

Some sub-classes of paranormal weighted conditional type operators

By YOUSEF ESTAREMI (Tehran)

Abstract. In this paper, some sub-classes of paranormal weighted conditional type operators of the form M_wEM_u , such as $*$ -paranormal, quasi- $*$ -paranormal and (n, k) -quasi- $*$ -paranormal weighted conditional type operators on $L^2(\Sigma)$ are investigated. As a consequence we get that some of these sub-classes coincide if and only if a reverse conditional type Hölder inequality holds for w, u . Finally, we get some results about the spectrum, point spectrum, joint point spectrum, approximate point spectrum and joint approximate point spectrum of WCT operators. The results of this paper and some other papers that we wrote about WCT operators show that the conditional type Hölder inequality for w, u plays a major role in the behavior of WCT operators.

1. Introduction and preliminaries

Theory of weighted conditional type operators (or briefly WCT Operators) is an important connection between operator theory and measure theory. WCT operators have been studied in an operator theoretic setting, by many authors, for example, DE PAGTER and GROBLER [7] and RAO [13, 14], as positive operators acting on L^p -spaces or Banach function spaces. In [11], S.-T. C. MOY characterized all operators on L^p of the form $f \rightarrow E(fg)$ for g in L^q with $E(|g|)$ bounded. Also, some results about these operators can be found in [1], [8], [9]. In [2] P. G. DODDS, C. B. HUIJSMANS and B. DE PAGTER showed that many classes of operators are WCT operators. In [7] a class of operators that factorizes

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through WCT operators is investigated. This class of operators includes operators such as kernel operators and order continuous Riesz homomorphisms. Also, we investigated some classical properties of these operators on L^p -spaces in [3], [4], [5].

Let (X, Σ, μ) be a σ -finite measure space. For a sub- σ -finite algebra $\mathcal{A} \subseteq \Sigma$, the conditional expectation operator associated with $\mathcal{A}$ is the mapping $f \rightarrow E^{\mathcal{A}}f$, defined for all non-negative measurable functions f as well as for all $f \in L^2(\Sigma)$, where $E^{\mathcal{A}}f$, by the Radon–Nikodym theorem, is the unique $\mathcal{A}$ -measurable function satisfying $\int_A f d\mu = \int_A E^{\mathcal{A}}f d\mu$, $\forall A \in \mathcal{A}$. As an operator on $L^2(\Sigma)$, $E^{\mathcal{A}}$ is idempotent and $E^{\mathcal{A}}(L^2(\Sigma)) = L^2(\mathcal{A})$. This operator will play a major role in our work. Let $f \in L^0(\Sigma)$, then f is said to be conditionable with respect to E if $f \in \mathcal{D}(E) := \{g \in L^0(\Sigma) : E(|g|) \in L^0(\mathcal{A})\}$ in which $L^0(\Sigma)$ is the vector space of all equivalence classes of almost everywhere finite valued measurable functions on X . Throughout this paper we take u and w in $\mathcal{D}(E)$. If there is no possibility of confusion, we write $E(f)$ in place of $E^{\mathcal{A}}(f)$. A detailed discussion about this operator may be found in [12]. All comparisons between two functions or two sets are to be interpreted as holding up to a μ -null set. The support of a measurable function f is defined as $S(f) = \{x \in X; f(x) \neq 0\}$.

Let $\mathcal{H}$ be an infinite complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded operators on $\mathcal{H}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is a partial isometry if $\|Th\| = \|h\|$ for h orthogonal to the kernel of T . It is known that an operator T on a Hilbert space is a partial isometry if and only if $TT^*T = T$. Every operator T on a Hilbert space $\mathcal{H}$ can be decomposed into $T = U|T|$ with a partial isometry U , where $|T| = (T^*T)^{\frac{1}{2}}$. U is determined uniquely by the kernel condition $\mathcal{N}(U) = \mathcal{N}(|T|)$. This decomposition is called the polar decomposition. The Aluthge transformation $\widehat{T}$ of the operator T is defined by $\widehat{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$. The operator T is said to be a positive operator and written as $T \geq 0$, if $\langle Th, h \rangle \geq 0$, for all $h \in \mathcal{H}$.

