, then M is a Hopf hypersurface. Furthermore, if $\eta(A\xi) \neq 0$, then M is locally congruent to a model space of type A .*

For the case of $\mathbb{C}P^n$ in order to determine real hypersurface of type A , the technical assumption $\eta(A\xi) \neq 0$ is needed. Actually, there is a non-homogeneous tube with $A\xi = 0$ (of radius $\frac{\pi}{4}$) over a certain Kaehler submanifold in $\mathbb{C}P^n$, when its focal map has constant rank on M [3]. For Hopf hypersurfaces in $\mathbb{C}H^n$, ($n > 2$) it is known that the associated principal curvature of ξ never vanishes [1]. However, in $\mathbb{C}H^2$ there exists a Hopf hypersurface with $A\xi = 0$ [6].

We must also notice that equation $(\nabla_\xi l)X = 0$, $X \in \ker(\eta)$, is equivalent to $(\nabla_\xi l)X = \mu\xi$, $X \in \ker(\eta)$ and μ is a real valued function. Indeed, $(\nabla_\xi l)X = 0$ implies $(\nabla_\xi l)X = \mu\xi$, where $\mu = 0$. Conversely if $(\nabla_\xi l)X = \mu\xi$ holds $\forall X \in \ker(\eta)$, then by putting $-X$ instead of X we have $-(\nabla_\xi l)X = \mu\xi$ which is combined with $(\nabla_\xi l)X = \mu\xi$ to give $\mu = 0$.

1. Preliminaries

Let M_n be a Kaehlerian manifold of real dimension $2n$, equipped with an almost complex structure J and a Hermitian metric tensor G . Then for any vector fields X and Y on $M_n(c)$, the following relations hold:

$$J^2X = -X, \quad G(JX, JY) = G(X, Y), \quad \tilde{\nabla}J = 0$$

where $\tilde{\nabla}$ denotes the Riemannian connection of G of M_n .

Now, let M_{2n-1} be a real $(2n-1)$ -dimensional hypersurface of $M_n(c)$, and denote by N a unit normal vector field on a neighborhood of a point in M_{2n-1} (from now on we shall write M instead of M_{2n-1}). For any vector field X tangent to M we have $JX = \phi X + \eta(X)N$, where ϕX is the tangent component of JX , $\eta(X)N$ is the normal component, and

$$\xi = -JN, \quad \eta(X) = g(X, \xi), \quad g = G|_M.$$

By properties of the almost complex structure J , and the definitions of η and g , the following relations hold [2]:

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta \circ \phi = 0, \quad \phi\xi = 0, \quad \eta(\xi) = 1 \quad (1.1)$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(X, \phi Y) = -g(\phi X, Y). \quad (1.2)$$

The above relations define an *almost contact metric structure* on M denoted by (ϕ, ξ, g, η) . For (ϕ, ξ, g, η) , we can define a local orthonormal basis $\{V_1, V_2, \dots, V_{n-1}, \phi V_1, \phi V_2, \dots, \phi V_{n-1}, \xi\}$, called a ϕ -basis on M . Furthermore, let A be the shape operator with respect to N , and denote by ∇ the Riemannian connection of g on M . Then, A is symmetric and the following equations are satisfied

$$\nabla_X \xi = \phi AX, \quad (\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi. \quad (1.3)$$

As the ambient space $M_n(c)$ is of constant holomorphic sectional curvature c , the equations of Gauss and Codazzi are respectively given by

$$\begin{aligned} R(X, Y)Z = \frac{c}{4}[g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\ - 2g(\phi X, Y)\phi Z] + g(AY, Z)AX - g(AX, Z)AY, \end{aligned} \quad (1.4)$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}[\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi]. \quad (1.5)$$

The tangent space $T_p M$, for every point $p \in M$, is decomposed as following

$$T_p M = \ker(\eta)^\perp \oplus \ker(\eta)$$

where $\ker(\eta)^\perp = \text{span}\{\xi\}$ and $\ker(\eta)$ is defined as following

$$\ker(\eta) = \{X \in T_p M : \eta(X) = 0\}.$$

Based on the above decomposition, by virtue of (1.3), we decompose the vector field $A\xi$ in the following way

$$A\xi = \alpha\xi + \beta U \quad (1.6)$$

where $\beta = |\phi \nabla_\xi \xi|$ and $U = -\frac{1}{\beta} \phi \nabla_\xi \xi \in \ker(\eta)$, provided that $\beta \neq 0$.

