

Functional equations in the spectral theory of random fields II.

By K. LAJKÓ (Debrecen)

1. Introduction

Let S_∞ denote the unit sphere in l^2 , that is, $S_\infty = \{t \mid t \in l^2; t = (t_1, \dots, t_n, \dots); \sum_{k=1}^\infty t_k^2 = 1\}$. A random field $X(t)$ on S_∞ is called isotropic in the wide sense if $MX^2(t) < +\infty$, or $MX(t)$ is independent of t , and if $MX(t)X(s) = B(\cos \Theta)$ depends on $\cos \Theta = \sum_{k=1}^\infty t_k s_k$ ($t = (t_k)$, $s = (s_k)$).

Next let us consider Markov random fields. Let $L(t_1^0)$ be the set of vectors from S_∞ , whose first component equals t_1^0 . A random field $X(t)$ on S_∞ is called Markov if for any $-1 \leq t_1^0 \leq 1$ and any $t', t'', t'_1 > t_1^0 > t''_1$, the random variables $X(t')$, $X(t'')$ are conditionally independent given $X(t)$ on $L(t_1^0)$.

The following theorem is known.

Theorem (see [3]). *The correlation function of a Gaussian isotropic random field of the Markov type on S_∞ is given by*

$$B(\cos \Theta) = (\cos \Theta)^m,$$

where m is an integer.

The proof based on the fact that, in this case, the correlation function B satisfies the functional equation

$$(1) \quad B(\cos \Theta'') B(\cos^2 \Theta) = B(\cos \Theta'' \cos \Theta) B(\cos \Theta),$$

$$\Theta'' > \Theta; \Theta'', \Theta \in [0, \pi],$$

and differentiable in $(0, 1)$.

The general solution of this functional equation is presented here.

2. The non zero solutions of (1)

Let $\mathbb{R}$ be the set of real numbers and let $\mathbb{R}_0 = \mathbb{R} \setminus \{0\}$.

Lemma 1. *If the function $B : [-1, 1] \rightarrow \mathbb{R}$ satisfies the functional equation (1) then B satisfies the functional equation*

$$(2) \quad B(t^2) B(s) = B(ts) B(t); \quad t > s; \quad t, s \in [-1, 1].$$

PROOF. By the transformation

$$(3) \quad s = \cos \Theta'', \quad t = \cos \Theta \quad (\Theta'' > \Theta; \quad \Theta'', \Theta \in [0, \pi]),$$

it follows immediately from (1) that the functional equation (3) is valid on the (t, s) domain of transformation (3), i.e., for all $(t, s) \in T = \{(t, s) \mid t > s; \quad t, s \in [-1, 1]\}$.

Before stating the main theorem of this section, we will prove the following lemma.

Lemma 2. *If the function $B : (0, 1) \rightarrow \mathbb{R}_0$ satisfies the functional equation*

$$(2') \quad B(t^2) B(s) = B(ts) B(t); \quad t > s, \quad t, s \in (0, 1),$$

then there exists an additive function $A : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$(4) \quad B(t) = C \exp[A(\ln t)], \quad t \in (0, 1),$$

where $C \in \mathbb{R}_0$ is an arbitrary constant.

PROOF. By the transformation

$$(5) \quad t = e^{-\frac{u}{2}}, \quad s = e^{-v} \quad (t > s, \quad t, s \in (0, 1)),$$

we obtain from (2') that the function

$$(6) \quad f : \mathbb{R}_+ \rightarrow \mathbb{R}_0, \quad f(u) = B(\exp(-u)),$$

satisfies the functional equation

$$(7) \quad f(u) f(v) = f\left(\frac{u}{2} + v\right) f\left(\frac{u}{2}\right) \quad \left(v > \frac{u}{2} > 0\right).$$

Using Lemma 3 in [2], it follows that the function

$$(8) \quad h : (-1, \infty) \rightarrow \mathbb{R}, \quad h(x) = \ln \frac{f(x+1)}{f(1)},$$

satisfies the functional equation

$$(9) \quad h(x+y) = h(x) + h(y) \quad (x \in (0, 1), \quad y > x-1),$$

i.e., h is additive on the open connected domain $\{(x, y) \mid x \in (0, 1), y > x - 1\}$.

Applying now Lemma 4 in [2] we get that there exists an additive function $\bar{A} : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$(10) \quad h(x) = \bar{A}(x), \quad x \in (-1, \infty).$$

From (6), (8) and (10) we obtain the form

$$(11) \quad B(t) = f(1) \exp[\bar{A}(-1)] \exp[\bar{A}(-\ln t)], \quad t \in (0, 1),$$

for the function B , where $f(1)$, $\bar{A}(-1)$ are arbitrary real constants. Finally we get from (11) the form (4) for B , where $C = f(1) \exp[\bar{A}(-1)] \in \mathbb{R}_0$ is arbitrary constant and the additive function A is defined by $A(x) = \bar{A}(-x)$ ($x \in \mathbb{R}$).

