

On C-automorphisms of finite groups admitting a strongly 2-embedded subgroup

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Abstract. Let G be a finite group with a strongly 2-embedded subgroup. It is proved that every C-automorphism of G is inner. In particular, the normalizer property holds for G . As a consequence, it is also obtained that every C-automorphism of finite (TI)-groups is inner.

1. Introduction

All groups considered in this paper are finite. Let G be a finite group, $\mathbb{Z}G$ its integral group ring and $U(\mathbb{Z}G)$ the unit group of $\mathbb{Z}G$. The normalizer problem ([16, Problem 43], see also [6, Question 3.7]) of integral group rings asks whether $N_{U(\mathbb{Z}G)}(G) = G \cdot Z(U(\mathbb{Z}G))$ for any group G , where $N_{U(\mathbb{Z}G)}(G)$ and $Z(U(\mathbb{Z}G))$ denote the normalizer of G in $U(\mathbb{Z}G)$ and the center of $U(\mathbb{Z}G)$ respectively. If the equality is valid for G , then G is said to have the normalizer property.

C-automorphisms of groups have intimate connection with the normalizer problem. Recall that an automorphism σ of G is called a C-automorphism if σ satisfies the following three conditions: (I) the restriction of σ to each Sylow subgroup of G equals the restriction of some inner automorphism of G ; (II) σ is a class-preserving automorphism; (III) $\sigma^2 \in \text{Inn}(G)$, where $\text{Inn}(G)$ is the inner

Mathematics Subject Classification: 16S34; 20D45.

Key words and phrases: C-automorphism; integral group ring; the normalizer property.

The work was supported by the National Natural Science Foundation of China (11401329), the Doctoral Fund of Shandong Province (BS2012SF003), the Project of Shandong Province Higher Educational Science and Technology Program (J14LI10) and the Project of Shandong Province Higher Educational Excellent Backbone Teachers for International Cooperation and Training(J20140608).

automorphism group of G . This definition was firstly introduced by MARCINIAK and ROGGENKAMP in [14]. Denote by $\text{Aut}_C(G)$ the group of all C -automorphisms of G . Set $\text{Out}_C(G) := \text{Aut}_C(G)/\text{Inn}(G)$. Additionally, denote by $\text{Aut}_{\text{Col}}(G)$ the group of all automorphisms of G which satisfy only condition (I) above. It is known that $\text{Out}_C(G) = 1$ implies that G has the normalizer property. In this direction, a lot of results on C -automorphisms have appeared in the literature, see [2]–[7], [9]–[15] for instance.

The aim of this paper is to study C -automorphisms of finite groups with a strongly 2-embedded subgroup. Recall that a subgroup M of a group G is said to be a strongly 2-embedded subgroup in G if the following conditions are satisfied: (i) $M < G$ and 2 divides $|M|$; (ii) If $g \in G \setminus M$, then 2 does not divide $|M \cap M^g|$. A characterization of groups with a strongly 2-embedded subgroup can be found in [1] or the appendix in [8]. Our main result is as follows.

Theorem A. *Let G be a group with a strongly 2-embedded subgroup. Then $\text{Out}_C(G) = 1$. In particular, the normalizer property holds for G .*

A group G of even order is called a (TI)-group if two different Sylow 2-subgroups contain only the identity element in common. This notion was introduced by SUZUKI [17]. As a consequence of Theorem A, we have the following result.

Corollary B. *Let G be a (TI)-group. Then $\text{Out}_C(G) = 1$. In particular, the normalizer property holds for G .*

2. Notation and preliminaries

Let σ be an automorphism of a group G , and $N \trianglelefteq G$ with $N^\sigma = N$. Write $\sigma|_N$ for the restriction of σ to N , and $\sigma|_{G/N}$ for the automorphism of G/N induced by σ in the natural way. For a fixed element $y \in G$, write $\text{conj}(y)$ for the inner automorphism of G induced by y via conjugation. Our other notation is standard and follows that in [8].