If the vector field $A\xi$ is expressed as $A\xi = \alpha\xi$, then ξ is called a principal vector field. Differentiation of a function f along a vector field X will be denoted by (Xf) . All manifolds and vector fields of this paper are assumed to be connected and of class C^∞ .

Finally, we mention the theorems of OKUMURA [14], and MONTIEL, ROMERO [12], who proved respectively the following theorems.

Theorem 1.1. *Let M be a real hypersurface of $\mathbb{C}P^n$, $n \geq 2$. If it satisfies*

$$g((A\phi - \phi A)X, Y) = 0$$

for any vector fields X and Y , then M is a tube of radius r over one of the following Kaehlerian submanifolds:

- (A₁) a hyperplane $\mathbb{C}P^{n-1}$, where $0 < r < \frac{\pi}{2}$,
 (A₂) a totally geodesic $\mathbb{C}P^k$ ($0 < k \leq n-2$), where $0 < r < \frac{\pi}{2}$.

Theorem 1.2. *Let M be a real hypersurface of $\mathbb{C}H^n$, $n \geq 2$. If it satisfies*

$$g((A\phi - \phi A)X, Y) = 0$$

for any vector fields X and Y , then M is locally congruent to one of the following:

- (A₀) a self-tube, that is, horosphere,
 (A₁) a geodesic hypersphere or a tube over a hyperplane $\mathbb{C}H^{n-1}$,
 (A₂) a tube over a totally geodesic $\mathbb{C}H^k$ ($1 \leq k \leq n-2$).

2. Auxiliary relations

In the study of real hypersurfaces in a complex space form $M_n(c)$, $c \neq 0$, it is a crucial condition that the structure vector field ξ is principal. The purpose of this paragraph is to prove this condition.

Let V be the open subset of points p of M , where $\alpha \neq 0$ in a neighborhood of p and V_0 be the open subset of points p of M such that $\alpha = 0$ in a neighborhood of p . Since α is a smooth function on M , then $V \cup V_0$ is an open and dense subset of M .

Lemma 2.1. *Let M be a real hypersurface in a complex space form $M_n(c)$ ($c \neq 0$), satisfying $\phi l = l\phi$ on $\ker(\eta)$. Then, $\beta = 0$ on V_0 .*

PROOF. From (1.6) we have $A\xi = \beta U$ on V_0 . Then (1.4) for $X = U$ and $Y = Z = \xi$ yields

$$\begin{aligned} lU &= \frac{c}{4}U + g(A\xi, \xi)AU - g(AU, \xi)A\xi = \frac{c}{4}U - g(U, A\xi)A\xi = \left(\frac{c}{4} - \beta^2\right)U \implies \\ \phi lU &= \left(\frac{c}{4} - \beta^2\right)\phi U. \end{aligned}$$

In the same way, from (1.4) for $X = \phi U$, $Y = Z = \xi$ we obtain

$$l\phi U = \frac{c}{4}\phi U.$$

The last two equations yield $\beta = 0$. □

Remark 1. We have proved that on V_0 , $A\xi = 0\xi$ i.e., ξ is a principal vector field on V_0 . Now we define on V the set V' of points p where $\beta \neq 0$ in a neighborhood of p and the set V'' of points p where $\beta = 0$ in a neighborhood of p . Obviously ξ is principal on V'' . In what follows we study the open subset V' of M and define the following classes

A = hypersurfaces satisfying $\phi l = l\phi$ and $lA = Al$ on $\ker(\eta)$,

B = hypersurfaces satisfying $\phi l = l\phi$ and $lA = Al$ on $\text{span}\{\xi\}$,

C = hypersurfaces satisfying $\phi l = l\phi$ and $\nabla_\xi l = \mu\xi$ on $\ker(\eta)$,

D = hypersurfaces satisfying $\phi l = l\phi$ and $\nabla_\xi l = \mu\xi$ on $\text{span}\{\xi\}$.