Theorem 1. *If the function $B : [-1, 1] \rightarrow \mathbb{R}$ satisfies the functional equation (1) and $B(t) \neq 0$ ($t \in (0, 1)$), then either*

$$(12) \quad B(t) = \begin{cases} 0 & t = 0 \\ c_1 \exp[A(\ln t)] & t \in (0, 1) \\ c_2 & t = 1 \\ c_3 \exp[A(\ln |t|)] & t \in [-1, 0), \end{cases}$$

or

$$(13) \quad B(t) = \begin{cases} c_3 & t \in [-1, 0] \\ c_1 & t \in (0, 1) \\ c_2 & t = 1, \end{cases}$$

where $c_1, c_3 \in \mathbb{R}_0$, $c_2 \in \mathbb{R}$ are arbitrary constants and $A : \mathbb{R} \rightarrow \mathbb{R}$ is an additive function on $\mathbb{R}^2$.

PROOF. If the function B satisfies the functional equation (1) and $B \neq 0$ on $(0, 1)$, then B satisfies (2) and (2') too. Thus, by Lemma 2,

$$(14) \quad B(t) = c_1 \exp[A(\ln(t))] \quad t \in (0, 1),$$

where $c_1 \in \mathbb{R}_0$ is an arbitrary constant and $A : \mathbb{R} \rightarrow \mathbb{R}$ is an additive function on $\mathbb{R}^2$.

Substitute $s = -1$ in (2), then we get

$$(15) \quad B(t^2) B(-1) = B(-t) B(t), \quad t \in (-1, 1].$$

Since $B(t) \neq 0$ for $t \in (0, 1)$, we obtain from (15) that

$$(16) \quad B(t) = B(-1) \frac{B(t^2)}{B(-t)}, \quad t \in (-1, 0).$$

If $t \in (-1, 0)$, then $-t, t^2 \in (0, 1)$ and (16) with (14) implies

$$(17) \quad B(t) = B(-1) \exp[A(\ln |t|)], \quad t \in (-1, 0).$$

Let us write in (2) $t = 1$, then we get the identity

$$B(1) B(s) = B(s) B(1), \quad s \in [-1, 1).$$

On the other hand if $t, s \in (-1, 1)$ then $t^2, ts \in (-1, 1)$. Thus the value of B on $[-1, 1)$ does not depend on $B(1)$ and so

$$(18) \quad B(1) = c_2 \quad (c_2 \in \mathbb{R} \text{ arbitrary}),$$

Put in (2) $t = 0$, then we get

$$(19) \quad B(0) B(s) = B^2(0), \quad s \in [-1, 0).$$

If $B(0) = 0$, then, according to (14), (17) and (18), we obtain the form (12) for B , where $c_3 = B(-1) \in \mathbb{R}_0$ is an arbitrary constant.

If $c_3 = B(0) \neq 0$, then (19) implies that

$$(20) \quad B(s) = c_3, \quad s \in [-1, 0].$$

But in this case (17) shows that $A = 0$ on $(0, 1)$ and so

$$B(t) = c_1, \quad t \in (0, 1),$$

which together with (18) and (20) implies the form (13) for B .

It is easy to see that the functions (12) and (13) satisfy the functional equation (1).

3. The not almost everywhere zero solutions of (1)

Lemma 3. *If the function $B : [-1, 1] \rightarrow \mathbb{R}$ satisfies the functional equation (1) and there exists a subset $E \subset (0, 1)$ of positive Lebesgue-measure, such that $B(t) \neq 0$ for all $t \in E$, then $B(t) \neq 0$ for all $t \in (0, 1)$.*

PROOF. Under the conditions of the Lemma we get that B satisfies the functional equation (2'). Let us write in (2') $\sqrt{t}$ for t , then we get the functional equation

$$(21) \quad B(t) B(s) = B(s\sqrt{t}) B(\sqrt{t}),$$

$$(t, s) \in D = \{(t, s) \mid \sqrt{t} > s; t, s \in (0, 1)\}.$$

Suppose $B(t) \neq 0$ for $t \in E$, where E has positive measure, then there exists a compact subset $E_1 \subset E$ of positive measure such that $E_1 \subset [\alpha, \beta] \subset (0, 1)$ for some closed interval $[\alpha, \beta]$. Further there exists a natural number n such that $\sqrt[n]{\alpha} > \beta$. It is obvious that one of the intervals

$(\alpha, \sqrt{\alpha}), \dots, (\sqrt[n-2]{\alpha}, \sqrt[n-1]{\alpha})$ contains a subset $E_2 \subset E_1$ of positive measure with $E_2 \times E_2 \subset D$ and then $B(t)B(s) \neq 0$ for $(t, s) \in E_2 \times E_2$.

Thus, by equation (21), $B(s\sqrt{t})B(\sqrt{t}) \neq 0$ if $t, s \in E_2$, which implies that $B(u) \neq 0$, whenever $u \in E_2\sqrt{E_2}$.

Since E_2 and $\sqrt{E_2}$ have positive Lebesgue-measure, by a theorem of Steinhaus (see [1]), the set $E_2\sqrt{E_2}$ contains an interval of positive length $[a, b] \subset (0, 1)$ and so $B(u) \neq 0$ ($u \in [a, b]$).

