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Applications of exact structures in abelian categories

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Abstract. In an abelian category $\mathscr{A}$ with small Ext groups, we show that there exists a one-to-one correspondence between any two of the following: balanced pairs, subfunctors $\mathcal{F}$ of $\mathrm{Ext}^1_{\mathscr{A}}(-,-)$ such that $\mathscr{A}$ has enough $\mathcal{F}$-projectives and enough $\mathcal{F}$-injectives and Quillen exact structures $\mathcal{E}$ with enough $\mathcal{E}$-projectives and enough $\mathcal{E}$-injectives. In this case, we get a strengthened version of the translation of the Wakamatsu lemma to the exact context, and also prove that subcategories which are $\mathcal{E}$-resolving and epimorphic precovering with kernels in their right $\mathcal{E}$-orthogonal class and subcategories which are $\mathcal{E}$-coresolving and monomorphic preenveloping with cokernels in their left $\mathcal{E}$-orthogonal class are determined by each other. Then we apply these results to construct some (pre)enveloping and (pre)covering classes and complete hereditary $\mathcal{E}$-cotorsion pairs in the module category.

1. Introduction

Throughout this paper, $\mathscr{A}$ is an abelian category and a subcategory of $\mathscr{A}$ means a full and additive subcategory closed under isomorphisms and direct summands.

The notion of exact categories was originally due to Quillen [17]. The theory of exact categories was developed by Bühler, Fu, Gillespie, Hovey, Keller, Krause, Neeman, Šťovíček and possibly others, see [4], [8], [10], [13], [14], [15], [16], [20], [21], and so on. In addition, Auslander and Solberg

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developed in [1], [2] the theory of relative homological algebra with respect to a subfunctor $\mathcal{F}$ of $\mathrm{Ext}^1_{\mathscr{A}}(-,-) : \mathscr{A}^{op} \times \mathscr{A} \to \mathrm{Ab}$, where Ab is the category of abelian groups and $\mathscr{A} = \mathrm{mod}\,\Lambda$ is the category of finitely generated modules over an artin algebra Λ. Then, as a special case, CHEN introduced and studied in [5] relative homology with respect to balanced pairs in an abelian category. On the other hand, the notion of cotorsion pairs was introduced by SALCE in [19], which is based on the functor $\mathrm{Ext}^1_{\mathscr{A}}(-,-)$. The theory of cotorsion pairs, studied by many authors, has played an important role in homological algebra and representation theory of algebras, see [7], [8], [9], [11], [13], [15], [20], [21] and references therein. Motivated by these, in this paper, for an abelian category $\mathscr{A}$ with small Ext groups, we first investigate the relations among exact structures, subfunctors of $\mathrm{Ext}^1_{\mathscr{A}}(-,-)$ and balanced pairs in $\mathscr{A}$, then we study cotorsion pairs with respect to exact structures in $\mathscr{A}$. The paper is organized as follows.

In Section 2, we recall the definitions of exact categories, balanced pairs and some notions in relative homological algebra. We prove that there exists a one-to-one correspondence between any two of the following: (1) Balanced pairs $(\mathscr{C},\mathscr{D})$ in $\mathscr{A}$; (2) Subfunctors $\mathcal{F} \subseteq \mathrm{Ext}^1_{\mathscr{A}}(-,-)$ such that $\mathscr{A}$ has enough $\mathcal{F}$-projectives and enough $\mathcal{F}$-injectives; (3) Quillen exact structures $\mathcal{E}$ in $\mathscr{A}$ with enough $\mathcal{E}$-projectives and enough $\mathcal{E}$-injectives.

Let $\mathcal{E}$ be an exact structure on $\mathscr{A}$ such that $\mathscr{A}$ has enough $\mathcal{E}$-projectives and enough $\mathcal{E}$-injectives. In Section 3, we get a strengthened version of the Wakamatsu lemma in the exact context, and also prove that subcategories which are $\mathcal{E}$-resolving and epimorphic precovering with kernels in their right $\mathcal{E}$-orthogonal class and subcategories which are $\mathcal{E}$-coresolving and monomorphic preenveloping with cokernels in their left $\mathcal{E}$-orthogonal class are determined by each other. As applications, we get some complete hereditary $\mathcal{E}$-cotorsion pairs.