Lemma 2.2. *Let M be a real hypersurface of a complex space form $M_n(c)$ ($c \neq 0$), satisfying $\phi l = l\phi$ on $\ker(\eta)$. Then the following relations hold on the set V' of classes A, B, C, D :*

$$AU = \left(\frac{\beta^2}{\alpha} - \frac{c}{4\alpha}\right)U + \beta\xi, \quad A\phi U = -\frac{c}{4\alpha}\phi U \quad (2.1)$$

$$\nabla_\xi \xi = \beta\phi U, \quad \nabla_U \xi = \left(\frac{\beta^2}{\alpha} - \frac{c}{4\alpha}\right)\phi U, \quad \nabla_{\phi U} \xi = \frac{c}{4\alpha}U \quad (2.2)$$

$$\nabla_\xi U = W_1, \quad \nabla_U U = W_2, \quad \nabla_{\phi U} U = W_3 - \frac{c}{4\alpha}\xi \quad (2.3)$$

$$\nabla_\xi \phi U = \phi W_1 - \beta\xi, \quad \nabla_U \phi U = \phi W_2 + \left(\frac{c}{4\alpha} - \frac{\beta^2}{\alpha}\right)\xi, \quad \nabla_{\phi U} \phi U = \phi W_3. \quad (2.4)$$

where W_1, W_2, W_3 are vector fields on $\ker(\eta)$ satisfying $W_1, W_2, W_3 \perp U$.

PROOF. From (1.4) we get

$$lX = \frac{c}{4}[X - \eta(X)\xi] + \alpha AX - g(AX, \xi)A\xi \quad (2.5)$$

which, for $X = U$ yields

$$lU = \frac{c}{4}U + \alpha AU - \beta A\xi. \quad (2.6)$$

The scalar products of (2.6) with U (resp. ϕU) yield

$$g(AU, U) = \frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha}, \quad (2.7)$$

$$g(AU, \phi U) = \frac{1}{\alpha}g(lU, \phi U) \quad (2.8)$$

respectively, where $\gamma = g(lU, U) = g(\phi lU, \phi U) = g(l\phi U, \phi U)$.

The second relation of (1.2) for $X = U$, $Y = lU$, the condition $\phi l = l\phi$ and the symmetry of the operator l imply:

$$g(lU, \phi U) = 0.$$

The above equation and (2.8) imply

$$g(AU, \phi U) = 0. \quad (2.9)$$

The symmetry of A and (1.6) imply

$$g(AU, \xi) = \beta. \quad (2.10)$$

From relations (2.7), (2.9) and (2.10), we obtain

$$AU = \left(\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha} \right) U + \beta\xi + \lambda W \quad (2.11)$$

where $W \in \text{span}\{U, \phi U, \xi\}^\perp$ and $\lambda = g(AU, W)$. Combining (2.11) with (2.6) we obtain $lU = \gamma U + \lambda\alpha W$. Acting on this relation with the tensor field ϕ and by virtue of $\phi l = l\phi$ we take $l\phi U = \gamma\phi U + \lambda\alpha\phi W$. On the other hand by virtue of (2.5) we have $l\phi U = \frac{c}{4}\phi U + \alpha A\phi U$. From the last two relations we obtain $A\phi U = \left(\frac{\gamma}{\alpha} - \frac{c}{4\alpha} \right) \phi U + \lambda\phi W$.

On class A

Since $lA = Al$ holds on $\ker(\eta)$ we have $lAW = AlW$. This relation because of (2.5) and (2.11) implies $\lambda\beta A\xi = 0$ and so $\lambda = 0$. Since $\lambda = 0$, equations $lAU = AlU$, (2.6) and (2.11) yield $\gamma = 0$, therefore we have the first of (2.1). Moreover from (2.5) we have $l\phi U = \frac{c}{4}\phi U + \alpha A\phi U$ which is written as $\phi lU = \frac{c}{4}\phi U + \alpha A\phi U$ ($\phi l = l\phi$). From $\phi lU = \frac{c}{4}\phi U + \alpha A\phi U$ and $\gamma = \lambda = 0$ we obtain the second of (2.1). Using (1.3) for $X \in \{\xi, U, \phi U\}$ and by virtue of (2.1) we obtain (2.2). It is well known that:

