

Continuum-wise expansive homoclinic classes for generic diffeomorphisms

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Abstract. Let $f : M \rightarrow M$ be a diffeomorphism on a closed smooth n ($n \geq 2$)-dimensional Riemannian manifold M . For C^1 generic f , if a homoclinic class $H_f(p)$ is continuum-wise expansive then it is hyperbolic. Moreover, we show that if a diffeomorphism $f : M \rightarrow M$ exhibiting a homoclinic tangency associated to a hyperbolic periodic point p , there is g C^1 close to f such that g is not continuum-wise expansive.

1. Introduction

Let M be a closed smooth n ($n \geq 2$)-dimensional Riemannian manifold without boundary, and let $f : M \rightarrow M$ be a diffeomorphism. Denote $\text{Diff}(M)$ the space of diffeomorphisms of M with the C^1 topology. Let d be the distance on M induced from the Riemannian metric $\|\cdot\|$ on the tangent bundle TM . For any closed f -invariant set $\Lambda \subset M$, we say that Λ is *expansive* for f if there is $e > 0$ such that for any $x, y \in \Lambda$ if $d(f^n(x), f^n(y)) \leq e$ for all $n \in \mathbb{Z}$ then $x = y$. If $\Lambda = M$ then f is expansive. Roughly speaking, a system is expansive if two points stay near for future and past iterates then they must be equal. This notion was introduced by UTZ [21]. In a dynamical system, the notion of expansiveness is a very useful tool to investigate of the stability theory. For instance, MAÑÉ [12] proved that a diffeomorphism f belongs to the C^1 -interior of the set of all expansive diffeomorphisms if and only if it is quasi-Anosov. We say that f is *quasi-Anosov* if for any $v \in TM \setminus \{0\}$, the set $\{\|Df^n(v)\| : n \in \mathbb{Z}\}$ is unbounded.

Mathematics Subject Classification: Primary: 37C50; Secondary: 34D10.

Key words and phrases: expansive, continuum-wise expansive, hyperbolic, homoclinic class, generic.

For expansivity, KATO [7] introduced the generalized concept of expansivity which is called *continuum-wise expansive*. A set Λ is *nondegenerate* if the set Λ is not reduced to one point. We say that $\Lambda \subset M$ is a *subcontinuum* if it is a compact connected nondegenerate subset Λ of M . A diffeomorphism f on M is said to be *continuum-wise expansive* if there is a constant $e > 0$ such that for any nondegenerate subcontinuum A there is an integer $n = n(A)$ such that $\text{diam } f^n(A) \geq e$, where $\text{diam } A = \sup\{d(x, y) : x, y \in A\}$ for any subset A of M . Such the constant e is called a *continuum-wise expansive constant* for f . Note that every expansive homeomorphism is continuum-wise expansive, but its converse is not true (see [6]). In fact, we consider that if for a unit $\mathbf{S}^2$, $f : \mathbf{S}^2 \rightarrow \mathbf{S}^2$ is a diffeomorphism, then it is well-known that f does not admit an expansive diffeomorphism, but it admits a continuum-wise expansive diffeomorphisms. Note that by Mañé's result, a robustly expansive diffeomorphism is a quasi-Anosov diffeomorphism. For continuum-wise expansiveness, SAKAI [18] proved that if a diffeomorphism f belongs to the C^1 -interior of the set of all continuum-wise expansive diffeomorphisms then it is quasi-Anosov. Thus we know that a robustly expansive diffeomorphism is a robustly continuum-wise expansive diffeomorphism.

2. Statement of the main results

A point $x \in M$ is *non-wandering point* of f if a neighborhood U of x there is $n > 0$ such that $f^n(U) \cap U \neq \emptyset$. Denote by $\Omega(f)$ the set of all non-wandering points of f . We say that Λ is *hyperbolic* for f if the tangent bundle $T_\Lambda M$ has a Df -invariant splitting $E^s \oplus E^u$ and there exist constants $C > 0$ and $0 < \lambda < 1$ such that

$$\|D_x f^n|_{E_x^s}\| \leq C\lambda^n \quad \text{and} \quad \|D_x f^{-n}|_{E_x^u}\| \leq C\lambda^n$$

for all $x \in \Lambda$ and $n \geq 0$. If $\Lambda = M$ then f is *Anosov*.

We say that f satisfies *Axiom A* if its periodic points are dense in the set of non-wandering points $\Omega(f)$, and f is hyperbolic on $\Omega(f)$.