Let R be an associative ring with identity, and let $\mathscr{A} = \mathrm{Mod}\,R$ be the category of right R-modules and $\mathcal{E}$ contain all pure short exact sequences. As applications of results obtained in Section 3, we prove in Section 4 that for any subcategory $\mathcal{X}$ of $\mathrm{Mod}\,R$ in which all modules are pure projective (resp. pure injective), the right (resp. left) orthogonal class with respect to the exact structure $\mathcal{E}$ of $\mathcal{X}$ is preenveloping (resp. covering). Moreover, we construct some complete hereditary $\mathcal{E}$-cotorsion pairs induced by the category of pure injective modules. Some results of GÖBEL and TRLIFAJ in [11] are obtained as corollaries.

2. Exact categories, balanced pairs and relative homology

The following two definitions are cited from [4], see also [17] and [14].

Definition 2.1. Let $\mathscr{B}$ be an additive category. A *kernel-cokernel pair* (i,p) in $\mathscr{B}$ is a pair of composable morphisms

$$L \xrightarrow{i} M \xrightarrow{p} N$$

such that i is a kernel of p and p is a cokernel of i. If a class $\mathcal{E}$ of kernel-cokernel pairs on $\mathscr{B}$ is fixed, an *admissible monic* (sometimes called *inflation*) is a morphism i for which there exists a morphism p such that $(i,p) \in \mathcal{E}$. *Admissible epics* (sometimes called *deflations*) are defined dually.

An *exact structure* on $\mathscr{B}$ is a class $\mathcal{E}$ of kernel-cokernel pairs which is closed under isomorphisms and satisfies the following axioms:

[E0] For any object B in $\mathscr{B}$, the identity morphism id_B is both an admissible monic and an admissible epic.

[E1] The class of admissible monics is closed under compositions.

[E1^{op}] The class of admissible epics is closed under compositions.

[E2] The push-out of an admissible monic along an arbitrary morphism exists and yields an admissible monic, that is, for any admissible monic $i : L \to M$ and any morphism $f : L \to M'$, there is a push-out diagram

$$\begin{tikzcd} L \arrow[r, "i"] \arrow[d, "f"'] & M \arrow[d, "f'"] \\ M' \arrow[r, "i'"] & {} \end{tikzcd}$$

with i' an admissible monic.

[E2^{op}] The pull-back of an admissible epic along an arbitrary morphism exists and yields an admissible epic, that is, for any admissible epic $p : M \to N$ and any morphism $g : M' \to N$, there is a pull-back diagram

$$\begin{tikzcd} X \arrow[r, "p'"] \arrow[d, "G'"'] & M' \arrow[d, "g"] \\ M \arrow[r, "p"] & N \end{tikzcd}$$

with p' an admissible epic.

An *exact category* is a pair $(\mathscr{B}, \mathcal{E})$ consisting of an additive category $\mathscr{B}$ and an exact structure $\mathcal{E}$ on $\mathscr{B}$. Elements of $\mathcal{E}$ are called *admissible short exact sequences* (or *conflations*).

Definition 2.2. Let $(\mathscr{B}, \mathcal{E})$ be an exact category.

(1) An object $P \in \mathscr{B}$ is an ($\mathcal{E}$-)*projective object* if for any admissible epic $p : M \to N$ and any morphism $f : P \to N$ there exists $f' : P \to M$ such that $f = pf'$; an object $I \in \mathscr{B}$ is an ($\mathcal{E}$-)*injective object* if for any admissible monic $i : L \to M$ and any morphism $g : L \to I$ there exists $g' : M \to I$ such that $g = g'i$.

(2) $(\mathscr{B}, \mathcal{E})$ is said to *have enough projective objects* if for any object $M \in \mathscr{B}$ there exists an admissible epic $p : P \to M$ with P a projective object of $\mathscr{B}$; $(\mathscr{B}, \mathcal{E})$ is said to *have enough injective objects* if for any object $M \in \mathscr{B}$ there exists an admissible monic $i : M \to I$ with I an injective object of $\mathscr{B}$.

We have the following standard observation.