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z). \quad (2.12)$$

Let us set $\nabla_\xi U = W_1$ and $\nabla_U U = W_2$. If we use (2.2) and (2.12), it is easy to verify that $g(\nabla_\xi U, U) = 0 = \eta(\nabla_\xi U)$ and $g(\nabla_U U, U) = 0 = \eta(\nabla_U U)$ which means $W_1 \perp \{\xi, U\}$ and $W_2 \perp \{\xi, U\}$.

On the other hand using (2.12) and the third of (2.2) we find $\eta(\nabla_{\phi U} U) = -\frac{c}{4\alpha}$ and $g(\nabla_{\phi U} U, U) = 0$ which means that $\nabla_{\phi U} U$ is decomposed as $\nabla_{\phi U} U = W_3 - \frac{c}{4\alpha}\xi$, $W_3 \perp \{U, \xi\}$. Now, by virtue of (1.3) and (2.3) for $X = \xi, Y = U$ and $X = Y = U$ and $X = \phi U, Y = U$, we get (2.4).

On class B

We analyze equation $lA\xi = Al\xi$ by virtue of (1.6), (2.6) and (2.11) and we have $\gamma U + \lambda\alpha W = 0$. Since $W \perp U$ we have $\gamma = \lambda = 0$. The rest of the proof is similar to the one in class A.

On class C

The scalar product of $(\nabla_\xi l)\phi U = 0$ with ξ , the symmetry of l and (2.12) yield $g(l\phi U, \phi U) = \gamma = 0$. In addition $(\nabla_\xi l)\phi W = 0$ holds. So, the scalar product of the previous equation with ξ , the symmetry of l and (2.12) yield $g(l\phi U, \phi W) = 0$, which, by virtue of (2.5), the second of (2.1) and $\gamma = 0$, yields $\lambda = 0$. The rest of the proof is similar to the one in class A.

On class D

We analyze $(\nabla_\xi l)\xi = 0$ and obtain $\beta l\phi U = 0 \Rightarrow l\phi U = 0$. We analyze $l\phi U = 0$ using (2.5) and $A\phi U = \left(\frac{\gamma}{\alpha} - \frac{c}{4\alpha}\right)\phi U + \lambda\phi W$, and we have $\gamma\phi U + \lambda\alpha\phi W = 0$. This relation and the linear independency of the vector fields ϕU and ϕW yield $\gamma = \lambda = 0$. The rest of the proof is similar to the one in class A. $\square$

Lemma 2.3. *Let M be a real hypersurface of a complex space form $M_n(c)$ ($c \neq 0$), of class A, B, C, or D. Then on V' we have $g(\nabla_\xi U, \phi U) = -4\alpha$ and $g(\nabla_U U, \phi U) = -4\beta + \frac{c}{4\alpha\beta}\left(\frac{c}{4\alpha} - \frac{\beta^2}{\alpha}\right)$.*

PROOF. Putting $X = U$, $Y = \xi$ in (1.5), we obtain

$$(\nabla_U A)\xi - (\nabla_\xi A)U = -\frac{c}{4}\phi U.$$

Combining the last equation with (1.6), and Lemma 2.2 it follows:

$$\begin{aligned} & (U\alpha)\xi + (U\beta)U + \beta W_2 + \left(-\frac{c}{4\alpha} + \frac{\beta^2}{\alpha}\right)\frac{c}{4\alpha}\phi U \\ & - \xi\left(-\frac{c}{4\alpha} + \frac{\beta^2}{\alpha}\right)U - \left(-\frac{c}{4\alpha} + \frac{\beta^2}{\alpha}\right)W_1 - (\xi\beta)\xi + AW_1 = 0. \end{aligned}$$

Taking the scalar products of the last relation with ξ and U respectively, we obtain

$$(U\alpha) = (\xi\beta) \tag{2.13}$$

and

$$(U\beta) = \left(\xi\left(\frac{\beta^2}{\alpha} - \frac{c}{4\alpha}\right)\right). \tag{2.14}$$