In this paper we will be concerned with characterizing WCT operators $W_{w,u}$ on $L^2(\Sigma)$, which are studied in [2], [5], [8], [11], in terms of membership of the partial paranormal classes. Also, we prove that the point spectrum and joint point spectrum of the WCT operators $W_{w,u}$ are the same, when u, w satisfy a mild condition. Here is a brief review of what constitutes membership for an operator T on a Hilbert space in each classes:

- (i) T is called paranormal if for all unit vectors x in H , $\|Tx\|^2 \leq \|T^2x\|$ or equivalently

$$T^{*2}T^2 - 2\lambda T^*T + \lambda^2 I \geq 0,$$

for all $\lambda > 0$;

- (ii) T is called M -paranormal, if there exists $M > 0$ such that for all unit vectors x in H , $\|Tx\|^2 \leq M\|T^2x\|$ or equivalently $M^2T^{*2}T^2 - 2\lambda T^*T + \lambda^2 I \geq 0$, for all $\lambda > 0$.

- (iii) T is called $*$ -paranormal, if $\|T^*x\|^2 \leq \|T^2x\|$ for all unit vector $x \in H$ or equivalently

$$T^{*2}T^2 - 2\lambda TT^* + \lambda^2 I \geq 0,$$

for all $\lambda > 0$;

- (iv) T is called quasi- $*$ -paranormal, if it satisfies the following inequality:

$$\|T^*Tx\|^2 \leq \|T^3x\| \cdot \|Tx\|$$

for all $x \in H$ or equivalently

$$*(T^{*2}T^2 - 2\lambda TT^* + \lambda^2 I)T \geq 0,$$

for all $\lambda > 0$;

- (v) is called (n, k) -quasi- $*$ -paranormal if

$$\|T^{1+n}(T^kx)\|^{\frac{1}{1+n}} \|T^kx\|^{\frac{n}{n+1}} \geq \|T^*T^kx\|$$

for all $x \in H$, or equivalently

$$T^{*k}T^{*(1+n)}T^{1+n}T^k - (1+n)\mu^n T^{*k}TT^*T^k + n\mu^{n+1}T^{*k}T^k \geq 0$$

for all $\mu > 0$;

- (vi) (vi) T is called absolute- k -paranormal for each $k > 0$ if

$$\||T|^kTx\| \geq \|Tx\|^{k+1}$$

for every unit vector $x \in H$ or equivalently

$$T^*|T|^{2k}T - (k+1)\lambda^k|T|^2 + k\lambda^{k+1}I \geq 0$$

for all $\lambda > 0$;

2. Partial paranormal WCT operators

We now define the class of operators under investigation.

Definition 2.1. Let (X, Σ, μ) be a σ -finite measure space and let $\mathcal{A}$ be a σ -subalgebra of Σ such that $(X, \mathcal{A}, \mu)$ is σ -finite. Let E be the corresponding conditional expectation operator on $L^2(\Sigma)$ relative to $\mathcal{A}$. If $w, u \in L^0(\Sigma)$ such

that uf is conditionable and $wE(uf) \in L^2(\Sigma)$ for all $f \in L^2(\Sigma)$, then the corresponding weighted conditional type operator (or WCT operator) is the linear transformation $W_{w,u} : L^2(\Sigma) \rightarrow L^2(\Sigma)$ defined by $f \rightarrow wE(uf)$.

The functions w, u are called the weight functions of $W_{w,u}$ and they are not assumed to be bounded (hence, $W_{w,u}$ need not be a bounded operator). In the event they are bounded, however, it is easy to show that $W_{w,u}$ is a bounded operator.

First we recall some results of [2] that state our results is valid for a large class of linear operators. Let (X, Σ, μ) be a finite measure space, then $L^\infty(\Sigma) \subseteq L^2(\Sigma) \subseteq L^1(\Sigma)$ and $L^2(\Sigma)$ is an order ideal of measurable functions on (X, Σ, μ) . Thus by Propositions 3.1, 3.3, 3.6 of [2] immediately we have Theorems 2.2, 2.3, 2.4:

Theorem 2.2. *If T is a linear operator on $L^2(\Sigma)$ for which*

- (i) $Tf \in L^\infty(\Sigma)$ whenever $f \in L^\infty(\Sigma)$.
- (ii) $\|Tf_n\|_1 \rightarrow 0$ for all sequences $\{f_n\}_{n=1}^\infty \subseteq L^2(\Sigma)$ such that $|f_n| \leq g$ ($n = 1, 2, 3, \dots$) for some $g \in L^2(\Sigma)$ and $f_n \rightarrow 0$ a.e.,
- (iii) $T(f.Tg) = Tf.Tg$ for all $f \in L^\infty(\Sigma)$ and all $g \in L^2(\Sigma)$, then there exists a σ -subalgebra $\mathcal{A}$ of Σ and there exists $w \in L^2(\Sigma)$ such that $Tf = E^{\mathcal{A}}(wf) = W_{1,w}(f)$ for all $f \in L^2(\Sigma)$.