Lemma 2.1. *Let $(\mathscr{B}, \mathcal{E})$ be an exact category with enough projective objects and enough injective objects, and let*

$$0 \to X \to Y \to Z \to 0 \tag{2.1}$$

be a sequence of morphisms in $\mathscr{B}$. Then the following statements are equivalent.

(1) (2.1) *is a conflation.*

(2) *For any projective object P of $\mathscr{B}$, the induced sequence of abelian groups*

$$0 \to \mathrm{Hom}_{\mathscr{B}}(P, X) \to \mathrm{Hom}_{\mathscr{B}}(P, Y) \to \mathrm{Hom}_{\mathscr{B}}(P, Z) \to 0$$

is exact.

(3) *For any injective object I of $\mathscr{B}$, the induced sequence of abelian groups*

$$0 \to \mathrm{Hom}_{\mathscr{B}}(Z, I) \to \mathrm{Hom}_{\mathscr{B}}(Y, I) \to \mathrm{Hom}_{\mathscr{B}}(X, I) \to 0$$

is exact.

Proof. The implications (1) $\Longrightarrow$ (2) and (1) $\Longrightarrow$ (3) are trivial. We just need to prove the implication (2) $\Longrightarrow$ (1) since the implication (3) $\Longrightarrow$ (1) is its dual.

Let

$$0 \to X \xrightarrow{f_2} Y \xrightarrow{f_1} Z \to 0$$

be a sequence in $\mathscr{B}$. One easily proves that f_2 is a monomorphism and that $f_1 f_2 = 0$. We claim that f_2 is a kernel of f_1. In fact, for any $K \in \mathscr{B}$, there exists an exact sequence

$$P_1 \to P_0 \to K \to 0$$

with P_0, P_1 projective in $\mathscr{B}$. By (2) and the snake lemma, we obtain the following commutative diagram with exact columns and rows:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r, dashed] & \mathrm{Hom}_{\mathscr{B}}(K,X) \arrow[r, dashed, "\mathrm{Hom}_{\mathscr{B}}(K,f_2)"] \arrow[d] & \mathrm{Hom}_{\mathscr{B}}(K,Y) \arrow[r, dashed, "\mathrm{Hom}_{\mathscr{B}}(K,f_1)"] \arrow[d] & \mathrm{Hom}_{\mathscr{B}}(K,Z) \arrow[d] & \\
0 \arrow[r] & \mathrm{Hom}_{\mathscr{B}}(P_0,X) \arrow[r, "\mathrm{Hom}_{\mathscr{B}}(P_0,f_2)"] \arrow[d] & \mathrm{Hom}_{\mathscr{B}}(P_0,Y) \arrow[r, "\mathrm{Hom}_{\mathscr{B}}(P_0,f_1)"] \arrow[d] & \mathrm{Hom}_{\mathscr{B}}(P_0,Z) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Hom}_{\mathscr{B}}(P_1,X) \arrow[r, "\mathrm{Hom}_{\mathscr{B}}(P_1,f_2)"] & \mathrm{Hom}_{\mathscr{B}}(P_1,Y) \arrow[r, "\mathrm{Hom}_{\mathscr{B}}(P_1,f_1)"] & \mathrm{Hom}_{\mathscr{B}}(P_1,Z) \arrow[r] & 0.
\end{tikzcd}$$

For any $u : K \to Y$, if $f_1 u = 0$, then

$$\mathrm{Hom}_{\mathscr{B}}(K, f_1)(u) = \mathrm{Hom}_{\mathscr{B}}(K, f_1 u) = 0$$

and $u \in \operatorname{Ker} \mathrm{Hom}_{\mathscr{B}}(K, f_1) = \operatorname{Im} \mathrm{Hom}_{\mathscr{B}}(K, f_2)$. So there exists a unique morphism $v : K \to X$ such that $u = \mathrm{Hom}_{\mathscr{B}}(K, f_2)(v) = f_2 v$. Thus f_2 is a kernel of f_1.