Combining the last three equations we have

$$AW_1 = \frac{c}{4\alpha}\left(\frac{c}{4\alpha} - \frac{\beta^2}{\alpha}\right)\phi U + \left(\frac{\beta^2}{\alpha} - \frac{c}{4\alpha}\right)W_1 - \beta W_2. \tag{2.15}$$

The scalar product of (2.15) with ϕW_1 yields:

$$\beta g(\phi W_1, W_2) = -g(AW_1, \phi W_1).$$

But from (2.5) we have

$$g(l\phi W_1, W_1) = g(\phi W_1, lW_1) = \alpha g(AW_1, \phi W_1).$$

Moreover $g(l\phi W_1, W_1) = g(\phi W_1, lW_1) = -g(W_1, \phi lW_1) = -g(W_1, l\phi W_1)$ which means that

$$g(l\phi W_1, W_1) = 0.$$

The above relations lead to $g(\phi W_1, W_2) = 0$ which, by virtue of (2.15) implies $g(AW_1, \phi W_2) = 0$.

In what follows we define the following functions:

$$\kappa_1 = g(W_1, \phi U) \quad \kappa_2 = g(W_2, \phi U), \quad \kappa_3 = g(W_3, \phi U).$$

Putting $X = \phi U$, $Y = \xi$ in (1.5), we obtain

$$\begin{aligned} A\phi W_1 = & \left[\frac{3\beta c}{4\alpha} + \alpha\beta - (\phi U\alpha) \right] \xi - \left[(\phi U\beta) + \frac{c}{4\alpha} \left(\frac{c}{4\alpha} - \frac{\beta^2}{\alpha} \right) - \beta^2 \right] U \\ & + \frac{c}{4\alpha^2} (\xi\alpha)\phi U - \frac{c}{4\alpha} \phi W_1 - \beta W_3. \end{aligned} \quad (2.16)$$

The scalar product of (2.16) with ξ implies

$$(\phi U\alpha) = \frac{3\beta c}{4\alpha} + \alpha\beta + \kappa_1\beta. \quad (2.17)$$

Using the A is symmetric and ϕ is skew-symmetric, by taking the scalar product of (2.16) with U we have

$$g(\phi W_1, AU) = -(\phi U\beta) - \frac{c}{4\alpha} \left(\frac{c}{4\alpha} - \frac{\beta^2}{\alpha} \right) + \beta^2 + \frac{c}{4\alpha} g(W_1, \phi U)$$

which, eventually (with the aid of Lemma 2.2 and the definition of κ_1) yields

$$(\phi U\beta) = \frac{c}{4\alpha} \left(\frac{\beta^2}{\alpha} - \frac{c}{4\alpha} \right) + \beta^2 + \kappa_1 \frac{\beta^2}{\alpha}. \quad (2.18)$$

Due to (2.5), (2.15) and (2.16) the condition $\phi lW_1 = l\phi W_1$ implies

$$\begin{aligned} \beta^2 \phi W_1 - \alpha\beta \phi W_2 + \alpha \left[(\phi U\alpha) - \frac{3\beta c}{4\alpha} - \alpha\beta \right] \xi + \alpha [(\phi U\beta) - \beta^2] U + \alpha\beta W_3 \\ = \kappa_1 \beta A\xi + \frac{c}{4\alpha} (\xi\alpha)\phi U. \end{aligned}$$

Taking the scalar product of the last relation with U we have

$$-2\kappa_1\beta^2 + \alpha\beta\kappa_2 + \alpha(\phi U\beta) - \alpha\beta^2 = 0.$$

If in the above relation we replace the term κ_1 using (2.17) we obtain

$$-2\beta(\phi U\alpha) + \frac{3\beta^2c}{2\alpha} + \alpha\beta^2 + \alpha\beta\kappa_2 + \alpha(\phi U\beta) = 0. \quad (2.19)$$