Theorem 2.3. *For a linear operator $T : L^2(\Sigma) \rightarrow L^2(\Sigma)$ the following statement are equivalent.*

- (i) T is positive and order continuous, $T^2 = T$, $T1 = 1$ and the range $\mathcal{R}(T)$ of T is a sublattice.
- (ii) There exist a σ -subalgebra $\mathcal{A}$ of Σ and a function $0 \leq w \in L^2(\Sigma)$ with $E^{\mathcal{A}}(w) = 1$ such that $Tf = E^{\mathcal{A}}(wf) = W_{1,w}(f)$ for all $f \in L^2(\Sigma)$.

Theorem 2.4. *For a linear operator $T : L^2(\Sigma) \rightarrow L^2(\Sigma)$ the following statement are equivalent.*

- (i) T is a positive and order continuous projection onto a sublattice such that $T1$ is strictly positive.
- (ii) There exist a σ -subalgebra $\mathcal{A}$ of Σ , $0 \leq w \in L^2(\Sigma)$ and a strictly positive function $k \in L^1(\Sigma)$ with $E^{\mathcal{A}}(wk) = 1$ such that $Tf = kE^{\mathcal{A}}(wf) = W_{k,w}(f)$ for all $f \in L^2(\Sigma)$. Moreover, if we choose k such that $E^{\mathcal{A}}(k) = 1$, then both w and k are uniquely determined by T .

Here, we recall some properties of WCT operators, that we have proved in [5].

$W_{w,u}$ is bounded on $L^2(\Sigma)$ if and only if $(E|w|^2)^{\frac{1}{2}}(E|u|^2)^{\frac{1}{2}} \in L^\infty(\mathcal{A})$, and in this case its norm is given by $\|W_{w,u}\| = \|(E(|w|^2))^{\frac{1}{2}}(E(|u|^2))^{\frac{1}{2}}\|_\infty$. The unique polar decomposition of a bounded WCT operator $W_{w,u}$ is $U|W_{w,u}|$, where

$$|W_{w,u}|(f) = \left(\frac{E(|w|^2)}{E(|u|^2)} \right)^{\frac{1}{2}} \chi_S W_{\bar{u},u}(f)$$

and

$$U(f) = \left(\frac{\chi_{S \cap G}}{E(|w|^2)E(|u|^2)} \right)^{\frac{1}{2}} W_{w,u}(f),$$

for all $f \in L^2(\Sigma)$, where $S = S(E(|u|^2))$ and $G = S(E(|w|^2))$.

Also, for all $f \in L^2(\Sigma)$ we have

$$\begin{aligned} \langle wE(uf), f \rangle &= \int_X wE(uf) \bar{f} d\mu = \int_X uf E(w\bar{f}) d\mu \\ &= \int_X f \bar{u} E(\overline{w\bar{f}}) d\mu = \langle f, \bar{u} E(\bar{w}f) \rangle. \end{aligned}$$

This implies that $(W_{w,u})^* = W_{\bar{u},\bar{w}}$.

Here we recall the definition of multiplication operators. Let $u : \Omega \rightarrow \mathbb{C}$ be a measurable function on Ω . Then the rule taking u to $u.f$, is a linear transformation on $L^0(\Omega)$ and we denote this transformation by M_u . In the case that M_u is continuous, it is called multiplication operator induced by u . In the sequel some necessary and sufficient conditions for a WCT operator $W_{w,u}$ to be M -paranormal, quasi- $*$ -paranormal, absolute- k -paranormal and (n, k) -quasi- $*$ -paranormal will be presented.

Theorem 2.5. *Let $W_{w,u}$ be bounded on $L^2(\Sigma)$. Then*

- (i) *If $M^2|E(uw)|^2 E(|w|^2) - 2\lambda E(|w|^2) \geq 0$ for all $\lambda > 0$, then $W_{w,u}$ is M -paranormal.*
- (ii) *If $W_{w,u}$ is M -paranormal, we have*

$$M^2|E(uw)|^2 E(|w|^2) - 2\lambda E(|w|^2)|E(u)|^2 + \lambda^2 \geq 0,$$

for all $\lambda > 0$.