Again by (2), there exists a deflation $\varphi : P_2 \to Z$ with P_2 projective in $\mathscr{B}$ such that $\varphi = f_1 s$ for some $s : P_2 \to Y$. Then f_1 is a deflation by [4, Proposition 2.16]. So

$$0 \to X \to Y \to Z \to 0$$

is a conflation, as required. □

Definition 2.3 ([1]). Let $\mathscr{A}$ be an abelian category with small Ext groups (that is, $\mathrm{Ext}^1_{\mathscr{A}}(X, Y)$ is a set for any $X, Y \in \mathscr{A}$). For a subfunctor $\mathcal{F} \subseteq \mathrm{Ext}^1_{\mathscr{A}}(-,-)$, an object C (resp. D) in $\mathscr{A}$ is called *$\mathcal{F}$-projective* (resp. *$\mathcal{F}$-injective*) if $\mathcal{F}(C,-) = 0$ (resp. $\mathcal{F}(-,D) = 0$). An exact sequence

$$0 \to A' \to A \to A'' \to 0$$

in $\mathscr{A}$ is called an *$\mathcal{F}$-sequence* if it is an element of $\mathcal{F}(A'', A')$. The category $\mathscr{A}$ is said to *have enough $\mathcal{F}$-projective objects* if for any $A \in \mathscr{A}$, there exists an $\mathcal{F}$-sequence

$$0 \to X \to C \to A \to 0$$

such that C is $\mathcal{F}$-projective; and $\mathscr{A}$ is said to *have enough $\mathcal{F}$-injective objects* if for any $A \in \mathscr{A}$, there exists an $\mathcal{F}$-sequence

$$0 \to A \to D \to Y \to 0$$

such that D is $\mathcal{F}$-injective.

Definition 2.4 ([6]). Let $\mathscr{C}$ be a subcategory of $\mathscr{A}$. A morphism $f : C \to D$ in $\mathscr{A}$ with $C \in \mathscr{C}$ is called a *$\mathscr{C}$-precover* of D if for any morphism $g : C' \to D$ in $\mathscr{A}$ with $C' \in \mathscr{C}$, there exists a morphism $h : C' \to C$ such that $g = fh$. The morphism $f : C \to D$ is called *right minimal* if an endomorphism $h : C \to C$ is an automorphism whenever $f = fh$. A *$\mathscr{C}$-precover* is called a *$\mathscr{C}$-cover* if it is right minimal; $\mathscr{C}$ is called a *(pre)covering subcategory* of $\mathscr{A}$ if every object in $\mathscr{A}$ has a $\mathscr{C}$-(pre)cover; $\mathscr{C}$ is called an *epimorphic (pre)covering subcategory* of $\mathscr{A}$ if every object in $\mathscr{A}$ has an epimorphic $\mathscr{C}$-(pre)cover. Dually, the notions of a *$\mathscr{C}$-(pre)envelope*, a *(pre)enveloping subcategory* and a *monomorphic (pre)enveloping subcategory* are defined.

Let $\mathscr{C}$ be a subcategory of $\mathscr{A}$. Recall that a sequence in $\mathscr{A}$ is called $\mathrm{Hom}_{\mathscr{A}}(\mathscr{C}, -)$*-exact* if it is exact after applying the functor $\mathrm{Hom}_{\mathscr{A}}(C, -)$ for any object $C \in \mathscr{C}$. Let $M \in \mathscr{A}$. An exact sequence (of finite or infinite length)

$$\cdots \xrightarrow{f_{n+1}} C_n \xrightarrow{f_n} \cdots \xrightarrow{f_2} C_1 \xrightarrow{f_1} C_0 \xrightarrow{f_0} M \to 0$$

in $\mathscr{A}$ with all $C_i \in \mathscr{C}$ is called a *$\mathscr{C}$-resolution* of M if it is $\mathrm{Hom}_{\mathscr{A}}(\mathscr{C}, -)$-exact, that is, each f_i is an epimorphic $\mathscr{C}$-precover of $\mathrm{Im}\, f_i$. We denote sometimes the $\mathscr{C}$-resolution of M by $\mathscr{C}^\bullet \to M$, where

$$\mathscr{C}^\bullet := \cdots \to C_2 \xrightarrow{f_2} C_1 \xrightarrow{f_1} C_0 \to 0$$

is the *deleted $\mathscr{C}$-resolution* of M. Note that by a version of the comparison theorem, the $\mathscr{C}$-resolution is unique up to homotopy ([7, p. 169]). Dually, the notions of a $\mathrm{Hom}_{\mathscr{A}}(-, \mathscr{C})$-exact sequence and a $\mathscr{C}$-coresolution are defined.