We say that a subset $\mathcal{G} \subset \text{Diff}(M)$ is *residual* if $\mathcal{G}$ contains the intersection of a countable family of open and dense subsets of $\text{Diff}(M)$; in this case $\mathcal{G}$ is dense in $\text{Diff}(M)$. A property "P" is said to be (C^1) *generic* if "P" holds for all diffeomorphisms which belong to some residual subset of $\text{Diff}(M)$. ARBIETO [2] proved that for C^1 generic f , if f is expansive then it satisfies both Axiom A and the no-cycle condition. In [10], LEE proved that for C^1 -generic f , if f is continuum-wise expansive then it satisfies both Axiom A and the no-cycle condition.

If f satisfies Axiom A then the non-wandering set $\Omega(f) = \Lambda_1 \cup \Lambda_2 \cup \dots \cup \Lambda_m$, where Λ_i are compact, disjoint, invariant sets, and each Λ_i contains dense periodic orbits. The sets $\Lambda_1, \dots, \Lambda_m$ are called the *basic sets*. It is well known that if p is a hyperbolic periodic point of f with period k then the sets $W^s(p) = \{x \in M : f^{kn}(x) \rightarrow p \text{ as } n \rightarrow \infty\}$ and $W^u(p) = \{x \in M : f^{-kn}(x) \rightarrow p \text{ as } n \rightarrow \infty\}$ are C^1 -injectively immersed submanifolds of M . A point $x \in W^s(p) \cap W^u(p)$ is called a *homoclinic point* of f associated to p . The closure of the homoclinic points of f associated to p is called the *homoclinic class* of f associated to p , and it is denoted by $H_f(p)$. It is well-known that the basic sets is a homoclinic class $H_f(p)$. This set like occurs for instance in Smale's horseshoe. Actually, in [14] we can see various examples. For homoclinic classes and expansivity, there are many results published in [6], [8], [9], [10], [11], [14], [15], [19], [20], [23]. Among that we introduce two results. First, DAS, LEE and LEE [6] proved that if the homoclinic class $H_f(p)$ is C^1 -persistently expansive and the chain condition then it is hyperbolic. Here a homoclinic class $H_f(p)$ satisfy the chain condition if for any g C^1 -close to f , the homoclinic class $H_g(p_g)$ is the chain component, say, $C_g(p_g)$. Finally, YANG and GAN [23] showed that for C^1 generic f , expansive homoclinic classes are hyperbolic. From the results, we have the following which is a main result of the paper. It is a general result of [23].

Theorem A. *For C^1 generic f , if a homoclinic class $H_f(p)$ is continuum-wise expansive then it is hyperbolic.*

Let p be a hyperbolic periodic point. We say that f is a *homoclinic tangency* if there is a hyperbolic periodic point p whose invariant manifolds $W^s(p)$ and $W^u(p)$ have a non-transverse intersection. Denote by $\mathcal{HT}$ the set of all homoclinic tangency diffeomorphisms. We say that Λ admits a *dominated splitting* if the tangent bundle $T_\Lambda M$ has a continuous Df -invariant splitting $E \oplus F$ and there exist constants $C > 0$ and $0 < \lambda < 1$ such that

$$\|D_x f^n|_{E(x)}\| \cdot \|D_x f^{-n}|_{F(f^n(x))}\| \leq C\lambda^n$$

for all $x \in \Lambda$ and $n \geq 0$. The set Λ is *partially hyperbolic* if the tangent bundle $T_\Lambda M$ has a dominated splitting $E^s \oplus E^c \oplus E^u$ and there exist $C > 0$, and $0 < \lambda < 1$ such that E^s is contracting, E^u is expanding, and for any vector in E^c is less expand than vector in E^u and less contracted than vectors in E^s . In [5], the authors proved that for C^1 generic f , if $f \in \text{Diff}(M) \setminus \overline{\mathcal{HT}}$ then f is partially hyperbolic, where $\overline{A}$ is the closure of A .

Let M be a compact smooth 2-dimensional manifold, and let $f : M \rightarrow M$ be a diffeomorphism. PACIFICO and VIEITES [17] proved that if f having a homoclinic

tangency associated to a hyperbolic periodic point p , then there is a g C^1 -close to f such that g is not measure expansive. Here, measure expansive was introduced by [13]. By ARTIGUE and CARRASCO-OLIVERA [3, Lemma 2.3], we know that continuum-wise expansive is a more general notion than measure expansive. Thus the following is a general result of [17, Theorem B].