PROOF. (i) It is easy to see that

$$W_{w,u}^* W_{w,u} = M_{E(|w|^2)} W_{\bar{u},u}, \quad W_{w,u}^{*2} W_{w,u}^2 = M_{|E(uw)|^2 E(|w|^2)} W_{\bar{u},u}.$$

So for every $\lambda > 0$ and $M > 0$

$$\begin{aligned} M^2 M_{|E(uw)|^2 E(|w|^2)} W_{\bar{u},u} - 2\lambda M_{E(|w|^2)} W_{\bar{u},u} + \lambda^2 I \\ = (M_{M^2|E(uw)|^2 E(|w|^2) - 2\lambda E(|w|^2)}) W_{\bar{u},u} + \lambda^2 I. \end{aligned}$$

Let $\alpha = M^2|E(uw)|^2 E(|w|^2) - 2\lambda E(|w|^2)$. Then for every $f \in L^2(\Sigma)$ we get

$$\langle M_\alpha W_{\bar{u},u}(f) + \lambda^2 f, f \rangle = \int_X \alpha |E(uf)|^2 d\mu + \int_X \lambda^2 |f|^2 d\mu.$$

This implies that if $\alpha \geq 0$ a.e, then $W_{w,u}$ is M -paranormal.

(ii) If $W_{w,u}$ is M -paranormal, then for all $f \in L^2(\mathcal{A})$

$$\begin{aligned} \langle M_\alpha W_{\bar{u},u}(f) + \lambda^2 f, f \rangle &= \int_X \alpha |E(uf)|^2 d\mu + \int_X \lambda^2 |f|^2 d\mu \\ &= \int_X (\alpha |E(u)|^2 + \lambda^2) |f|^2 d\mu \geq 0. \end{aligned}$$

Therefore $\alpha |E(u)|^2 + \lambda^2 \geq 0$ a.e., □

Corollary 2.6. *Let $W_{w,u}$ be bounded on $L^2(\Sigma)$. Then*

(i) *If $W_{w,u}$ is paranormal, we have*

$$(|E(uw)|^2 E(|w|^2) - 2k E(|w|^2)) |E(u)|^2 + k^2 \geq 0,$$

(ii) *If $|E(uw)|^2 E(|w|^2) - 2k E(|w|^2) \geq 0$, then $W_{w,u}$ is paranormal.*

Corollary 2.7. *Let $u \in L^0(\mathcal{A})$ and $W_{w,u}$ be bounded on $L^2(\Sigma)$. Then:*

(i) *$W_{w,u}$ is M -paranormal if and only if*

$$(M^2 |u E(w)|^2 E(|w|^2) - 2\lambda E(|w|^2)) |u|^2 + \lambda^2 \geq 0,$$

for all $\lambda > 0$.

(ii) *$W_{w,u}$ is paranormal if and only if*

$$(|u E(w)|^2 E(|w|^2) - 2\lambda E(|w|^2)) |u|^2 + \lambda^2 \geq 0,$$

for all $\lambda > 0$.

PROOF. Since $|E(f)|^2 \leq E(|f|^2)$ for every $f \in L^2(\Sigma)$, then by similar method of Theorem 2.5 we get the proof. □

The definition of quasi- $*$ -paranormal and $*$ -paranormal operators shows that, if T is quasi- $*$ -paranormal, then $T|_{\overline{\mathcal{R}(T)}}$ is $*$ -paranormal. Therefore, if T has dense range, then T is quasi- $*$ -paranormal if and only if it is $*$ -paranormal. In the next theorem we give a necessary and sufficient condition for $W_{w,u}$ to be quasi- $*$ -paranormal.

Theorem 2.8. *Let $W_{w,u}$ be bounded on $L^2(\Sigma)$. Then $W_{w,u}$ is quasi- $*$ -paranormal if and only if*

$$E(|u|^2)E(|w|^2) \leq |E(uw)|^2 \text{ a.e., on } G.$$

Where $G = S(E(|w|^2))$.