Substitute $s = t$ in (21) (this is possible because of $\sqrt{t} > t$ for $t \in (0, 1)$), then we get

$$(22) \quad B^2(t) = B(t^{3/2})B(\sqrt{t}), \quad t \in (0, 1),$$

which implies $B(t^{3/2}) \neq 0$, $B(\sqrt{t}) \neq 0$ for all $t \in [a, b]$. Thus $B(u) \neq 0$ if $u \in [a^{3/2}, \sqrt{b}]$. It follows by induction that

$$(23) \quad B(u) \neq 0, \quad u \in \left[a^{(\frac{3}{2})^n}, b^{(\frac{1}{2})^n} \right], \quad n \in N,$$

where N is the set of natural numbers.

Since $a^{(\frac{3}{2})^n} \rightarrow 0$ and $b^{(\frac{1}{2})^n} \rightarrow 1$, we get from (23) that $B(u) \neq 0$ for all $u \in (0, 1)$.

This completes the proof of Lemma 3.

Now it is easy to prove our main result.

Theorem 2. *If the function $B : [-1, 1] \rightarrow \mathbb{R}$ satisfies the functional equation (1) and there exists a subset $E \subset (0, 1)$ of positive Lebesgue-measure such that $B(t) \neq 0$ for all $t \in E$, then B has the form (12) or (13), where $c_1, c_3 \in \mathbb{R}_0$, $c_2 \in \mathbb{R}$ arbitrary constants and $A : \mathbb{R} \rightarrow \mathbb{R}$ additive function on $\mathbb{R}^2$.*

PROOF. By Lemma 3, $B(t) \neq 0$ for all $t \in (0, 1)$. Then Theorem 1 implies the statement of our Theorem 2.

4. Remarks

Remark 1. It is not too difficult to determine the continuous solution of (1). We have the following

Theorem 3. *A function $B : [-1, 1] \rightarrow \mathbb{R}$ is a continuous solution of equation (1) if and only if either B is a constant function, or f has the form*

$$(24) \quad B(t) = \begin{cases} c_1 t^a & t \in [0, 1] \\ c_3 t^a & t \in [-1, 0), \end{cases}$$

where $c_1, c_3 \in \mathbb{R}_0$ and $a \in \mathbb{R}_+ = \{x \mid x \in \mathbb{R}, x > 0\}$ is an arbitrary constants.

PROOF. If B is continuous in $[-1, 1]$ and there exists a value $t_0 \in (0, 1)$, such that $B(t_0) \neq 0$, then there exists an interval $I = [t_0 - r, t_0 + r] \subset (0, 1)$, such that $B(t) \neq 0$ for all $t \in I$ and $I \times I \subset D$.

Then, using the last part of the proof of Lemma 3, it follows that $B(t) \neq 0$ for all $t \in (0, 1)$.

Since B is continuous, functions f, h and A in Lemma 2 are continuous, thus $A(t) = at$ with some $a \in \mathbb{R}$.

Then we get from (12) and (13) that either

$$(12') \quad B(t) = \begin{cases} 0 & t = 0 \\ c_1 t^a & t \in (0, 1) \\ c_2 & t = 1 \\ c_3 |t|^a & t \in [-1, 0) \end{cases}$$

or

$$(13') \quad B(t) = \begin{cases} c_3 & t \in [-1, 0] \\ c_1 & t \in (0, 1) \\ c_2 & t = 1 \end{cases}$$

respectively.

Using again the continuity of B , $B = c$ results from (13') and from (12') for $a = 0$.

If $a < 0$, then B given by (12') satisfies $\lim_{x \rightarrow 0+0} B(t) = \infty$ and thus B cannot be continuous at zero.

If $a > 0$, then (24) follows from (12').

Remark 2. A similar calculation shows, that if B is continuous on the set $[-1, 1] \setminus \{0\}$, then either

$$B(t) = \begin{cases} 0 & t = 0 \\ c_1 t^a & t \in (0, 1] \\ c_3 |t|^a & t \in [-1, 0), \end{cases}$$

or

$$B(t) = \begin{cases} c_3 & t \in [-1, 0] \\ c_1 & t \in (0, 1], \end{cases}$$

where $a \in \mathbb{R}$, $c_1, c_3 \in \mathbb{R}_0$ arbitrary constants.

Remark 3. For measurable solutions of (1) the situation is more complicated. For example the function (12') and (13') are measurable, satisfy equation (1) but usually are not continuous on the interval $[-1, 1]$.

References

- [1] E. HEWITT and K. A. ROSS, Abstract harmonic analysis, Vol. I, *Academic Press, New York*, 1963.
- [2] K. LAJKÓ, Functional equations in the spectral theory of random fields I., *Publ. Math. Debrecen* **44** (1994), 395–399.
- [3] M. I. YADRENKO, Spectral Theory of Random Fields, Opt. Software, *INC., Publications Division, New York*, 1983.

KÁROLY LAJKÓ
LAJOS KOSSUTH UNIVERSITY
INSTITUTE OF MATHEMATICS AND INFORMATICS
4010 DEBRECEN P.O.B. 12.

(Received March 28, 1994)