The relation $(\nabla_U A)\phi U - (\nabla_{\phi U} A)U = -\frac{c}{2}\xi$, using Lemma 2.2 implies

$$\begin{aligned} & \frac{c}{4\alpha^2}(U\alpha)\phi U + \left[\frac{c}{2\alpha} \left(\frac{\beta^2}{\alpha} - \frac{c}{4\alpha} \right) + \beta^2 - (\phi U\beta) \right] \xi \\ & + \left[-\frac{3\beta c}{4\alpha} + \frac{\beta^3}{\alpha} + \left(\phi U \left(\frac{c}{4\alpha} - \frac{\beta^2}{\alpha} \right) \right) \right] U - \frac{c}{4\alpha} \phi W_2 - A\phi W_2 \\ & + AW_3 + \left(\frac{c}{4\alpha} - \frac{\beta^2}{\alpha} \right) W_3 = 0. \end{aligned} \quad (2.20)$$

The scalar product of the above relation with U yields

$$\frac{\kappa_2\beta^2}{\alpha} - \frac{3\beta c}{4\alpha} + \frac{\beta^3}{\alpha} + \phi U \left(\frac{c}{4\alpha} - \frac{\beta^2}{\alpha} \right) = 0.$$

Expanding the last relation and by virtue of (2.19) we get

$$\left(-\frac{3\beta^2}{\alpha^2} + \frac{c}{4\alpha^2} \right) (\phi U\alpha) + \frac{3\beta}{\alpha} (\phi U\beta) + \frac{3\beta^3c}{2\alpha^3} + \frac{3\beta c}{4\alpha} = 0.$$

Combining the last equation with (2.17) and (2.18) we obtain $\kappa_1 = -4\alpha$. The scalar product of (2.15) with ϕU because of $\kappa_1 = -4\alpha$, yields $\kappa_2 = -4\beta + \frac{c}{4\alpha\beta} \left(\frac{c}{4\alpha} - \frac{\beta^2}{\alpha} \right)$. $\square$

Lemma 2.4. *Let M be a real hypersurface of a complex space form $M_n(c)$ ($c \neq 0$), of class A , B , C , or D . Then the structure vector field ξ is principal on M .*

PROOF. The scalar products of (2.16) and (2.20) with ϕU , yield $(\xi\alpha) = \frac{4\alpha^2\beta}{c}\kappa_3$ and $(U\alpha) = \frac{4\alpha\beta^2}{c}\kappa_3$. Combining the last two relations with (2.13) and (2.14) we have

$$(\xi\alpha) = \frac{4\alpha^2\beta}{c}\kappa_3, \quad (U\alpha) = (\xi\beta) = \frac{4\alpha\beta^2}{c}\kappa_3, \quad (U\beta) = \left(\beta + \frac{4\beta^3}{c} \right) \kappa_3. \quad (2.21)$$

Using (1.5) for $X = \phi W_2$, $Y = \xi$ we have

$$\nabla_{\phi W_2} A\xi - A\nabla_{\phi W_2} \xi - \nabla_{\xi} A\phi W_2 + A\nabla_{\xi} \phi W_2 = \frac{c}{4} W_2,$$

which, from (1.6) is further decomposed as

$$\begin{aligned} (\phi W_2 \alpha) \xi + \alpha \phi A \phi W_2 + (\phi W_2 \beta) U + \beta \nabla_{\phi W_2} U - A \phi A \phi W_2 \\ - \nabla_{\xi} A \phi W_2 + A \nabla_{\xi} \phi W_2 = \frac{c}{4} W_2. \end{aligned}$$

Taking the scalar product with ξ and by using (1.6), (2.12), (2.21), Lemmas 2.2, 2.3 and $W_1 \perp \phi W_2$ we obtain

$$(\phi W_2 \alpha) = \kappa_3 \left(\frac{16\alpha\beta^3}{c} + \beta \left(\frac{\beta^2}{\alpha} - \frac{c}{4\alpha} \right) \right). \quad (2.22)$$

On the other hand from (1.5) we get

$$\nabla_{W_3} A\xi - A\nabla_{W_3} \xi - \nabla_{\xi} A W_3 + A\nabla_{\xi} W_3 = -\frac{c}{4} \phi W_3$$

which, by virtue of (1.6) is further decomposed as

$$(W_3 \alpha) \xi + \alpha \phi A W_3 + (W_3 \beta) U + \beta \nabla_{W_3} U - A \nabla_{W_3} \xi - \nabla_{\xi} A W_3 + A \nabla_{\xi} W_3 = -\frac{c}{4} \phi W_3.$$