PROOF. By direct computations we have

$$W_{w,u}^n f = (E(uw))^{n-1} w E(uf), \quad W_{w,u}^{*n} f = (\overline{E(uw)})^{n-1} \bar{u} E(\bar{w}f),$$

for all $f \in L^2(\Sigma)$ and $n \in \mathbb{N}$. So we get that

$$\begin{aligned} (W_{w,u}^* W_{w,u})^2 &= M_{E(|u|^2)(E(|w|^2))^2} W_{\bar{u},u}, \\ W_{w,u}^* W_{w,u} &= M_{E(|w|^2)} W_{\bar{u},u}, \\ W_{w,u}^{*3} W_{w,u}^3 &= M_{|E(uw)|^4 E(|w|^2)} W_{\bar{u},u}. \end{aligned}$$

Therefore $W_{w,u}$ is quasi- $*$ -paranormal if and only if

$$\begin{aligned} &M_{|E(uw)|^4 E(|w|^2)} W_{\bar{u},u} - 2\lambda M_{E(|u|^2)(E(|w|^2))^2} W_{\bar{u},u} + \lambda^2 M_{E(|w|^2)} W_{\bar{u},u} \\ &= (M_{|E(uw)|^4 E(|w|^2)} - 2\lambda M_{E(|u|^2)\chi_S(E(|w|^2))^2} + \lambda^2 M_{E(|w|^2)}) W_{\bar{u},u} \geq 0. \end{aligned}$$

This implies that $W_{w,u}$ is quasi- $*$ -paranormal if and only if for all $f \in L^2(\Sigma)$ and $\lambda > 0$

$$\begin{aligned} 0 &\leq \langle M_{|E(uw)|^4 E(|w|^2)} f - 2\lambda M_{E(|u|^2)\chi_S(E(|w|^2))^2} f + \lambda^2 M_{E(|w|^2)} f, f \rangle \\ &= \langle M_{(|E(uw)|^4 E(|w|^2) - 2\lambda E(|u|^2)\chi_S(E(|w|^2))^2 + \lambda^2 E(|w|^2))} f, f \rangle, \\ &\iff |E(uw)|^4 E(|w|^2) - 2\lambda E(|u|^2)\chi_S(E(|w|^2))^2 + \lambda^2 E(|w|^2) \geq 0 \\ &\iff (E(|u|^2))^2 (E(|w|^2))^4 - (E(|w|^2))^2 |E(uw)|^4 \geq 0 \\ &\iff E(|u|^2)E(|w|^2) \leq |E(uw)|^2 \text{ on } G, \end{aligned}$$

where we have used the fact that $T_1 T_2 \geq 0$ if $T_1 \geq 0$, $T_2 \geq 0$ and $T_1 T_2 = T_2 T_1$ for all $T_i \in \mathcal{B}(\mathcal{H})$ (the algebra of all bounded operators on the Hilbert space $\mathcal{H}$), positivity of $W_{\bar{u},u}$ and $M_\alpha W_{\bar{u},u} = W_{\bar{u},u} M_\alpha$ with

$$\alpha = |E(uw)|^4 E(|w|^2) - 2\lambda E(|u|^2) \chi_S(E(|w|^2))^2 + \lambda^2 E(|w|^2).$$

An operator $T \in \mathcal{L}(\mathcal{H})$ is a quasi- $*$ - A -class operator if $T^*|T|^2 T \geq T^*|T^*|^2 T$. Also T is an A -class operator if $|T|^2 \leq |T^*|^2$. One can see [6] for more details. In [4] we studied quasi- $*$ - A -class weighted conditional type operators. Here we get that quasi- $*$ - A -class and quasi- $*$ -paranormal weighted conditional type operators coincide. $\square$

Theorem 2.9. *Let $W_{w,u}$ be bounded on $L^2(\Sigma)$ and $G = X$. Then the following are mutually equivalent:*

- (a) $W_{w,u}$ is quasi- $*$ -paranormal;
- (b) $W_{w,u}$ is a quasi- $*$ - A -class operator;
- (c) $E(|u|^2)E(|w|^2) \leq |E(uw)|^2$ a.e.

Moreover, if $S(E(u)) = X = G$, then (a), (b), (c) and (d) are mutually equivalent, where

- (d) $W_{w,u}$ is an A -class operator.

PROOF. This is a direct consequence of Theorems 2.6 and 2.8 of [4] and Theorem 2.8. $\square$

Theorem 2.10. *Let $W_{w,u}$ be bounded on $L^2(\Sigma)$. Then,*

- (i) *If $W_{w,u}$ is absolute- k -paranormal, we have*

$$\begin{aligned} & (|E(uw)|^2 (E(|u|^2))^{k-1} \cdot \chi_S(E(|w|^2))^k |E(u)|^2 \\ & \quad - (k+1)\lambda^k E(|w|^2) |E(u)|^2 + k\lambda^{k+1} \geq 0, \end{aligned}$$

for all $\lambda > 0$.