Lemma 2.1 ([15, Theorem 7]). *If G is a group with a Sylow 2-subgroup of order 2, then $\text{Out}_C(G) = 1$.*

Lemma 2.2. *Let G be a group of odd order. Then $\text{Out}_C(G) = 1$.*

PROOF. This is a consequence of Proposition 1 in [5]; see also Theorem 3.4 in [6]. $\square$

Lemma 2.3. *Let G be a simple group. Then $\text{Out}_C(G) = 1$.*

PROOF. This is a consequence of Theorem 14 in [5]. $\square$

Lemma 2.4 ([5, Lemma 6]). *Let $\sigma \in \text{Aut}(G)$ and $M \trianglelefteq G$ with $M^\sigma = M$. Suppose that $\sigma|_Q = \text{conj}(h)|_Q$ for some $h \in G$, where Q is a Sylow subgroup of M . Then σ fixes $H = MC_G(Q) \trianglelefteq G$ and $\sigma|_{G/H} = \text{conj}(h)|_{G/H}$.*

Lemma 2.5 ([2, Lemma 2]). *Let $N \trianglelefteq G$ and let $\sigma \in \text{Aut}(G)$ be of p -power order, where p is a prime. Suppose that $\sigma|_N = \text{id}|_N$ and $\sigma|_{G/N} = \text{id}|_{G/N}$. Then $\sigma|_{G/O_p(Z(N))} = \text{id}|_{G/O_p(Z(N))}$. Furthermore, if σ fixes a Sylow p -subgroup of G element-wise, then $\sigma \in \text{Inn}(G)$.*

Lemma 2.6. *Let N be a normal subgroup of a group G such that G/N is of odd order. If $\text{Out}_C(N) = 1$, then $\text{Out}_C(G) = 1$. In particular, this is the case when G has a normal Sylow 2-subgroup.*

PROOF. This is a consequence of Corollary 3 in [5]; see also Theorem 3.6 in [6]. $\square$

Lemma 2.7. *Suppose that G possesses a strongly 2-embedded subgroup. Then one of the following statements holds.*

- (i) *The Sylow 2-subgroups of G are either cyclic or quaternion groups.*
- (ii) *G possesses a normal series $1 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq G$ such that G_1 and G/G_2 are of odd order, and G_2/G_1 is isomorphic to $\text{PSL}_2(2^n)$, $\text{Sz}(2^{2n-1})$ or $\text{PSU}_3(2^n)$ with $n \geq 2$.*

Lemma 2.8 ([3, Theorem] and [4, Proposition 4.7]). *Let G be a group whose Sylow 2-subgroups are either cyclic, dihedral or generalized quaternion. Then $\text{Out}_C(G) = 1$.*

Lemma 2.9. *Let G be a (TI)-group. Then G has either a normal Sylow 2-subgroup or a strongly 2-embedded subgroup.*

PROOF. Assume that G has no normal Sylow 2-subgroups. Let P be a Sylow 2-subgroup of G . We claim that $M := N_G(P)$ is a strongly 2-embedded subgroup of G . It is clear that 2 divides $|M|$ and $M < G$. In addition, since by hypothesis G is a (TI)-group, it follows that 2 does not divide $|M \cap M^g|$ for any $g \in G \setminus M$. We are done. $\square$

Lemma 2.10 ([16, Lemma 1]). *Let G be a (TI)-group. Then any subgroup of even order of G is a (TI)-group. If N is a normal subgroup of odd order of G , then G/N is a (TI)-group.*

3. Proof of Theorem A

In this section, we shall present a proof of Theorem A. To do this, we first prove the following result.

Theorem 3.1. *Let G be an extension of an odd order group by a simple group. Then $\text{Out}_C(G) = 1$. In particular, the normalizer property holds for G .*

PROOF. Let M be a normal subgroup of odd order of G such that G/M is a simple group. According to whether G/M is abelian or not, we divide the proof of Theorem 3.1 into two cases.

Case 1. G/M is abelian.

If the quotient G/M is of order 2, then G must have a Sylow 2-subgroup of order 2, and thus the assertion follows from Lemma 2.1. If the quotient G/M is of odd prime order, then G is of odd order, and thus the assertion follows from Lemma 2.2.