Taking the scalar product of the last equation with ξ and by making use of Lemma 2.2, (2.12) and (2.21) we obtain

$$(W_3 \alpha) = 3\beta \left(\frac{c}{4\alpha} - \alpha \right) \kappa_3. \quad (2.23)$$

In a similar way equation (1.5) yields $(\nabla_{\phi W_1} A)U - (\nabla_U A)\phi W_1 = 0$, which by virtue of Lemma 2.2 is further analyzed as

$$\begin{aligned} \left(\phi W_1 \left(\frac{\beta^2}{\alpha} - \frac{c}{4\alpha} \right) \right) U + \left(\frac{\beta^2}{\alpha} - \frac{c}{4\alpha} \right) \nabla_{\phi W_1} U \\ + (\phi W_1 \beta) \xi + \beta \phi A \phi W_1 - A \nabla_{\phi W_1} U - \nabla_U A \phi W_1 + A \nabla_U \phi W_1 = 0. \end{aligned}$$

The scalar product of the above equation with ξ and using $g(\phi W_1, W_2) = 0$, (2.21) and Lemmas 2.2, 2.3, leads to

$$(\phi W_1 \beta) = 4\alpha \kappa_3 \left(\beta + \frac{4\beta^3}{c} \right). \quad (2.24)$$

Now the calculation of Lie bracket $[\phi U, \xi]\beta$, by virtue of (2.18), Lemma 2.3 and (2.21), results to

$$[\phi U, \xi]\beta = \phi U(\xi\beta) + \kappa_3 \left[-\frac{\beta c}{2\alpha} + \frac{24\alpha\beta^3}{c} \right].$$

On the other hand from Lemma 2.2, (2.21) and (2.24) we obtain

$$[\phi U, \xi]\beta = (\nabla_{\phi U}\xi - \nabla_{\xi}\phi U)\beta = \beta\kappa_3 \left[\frac{c}{4\alpha} + \frac{\beta^2}{\alpha} - 4\alpha - \frac{12\alpha\beta^2}{c} \right].$$

Equalizing the above two relations we get

$$\phi U(\xi\beta) = \beta\kappa_3 \left[\frac{3c}{4\alpha} + \frac{\beta^2}{\alpha} - 4\alpha - \frac{36\alpha\beta^2}{c} \right]. \quad (2.25)$$

In a similar way, combining (2.18), (2.21), (2.22), (2.23) and Lemmas 2.2, 2.3, the Lie bracket $[\phi U, U]\alpha$ yields

$$\begin{aligned} [\phi U, U]\alpha &= \phi U(U\alpha) + 3\beta\kappa_3 \left[\alpha + \frac{8\alpha\beta^2}{c} - \frac{c}{4\alpha} \right] \\ [\phi U, U]\alpha &= (\nabla_{\phi U}U - \nabla_U\phi U)\alpha = \beta\kappa_3 \left[\frac{c}{\alpha} - 5\alpha - \frac{12\alpha\beta^2}{c} - \frac{\beta^2}{\alpha} \right]. \end{aligned}$$

From the above equations we obtain

$$\phi U(U\alpha) = \beta\kappa_3 \left[\frac{7c}{4\alpha} - 8\alpha - \frac{36\alpha\beta^2}{c} - \frac{\beta^2}{\alpha} \right]. \quad (2.26)$$

Because of (2.13) from (2.25) and (2.26) we obtain

$$\frac{\beta}{\alpha}[c - 4\alpha^2 - 2\beta^2]\kappa_3 = 0.$$

Let us assume there is a point $p \in V'$ such that $\kappa_3 \neq 0$ in a neighborhood around p . Then we have $c = 4\alpha^2 + 2\beta^2$. Differentiating the last equation along ξ and by virtue of (2.21) and $\kappa_3 \neq 0$ we take $2\alpha^2 + \beta^2 = 0$ which is a contradiction. So $\kappa_3 = 0 \Rightarrow (U\alpha) = (\xi\alpha) = 0 \Rightarrow [U, \xi]\alpha = 0$. But the last equation, because of Lemma 2.2 yields

$$\left(\frac{\beta^2}{\alpha} - \frac{c}{4\alpha} \right) (\phi U\alpha) - (W_1\alpha) = 0. \quad (2.27)$$