- (ii) *If $|E(uw)|^2 (E(|u|^2))^{k-1} \cdot \chi_S(E(|w|^2))^k - (k+1)\lambda^k E(|w|^2) |E(u)|^2 \geq 0$ for all $\lambda > 0$, then $W_{w,u}$ is absolute- k -paranormal.*

PROOF. (i) By methods similar to those used in the proofs of the last theorems, we have

$$\begin{aligned} W_{w,u}^* |W_{w,u}|^{2k} W_{w,u} &= M_{(E(|u|^2))^{k-1} \chi_S(E(|w|^2))^k |E(uw)|^2} W_{\bar{u},u}, \\ W_{w,u}^* W_{w,u} &= M_{E(|w|^2)} W_{\bar{u},u}, \end{aligned}$$

If $W_{w,u}$ is absolute- k -paranormal, then for all $f \in L^2(\mathcal{A})$

$$\begin{aligned} 0 &\leq \langle M_{(E(|u|^2))^{k-1}\chi_S(E(|w|^2))^k|E(uw)|^2} W_{\bar{u},u} f - (k+1)\lambda^k M_{E(|w|^2)} W_{\bar{u},u} f + k\lambda^{k+1} f, f \rangle \\ &= \int_X ((E(|u|^2))^{k-1}\chi_S(E(|w|^2))^k|E(uw)|^2|E(u)|^2 \\ &\quad - (k+1)\lambda^k E(|w|^2)|E(u)|^2 + k\lambda^{k+1})|f|^2 d\mu. \end{aligned}$$

So we get that

$$(E(|u|^2))^{k-1}\chi_S(E(|w|^2))^k|E(uw)|^2|E(u)|^2 - (k+1)\lambda^k E(|w|^2)|E(u)|^2 + k\lambda^{k+1} \geq 0.$$

(ii) The operator $W_{w,u}$ is absolute- k -paranormal if for all $f \in L^2(\Sigma)$

$$\begin{aligned} 0 &\leq W_{w,u}^* |W_{w,u}|^{2k} W_{w,u} - (k+1)\lambda^k |W_{w,u}|^2 + k\lambda^{k+1} I \\ &= \int_X ((E(|u|^2))^{k-1}\chi_S(E(|w|^2))^k|E(uw)|^2 \\ &\quad - (k+1)\lambda^k E(|w|^2)|E(u)|^2 + k\lambda^{k+1})|f|^2 d\mu. \end{aligned}$$

This implies that if $|E(uw)|^2(E(|u|^2))^{k-1}\chi_S(E(|w|^2))^k - (k+1)\lambda^k E(|w|^2) \geq 0$, then $W_{w,u}$ is absolute- k -paranormal. $\square$

Corollary 2.11. *Let $u \in L^0(\mathcal{A})$ and $W_{w,u}$ be bounded on $L^2(\Sigma)$. Then $W_{w,u}$ is absolute- k -paranormal if and only if*

$$(|uE(w)|^2|u|^{2k-2}\chi_S(E(|w|^2))^k|u|^2 - (k+1)\lambda^k E(|w|^2)|u|^2 + k\lambda^{k+1}) \geq 0,$$

for all $\lambda > 0$.

PROOF. Since $|E(f)|^2 \leq E(|f|^2)$ for every $f \in L^2(\Sigma)$, then by similar method of Theorem 2.10 we get the proof. $\square$

Theorem 2.12. *Let $W_{w,u}$ be bounded on $L^2(\Sigma)$. Then $W_{w,u}$ is (n, k) -quasi- $*$ -paranormal if and only if $\alpha_\phi \geq 0$ for all $\phi > 0$, where*

$$\begin{aligned} \alpha_k &= |E(uw)|^{2(n+k)} E(|w|^2) - (1+n)\phi^n |E(uw)|^{2(k-1)} \chi_{S_0}(E(|w|^2))^2 E(|u|^2) \\ &\quad + \phi^{1+n} |E(uw)|^{2(k-1)} \chi_{S_0} E(|w|^2), \end{aligned}$$

and $S_0 = S(E(uw))$.