Case 2. G/M is non-abelian.

Let $\sigma \in \text{Aut}_C(G)$. Next we shall show that $\sigma \in \text{Inn}(G)$. Since σ^2 is inner, it follows that σ is inner provided that an odd power of σ is inner. Noticing this, by replacing σ with an odd power of it (if necessary), we may assume without loss that σ is of 2-power order.

Since $\sigma \in \text{Aut}_C(G)$, it follows that $\sigma|_{G/M} \in \text{Aut}_C(G/M)$. By hypothesis G/M is non-abelian simple, so by Lemma 2.3 $\sigma|_{G/M} \in \text{Inn}(G/M)$, and thus $\sigma|_{G/M} = \text{conj}(x)|_{G/M}$ for some $x \in G$. Modifying σ with a suitable inner automorphism, we may assume that

$$\sigma|_{G/M} = \text{id}|_{G/M}. \quad (3.1)$$

Next we shall show that $\sigma|_M \in \text{Aut}_{\text{Col}}(M)$. Let P be an arbitrary Sylow subgroup of M . Since $\sigma \in \text{Aut}_C(G)$, it follows that there exists some $y \in G$ such that

$$\sigma|_P = \text{conj}(y)|_P. \quad (3.2)$$

Write $H = MC_G(P)$. Then by Lemma 2.4 H is normal in G and $H^\sigma = H$. Moreover,

$$\sigma|_{G/H} = \text{conj}(y)|_{G/H}. \quad (3.3)$$

Note that $H/M \trianglelefteq G/M$ and G/M is simple, so either $H/M = G/M$ or $H/M = 1$. It follows that $G = H$ or $H = M$.

If $G = H$, then $G = MC_G(P)$, and thus we may write y as $y = cm$ with $c \in C_G(P)$ and $m \in M$. It follows from (3.2) that

$$\sigma|_P = \text{conj}(y)|_P = \text{conj}(m)|_P. \quad (3.4)$$

If $H = M$, then (3.3) may be rewritten as

$$\sigma|_{G/M} = \text{conj}(y)|_{G/M}. \quad (3.5)$$

Combining (3.1) with (3.5), we obtain

$$\text{conj}(y)|_{G/M} = \sigma|_{G/M} = \text{id}|_{G/M}, \quad (3.6)$$

which implies that $yM \in Z(G/M)$. Since by hypothesis G/M is non-abelian simple, it follows that $Z(G/M) = \bar{1}$, and thus $yM = M$, i.e., $y \in M$.

Consequently, no matter which case appears, by what we have just proved, the element y in the equality (3.2) can always be assumed to lie in M . As P is an arbitrary Sylow subgroup of M , we obtain $\sigma|_M \in \text{Aut}_{\text{Col}}(M)$. Note that M is of odd order, but $\sigma|_M$ is of even order. So by Proposition 1 in [5], we obtain that

$$\sigma|_M = \text{id}|_M. \quad (3.7)$$

Then Lemma 2.6, (3.1) and (3.7) yield that

$$\sigma|_{G/O_2(Z(M))} = \text{id}|_{G/O_2(Z(M))}. \quad (3.8)$$

But note that M is of odd order, so (3.8) implies that $\sigma = \text{id}$.

This completes the proof of Theorem 3.1. $\square$

PROOF OF THEOREM A. In view of Lemma 2.7, we divide the proof of Theorem A into two cases.

Case 1. G has either a cyclic or a quaternion Sylow 2-subgroup.

The assertion follows from Lemma 2.8.

Case 2. G possesses a normal series $1 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq G$ such that G_1 and G/G_2 are of odd order, and G_2/G_1 is isomorphic to $\text{PSL}_2(2^n)$, $\text{Sz}(2^{2n-1})$ or $\text{PSU}_3(2^n)$ with $n \geq 2$.

By Lemma 2.6, it suffices to show that $\text{Out}_C(G_2) = 1$. But note that G_2 is an extension of an odd order group G_1 by a simple group, so by Theorem 3.1 $\text{Out}_C(G_2) = 1$.