On the other hand from (1.5) for $X = W_1$, $Y = \xi$, taking the scalar product with ξ , using the Lemmas 2.2, 2.3 we have $(W_1\alpha) = \beta|W_1|^2 - \beta(4\alpha^2 + 3c)$. The last relation, (2.27), (2.17) and Lemma 2.3 lead to

$$12(5\alpha^2 + \beta^2)c + 64\alpha^4 = 16\alpha^2(|W_1|^2 + 3\beta^2) + 3c^2. \quad (2.28)$$

Because of (2.28) $f(\omega) = 64\omega^2 + 60c\omega + 12c\beta^2$, where $\omega = \alpha^2$, is positive for every ω, β . This holds if and only if the discriminant of $f(\omega)$ is negative for all β, c . But this is not true, hence we have a contradiction. Therefore V' is empty and the real hypersurface M consists only of V_0 and V'' i.e., the Reeb vector field ξ is principal and M is a Hopf hypersurface. $\square$

3. Proof of theorems

From Lemma 2.4:

$$A\xi = \alpha\xi, \quad \alpha = g(A\xi, \xi). \quad (3.1)$$

We consider a ϕ -basis $\{V_i, \phi V_i, \xi\}$, ($i = 1, 2, \dots, n-1$). From (2.5) and (3.1) we obtain

$$lX = \frac{c}{4}[X - \eta(X)\xi] + \alpha AX - \eta(X)\alpha^2\xi. \quad (3.2)$$

(3.2) for $X = V_i$ implies

$$lV_i = \frac{c}{4}V_i + \alpha AV_i. \quad (3.3)$$

Applying ϕ to (3.3) we obtain

$$\phi lV_i = \frac{c}{4}\phi V_i + \alpha\phi AV_i, \quad i = 1, \dots, n-1. \quad (3.4)$$

The relation (3.2) for $X = \phi V_i$ yields

$$l\phi V_i = \frac{c}{4}\phi V_i + \alpha A\phi V_i. \quad (3.5)$$

Comparing (3.4) with (3.5), and by making use of the condition $\phi l = l\phi$ we have

$$(A\phi - \phi A)V_i = 0, \quad i = 1, \dots, n-1. \quad (3.6)$$

On the other hand the action of ϕ on (3.5) yields

$$\phi(l\phi V_i) = \frac{c}{4}\phi^2 V_i + \alpha\phi A\phi V_i, \quad (3.7)$$

which, by virtue of (1.1), is written in the form

$$(\phi l)\phi V_i = -\frac{c}{4}V_i + \alpha(\phi A)\phi V_i, \quad (3.8)$$

Moreover, the calculation of $(l\phi)\phi V_i$ by virtue of (1.1) and (3.3) yields:

$$\begin{aligned} (l\phi)\phi V_i &= l\phi^2 V_i = -lV_i = -\frac{c}{4}V_i - \alpha AV_i = -\frac{c}{4}V_i + \alpha A\phi^2 V_i \\ &= -\frac{c}{4}V_i + \alpha A\phi\phi V_i \iff (l\phi)\phi V_i = -\frac{c}{4}V_i + \alpha(A\phi)\phi V_i. \end{aligned} \quad (3.9)$$

Comparing (3.8) and (3.9), and by making use of the condition $\phi l = l\phi$ we have

$$(A\phi - \phi A)\phi V_i = 0 \quad (3.10)$$

for every $i = 1, \dots, n-1$. But from (1.1) and (3.1) we also have

$$(A\phi - \phi A)\xi = 0. \quad (3.11)$$

So, (3.6), (3.10) and (3.11) imply that $A\phi = \phi A$. This result and the Theorems 1.1 and 1.2 complete the proof of Theorems 0.1 and 0.2. $\square$

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(Received July 22, 2014; revised March 24, 2015)