PROOF. By similar methods of last theorems we have

$$\begin{aligned} W_{w,u}^{*k} W_{w,u}^{*(1+n)} W_{w,u}^{1+n} W_{w,u}^k &= M_{|E(uw)|^{2(n+k)} E(|w|^2)} W_{\bar{u}u}, \\ W_{w,u}^{*k} W_{w,u} W_{w,u}^* W_{w,u}^k &= M_{|E(uw)|^{2(k-1)} (E(|w|^2))^2 E(|u|^2)} W_{\bar{u}u}, \\ W_{w,u}^{*k} W_{w,u}^k &= M_{|E(uw)|^{2(k-1)} E(|w|^2)} W_{\bar{u}u}. \end{aligned}$$

This implies that $W_{w,u}$ is (n, k) -quasi- $*$ -paranormal if and only if

$$\begin{aligned} (M_{|E(uw)|^{2(n+k)} E(|w|^2)} - (1+n)\phi^n M_{|E(uw)|^{2(k-1)} (E(|w|^2))^2 E(|u|^2)} \\ + n\phi^{n+1} M_{|E(uw)|^{2(k-1)} E(|w|^2)}) W_{\bar{u}u} \geq 0. \end{aligned}$$

This inequality holds if and only if

$$\begin{aligned} M_{|E(uw)|^{2(n+k)} E(|w|^2)} - (1+n)\phi^n M_{|E(uw)|^{2(k-1)} (E(|w|^2))^2 E(|u|^2)} \\ + n\phi^{n+1} M_{|E(uw)|^{2(k-1)} E(|w|^2)} \geq 0. \end{aligned}$$

We have the same remark here as in where we have used the fact that $T_1 T_2 \geq 0$ if $T_1 \geq 0$, $T_2 \geq 0$ and $T_1 T_2 = T_2 T_1$ for all $T_i \in \mathcal{B}(\mathcal{H})$ where

$$\begin{aligned} \alpha_\phi |E(uw)|^{2(n+k)} E(|w|^2) - (1+n)\phi^n |E(uw)|^{2(k-1)} \chi_{S_0} (E(|w|^2))^2 E(|u|^2) \\ + n\phi^{n+1} |E(uw)|^{2(k-1)} \chi_{S_0} E(|w|^2). \end{aligned}$$

Therefore the operator $W_{w,u}$ is (n, k) -quasi- $*$ -paranormal if and only if $M_{\alpha_\phi} \geq 0$ if and only if $\alpha_\phi \geq 0$ for all $\phi > 0$.

Recall that an operator $T \in \mathcal{L}(\mathcal{H})$ is said to be n - $*$ -paranormal if $\|T^{n+1}x\|^{\frac{1}{n+1}} \|x\|^{\frac{n}{n+1}} \geq \|T^*x\|$ for all $x \in \mathcal{H}$. Also, T is called k -quasi- $*$ -paranormal if $\|T^2(T^k x)\|^{\frac{1}{2}} \|T^k x\|^{\frac{1}{2}} \geq \|T^*(Tx)\|$ for all $x \in \mathcal{H}$. So, we get the following corollaries. $\square$

Corollary 2.13. *Let $W_{w,u}$ be bounded on $L^2(\Sigma)$. Then $W_{w,u}$ is n - $*$ -paranormal if and only if $\alpha_\phi \geq 0$ for all $\phi > 0$. Same remark as above about ϕ .*

Corollary 2.14. *Let $W_{w,u}$ be bounded on $L^2(\Sigma)$. Then $W_{w,u}$ is k -quasi- $*$ -paranormal if and only if $\alpha_\phi \geq 0$ for all $\phi > 0$. Same remark as above about ϕ .*

3. Some applications

For $T \in \mathcal{L}(\mathcal{H})$, let $\sigma_p(T)$, $\sigma_{jp}(T)$, $\sigma_a(T)$ and $\sigma_{ja}(T)$ denote the point spectrum, joint point spectrum, approximate point spectrum and joint approximate

point spectrum of T . T is called isoloid if every isolated point of $\sigma(T)$ is an eigenvalue of T . Let $\phi \in \mathbb{C}$ be an isolated point of $\sigma(T)$. Then the Riesz idempotent E_ϕ of T with respect to ϕ is defined by

$$E_\phi := \frac{1}{2\pi i} \int_{\partial D_\phi} (\phi I - T)^{-1} d\mu,$$

where D_ϕ is the closed disk centered at ϕ which contains no other points of $\sigma(T)$.