This completes the proof of Theorem A. $\square$

PROOF OF COROLLARY B. Let G be a TI-group. Then by Lemma 2.9 G has either a normal Sylow 2-subgroup or a strongly 2-embedded subgroup. In the former case, the result follows from Lemma 2.6. In the latter case, the result follows from Theorem A. This completes the proof of Corollary B. $\square$

As a direct consequence of Corollary B, we have the following more general result.

Corollary 3.2. *Let G be a (TI)-group. Then the following statements hold.*

- (i) *Let H be an arbitrary subgroup of G . Then $\text{Out}_C(H) = 1$.*
- (ii) *Let K be an arbitrary homomorphic image of G . Then $\text{Out}_C(K) = 1$.*

PROOF. (i) Let H be an arbitrary subgroup of G . We have to show that $\text{Out}_C(H) = 1$. If H is of odd order, then the assertion follows immediately from Lemma 2.2. If H is of even order, then the assertion follows from Lemma 2.10 and Corollary B.

(ii) Let N be a normal subgroup of G . We have to show that $\text{Out}_C(G/N) = 1$. If N is of odd order, then by Lemma 2.10 G/N is a (TI)-group, and hence the assertion follows from Corollary B. Hereafter, we assume that N is of even order. According to Lemmas 2.7 and 2.9, we may divide the proof into three cases.

Case 1. G has a normal Sylow 2-subgroup.

It is easy to see that G/N also has a normal Sylow 2-subgroup and hence by Lemma 2.6 $\text{Out}_C(G/N) = 1$.

Case 2. G has a cyclic or generalized quaternion Sylow 2-subgroup.

Note that a factor group of a cyclic 2-group or a generalized quaternion 2-group is either cyclic, or a generalized 2-group, or $C_2 \times C_2$ (the dihedral group of order 4). Thus by Lemma 2.8 $\text{Out}_C(G/N) = 1$.

Case 3. G contains a normal series: $1 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq G$, where G/G_2 and G_1 are of odd order and G_2/G_1 is isomorphic to one of the groups $L_2(q)$, $U_3(q)$ or $G(q)$. Here the notation of simple groups follows that of [15].

We claim that G/N is of odd order. Considering all possible relationships between N and G_2 , we divide the proof of the preceding claim into three subcases.

Subcase 3.1. $N \geq G_2$

It is clear that G/N is of odd order since by hypothesis G/G_2 is.

Subcase 3.2. $N \leq G_2$.

Since G_2/G_1 is a simple group, it follows that either $NG_1/G_1 = 1$ or $NG_1/G_1 = G_2/G_1$. In the former case, we have $NG_1 = G_1$, and thus $N \leq G_1$, which is impossible since N is of even order and G_1 is of odd order. So

we must have $NG_1/G_1 = G_2/G_1$, i.e., $G_2 = NG_1$. It follows that $G_2/N = NG_1/N \cong G_1/N \cap G_1$. By hypothesis G_1 is of odd order, so is G_2/N . Note that $G/N/G_2/N \cong G/G_2$ and G/G_2 is of odd order, so G/N is of odd order.

Subcase 3.3. $N \not\leq G_2$ and $N \not\cong G_2$.

Write $M = N \cap G_2$. Then $1 \neq M \trianglelefteq G$. Note that $N/M = N/N \cap G_2 \cong NG_2/G_2 \leq G/G_2$ and G/G_2 is of odd order, so M must be of even order. Note further that $M \leq G_2$. Then by Subcase 3.2 G/M is of odd order. Note that $G/N \cong G/M/N/M$, so G/N is of odd order.

Thus in any subcase G/N is of odd order, as claimed. It follows from Lemma 2.2 that $\text{Out}_C(G/N) = 1$.

This completes the proof of Corollary 3.2. $\square$

ACKNOWLEDGMENTS. The author would like to thank the referees for his/her helpful comments and valuable suggestion which made the paper more readable.

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(Received October 18, 2014; revised April 14, 2015)