Theorem B. *Let M be a compact smooth $n(\geq 2)$ -dimensional manifold, and let $f : M \rightarrow M$ be a diffeomorphism. If f has a homoclinic tangency associated to a hyperbolic periodic point p , then there is a g C^1 -close to f such that g is not continuum-wise expansive.*

3. Proof of Theorem A

Let p and q be hyperbolic periodic points. We write $p \sim q$ if $W^s(p) \cap W^u(q) \neq \emptyset$ and $W^u(p) \cap W^s(q) \neq \emptyset$. We say that p and q are *homoclinic related* if $p \sim q$.

By Oseledec's theorem, any f -invariant probability μ , almost every point admits a splitting of tangent space

$$T_x M = E_x^1 \oplus \cdots \oplus E_x^k, \quad k = k(x)$$

and real numbers $\chi_1(x, v) \leq \chi_2(x, v) \leq \cdots \chi_k(x, v)$ such that

$$\chi(x, v) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \|Df^n(x)v\|,$$

for every non-zero $v_i \in E_x^i$. These objects are uniquely defined and they vary measurably with the point x . If μ is ergodic then the Lyapunov exponents $\chi_i(x, v)$ are constant on orbits. Thus they are constant μ -almost everywhere if μ is ergodic.

For a homoclinic class $H_f(p)$, WANG [22] proved the following.

Theorem 3.1. *For C^1 generic f , a homoclinic class $H_f(p)$ either is hyperbolic, or contains periodic orbits with arbitrarily long periods that are homoclinically related to p and have a Lyapunov exponent arbitrarily close to 0.*

Let p be a periodic point of f . For any $\delta \in (0, 1)$, we say that p has a δ -weak eigenvalue if $D_p f^{\pi(p)}$ has an eigenvalue λ such that $(1 - \delta)^{\pi(p)} < |\lambda| < (1 + \delta)^{\pi(p)}$.

Remark 3.2. A periodic point p of f is hyperbolic if and only if all the Lyapunov exponents of p are nonzero. Thus a periodic point q has a Lyapunov exponent arbitrarily close to 0 means that q has a δ -weak eigenvalue.

The following notion was introduced by YANG and GAN [23]. For any $\gamma > 0$, a C^1 curve ζ is called γ -*simply periodic curve* of f if (i) ζ is diffeomorphic to $[0, 1]$ and its two endpoints are hyperbolic periodic points of f , (ii) ζ is periodic with period $\pi(\zeta)$ and the length of ζ , that is, $L(f^i(\zeta)) < \gamma$ for any $i \in \{1, 2, \dots, \pi(\zeta)\}$, where $L(\zeta)$ denotes the length of ζ , and (iii) ζ is normally hyperbolic. For the γ -simply periodic curve and δ -weak eigenvalue, we have the following which was proved by YANG and GAN [23, Lemma 2.1].

Lemma 3.3. *For C^1 generic f , and any hyperbolic periodic point p of f , we have the following:*

- (a) *for any $\gamma > 0$, if for any C^1 -neighborhood $\mathcal{U}(f)$ of f some $g \in \mathcal{U}(f)$ has a γ -simply periodic curve ς such that two endpoints of ς are homoclinically related with p_g then f has an 2γ -simply periodic curve ζ such that the two endpoints of ζ are homoclinically related to p .*
- (b) *for any $\delta > 0$, if for any C^1 -neighborhood $\mathcal{U}(f)$ of f , some $g \in \mathcal{U}(f)$ has a periodic $q \sim p_g$ with δ -weak eigenvalue, then f has a periodic point $q_f \sim p$ with 2δ -weak eigenvalue and every eigenvalue of q_f is real.*

Lemma 3.4. *For C^1 generic $f \in \text{Diff}(M)$, if a homoclinic class $H_f(p)$ is continuum-wise expansive then a periodic point q contained in $H_f(p)$ with homoclinically related to p has no a δ -weak eigenvalue.*

PROOF. Suppose, by contradiction, that there is a periodic point $q \in H_f(p)$ with homoclinically related to p such that q has a δ -weak eigenvalue. For any $\gamma > 0$, there is g C^1 close to f such that g has a $\gamma/2$ -simply periodic curve ς such that two endpoints of ς are homoclinically related with p_g , where p_g is the continuation of p (see [19, Theorem 2]). By Lemma 3.3, f has an γ -simply periodic curve ζ such that the two endpoints of ζ are homoclinically related to p . By [4], $H_f(p) = C_f(p)$, and so, we know $\zeta \subset H_f(p)$. Take $e \geq \gamma$. Then we have

$$\text{diam}(f^{\pi(\zeta)^i}(\zeta)) \leq e, \quad \text{for all } i \in \mathbb{Z}. \quad (1)$$