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have single valued extension property at $\lambda_0 \in \mathbb{C}$ (SVEP at λ_0 for brevity) if for every open neighborhood U of λ_0 , the only analytic function $f : U \rightarrow H$ which satisfies the equation $(\lambda I - T)f(\lambda) = 0$ for all $\lambda \in U$ is the constant function $f \equiv 0$. Here we recall some spectral results about $M_w E M_u$. Then we get some conclusions for quasi- $*$ -paranormal and

Theorem 3.1 ([4]). *Let $W_{w,u} : L^2(\Sigma) \rightarrow L^2(\Sigma)$ be bounded. Then*

- (a) $\sigma(W_{w,u}) \setminus \{0\} = \text{ess range}(E(uw)) \setminus \{0\}$.
- (b) *If $S \cap G = X$, then*

$$\sigma(W_{w,u}) = \text{ess range}(E(uw)),$$

where $S = S(E(|u|^2))$ and $G = S(E(|w|^2))$.

- (c) $\sigma_p(W_{w,u}) \setminus \{0\} = \{\lambda \in \mathbb{C} \setminus \{0\} : \mu(A_{\lambda,w}) > 0\},$

where $A_{\lambda,w} = \{x \in X : E(uw)(x) = \lambda\}$.

Proposition 3.2. *If $|E(uw)|^2 \geq E(|u|^2)E(|w|^2)$, then*

$$\sigma_p(W_{w,u}) = \sigma_{jp}(W_{w,u}).$$

PROOF. This is a direct consequence of theorems 2.8 and 3.4 of [4]. □

Corollary 3.3. *If $W_{w,u}$ is quasi- $*$ -paranormal and $G = X$ or is an A -class operator and $S(E(u)) = X$ or is a quasi- $*$ - A -class operator, then $\sigma_p(W_{w,u}) = \sigma_{jp}(W_{w,u})$.*

PROOF. This is a direct consequence of Proposition 3.1, Theorems 2.8 and 2.9. □

Proposition 3.4. *Let $|E(uw)|^2 \geq E(|u|^2)E(|w|^2)$ on G . Then*

- (a) *Every non-zero isolated point of $\text{ess range}(E(uw))$ is a simple pole of the resolvent of $W_{w,u}$.*

- (b) If ϕ is a non-zero isolated point of $\text{ess range}(E(uw))$ and E_ϕ is the Riesz idempotent of $W_{w,u}$ with respect to ϕ . Then E_ϕ is self-adjoint if and only if $N(W_{w,u} - \phi) \subseteq N(W_{\bar{u},\bar{w}} - \bar{\phi})$.

PROOF. By using the results of [10] and Theorem 2.8 we get the proof. $\square$

The next corollary is a direct consequence of Theorem 2.12 and the results of [16].

Corollary 3.5. Let $W_{w,u}$ be a bounded operator on $L^2(\Sigma)$. If $\alpha_\phi \geq 0$ for all $\phi > 0$, where

$$\begin{aligned} \alpha_k = & |E(uw)|^{2(n+k)} E(|w|^2) - (1+n)\phi^n |E(uw)|^{2(k-1)} \chi_{S_0}(E(|w|^2))^2 E(|u|^2) \\ & + \phi^{1+n} |E(uw)|^{2(k-1)} \chi_{S_0} E(|w|^2), \end{aligned}$$

and $S_0 = S(E(uw))$, then

- (a) For every $0 \neq \lambda \in \mathbb{C}$ we have

$$\ker(W_{w,u} - \lambda) \subseteq \ker(W_{\bar{u},\bar{w}} - \bar{\lambda}).$$

- (b) $\sigma_{jp}(W_{w,u}) \setminus \{0\} = \sigma_p(W_{w,u}) \setminus \{0\}$,

and

$$\sigma_{ja}(W_{w,u}) \setminus \{0\} = \sigma_a(W_{w,u}) \setminus \{0\}.$$

- (c) If $\lambda \neq \phi$, then

$$\ker(W_{w,u} - \lambda) \perp \ker(W_{w,u} - \phi).$$

- (d) For every $0 \neq \lambda \in \mathbb{C}$ we have

$$\ker(W_{w,u} - \lambda) \ker(W_{w,u} - \lambda)^2,$$

and

$$\ker((W_{w,u})^{k+1}) = \ker((W_{w,u})^{k+2}).$$

- (e) The operator $W_{w,u}$ has SVEP.

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YOUSSEF ESTAREMI
 DEPARTMENT OF MATHEMATICS
 PAYAME NOOR UNIVERSITY
 P.O. BOX: 19395-3697, TEHRAN
 IRAN
 E-mail: yestaremi@pnu.ac.ir

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