Note that f is continuum-wise expansive if and only if f^n is continuum-wise expansive $n \in \mathbb{Z} \setminus \{0\}$ (see [7, Proposition 2.6]). Since ζ is γ -simply periodic curve, we know

$$f^{\pi(\zeta)}(\zeta) = f^{\pi(\zeta)^i}(\zeta) = \zeta,$$

for all $i \in \mathbb{Z}$. Then (1) should be continuum-wise expansive which is a contradiction since ζ is not one point set. $\square$

PROOF OF THEOREM A. By Theorem 3.1, we will prove that for C^1 generic f , if a homoclinic class $H_f(p)$ is measure expansive then there is $\delta > 0$ such that for any $q \sim p$, q has no δ -weak eigenvalue. Suppose, by contradiction, that for any $\delta > 0$, $H_f(p)$ contains a periodic point $q \sim p$ with δ -weak eigenvalue. Then by Lemma 3.4, this is a contradiction. Thus for C^1 generic f , if a homoclinic class $H_f(p)$ is measure expansive then there is no periodic point $q \in H_f(p)$ with homoclinically related to p such that q has a δ -weak eigenvalue, and so, a measure expansive homoclinic class $H_f(p)$ is hyperbolic. $\square$

Let Λ be a closed f -invariant set. We say that Λ is *transitive* if there is a point $x \in \Lambda$ such that $\omega(x) = \Lambda$, where $\omega(x)$ is the omega-limit set of x . If $\Lambda = M$ then f is transitive. Note that the homoclinic class $H_f(p)$ is a closed, invariant and transitive. We say that Λ is *locally maximal* if there is a neighborhood U of Λ such that $\Lambda = \bigcap_{n \in \mathbb{Z}} f^n(U)$.

Corollary 3.5. *For C^1 generic f , if a transitive diffeomorphism f is continuum-wise expansive then it is Anosov.*

PROOF. By [1, Theorem 4.10], for C^1 generic f , a locally maximal homoclinic class $H_f(p)$ is a transitive set. Thus C^1 generic f , if f is transitive then it is a homoclinic class $H_f(p)$. By Theorem A, f is Anosov. $\square$

4. Proof of Theorem B

In this section, we prove that for a diffeomorphism $f : M \rightarrow M$ with a homoclinic tangency associated to a hyperbolic point p , if there is a g C^1 -close to f such that g exhibits a homoclinic tangency then it is not continuum-wise expansive. To show that we need the following lemma which was proved by PACIFICO and VIEITEZ [16] and also founded in [17, Lemma 4.2].

Lemma 4.1 ([16, Proposition 2.6]). *Let $f : M \rightarrow M$ be a diffeomorphism with a homoclinic tangency associated to a hyperbolic periodic point p . Then there is a g C^1 -close to f such that g has a small arc $\mathcal{J}$ contained in $W^s(p_g, g) \cap W^u(p_g, g)$, where p_g is the continuation of p .*

PROOF OF THEOREM B. Let f having a homoclinic tangency associated to a hyperbolic periodic point p , and let $\mathcal{U}(f)$ be a C^1 -neighborhood of f . Suppose that for any $g \in \mathcal{U}(f)$, g is continuum-wise expansive. Since f has a homoclinic tangency associated to a hyperbolic periodic point p , by Lemma 4.1, there is $h \in \mathcal{U}(f)$ which has a small arc $\mathcal{J}$ contained in $W^s(p_h, h) \cap W^u(p_h, h)$. Clearly,

it is not an one point set. Put $\text{diam}(\mathcal{J}) = \alpha$. Let $e = \alpha/4$ be a continuum-wise expansive constant. Since $\mathcal{J} \subset W^s(p_h, h) \cap W^u(p_h, h)$, there is $N > 0$ such that (i) $\text{diam } h^i(\mathcal{J}) \leq e/2$ for $-N \leq i \leq N$, and (ii) $h^i(\mathcal{J}) \subset W_{e/2}^s(p_h, h) \cap W_{e/2}^u(p_h, h)$, for $|i| > N$. This means that $\text{diam } h^i(\mathcal{J}) \leq e$ for all $i \in \mathbb{Z}$. Since $\mathcal{J}$ is not an one point set, this is a contradiction. $\square$

ACKNOWLEDGEMENT. The author would like to thank K. LEE for suggestions given during the preparation of this work. This work is supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Science, ICT & Future Planning (No. 2014R1A1A1A05002124).

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(Received March 23, 2015)