Definition 2.5 ([5]). A pair $(\mathscr{C}, \mathscr{D})$ of additive subcategories in $\mathscr{A}$ is called a *balanced pair* if the following conditions are satisfied.

(1) $\mathscr{C}$ is epimorphic precovering and $\mathscr{D}$ is monomorphic preenveloping.

(2) For any $M \in \mathscr{A}$, there is a $\mathscr{C}$-resolution $\mathscr{C}^\bullet \to M$ such that it is $\mathrm{Hom}_{\mathscr{A}}(-, \mathscr{D})$-exact.

(3) For any $N \in \mathscr{A}$, there is a $\mathscr{D}$-coresolution $N \to \mathscr{D}^\bullet$ such that it is $\mathrm{Hom}_{\mathscr{A}}(\mathscr{C}, -)$-exact.

Remark 2.1. By [5, Proposition 2.2], in Definition 2.5, keeping condition (1) unaltered, the conditions (2) and (3) can be replaced by a common condition $(2' + 3')$: A short exact sequence

$$0 \to Y \to Z \to X \to 0$$

in $\mathscr{A}$ is $\mathrm{Hom}_{\mathscr{A}}(\mathscr{C}, -)$-exact if and only if it is $\mathrm{Hom}_{\mathscr{A}}(-, \mathscr{D})$-exact.

Some examples of balanced pairs are as follows.

Example 2.1. Let R be an associative ring with identity and $\operatorname{Mod} R$ the category of right R-modules.

(1) $(\mathcal{P}_0, \mathcal{I}_0)$ is the standard balanced pair in $\operatorname{Mod} R$, where $\mathcal{P}_0$ and $\mathcal{I}_0$ are the subcategories of $\operatorname{Mod} R$ consisting of projective modules and injective modules respectively.

(2) $(\mathcal{PP}, \mathcal{PI})$ is a balanced pair in $\operatorname{Mod} R$ ([7, Example 8.3.2]), where $\mathcal{PP}$ and $\mathcal{PI}$ are the subcategories of $\operatorname{Mod} R$ consisting of pure projective modules and pure injective modules respectively.

(3) If R is a Gorenstein ring, then $(\mathcal{GP}, \mathcal{GI})$ is a balanced pair in $\operatorname{Mod} R$ ([7, Theorem 12.1.4]), where $\mathcal{GP}$ and $\mathcal{GI}$ are the subcategories of $\operatorname{Mod} R$ consisting of Gorenstein projective modules and Gorenstein injective modules respectively.

(4) Let $\mathscr{A}$ be an abelian category with enough projective and injective objects. If both $(\mathcal{B}, \mathcal{C})$ and $(\mathcal{C}, \mathcal{D})$ are complete hereditary classical cotorsion pairs in $\mathscr{A}$, then $(\mathcal{B}, \mathcal{D})$ is a balanced pair in $\mathscr{A}$ ([5, Proposition 2.6]).

The main result in this section is the following

Theorem 2.2. *Let $\mathscr{A}$ be an abelian category with small* Ext *groups. Then there exists a one-to-one correspondence between any two of the following.*

(1) *Balanced pairs $(\mathscr{C}, \mathscr{D})$ in $\mathscr{A}$.*

(2) *Subfunctors $\mathcal{F} \subseteq \operatorname{Ext}^1_{\mathscr{A}}(-,-)$ such that $\mathscr{A}$ has enough $\mathcal{F}$-projectives and enough $\mathcal{F}$-injectives.*

(3) *Quillen exact structures $\mathcal{E}$ in $\mathscr{A}$ with enough $\mathcal{E}$-projectives and enough $\mathcal{E}$-injectives (that is, such that the resulting exact category $(\mathscr{A}, \mathcal{E})$ has enough projective and enough injective objects).*

Proof. We first show that there exists a one-to-one correspondence between (1) and (2). Let $(\mathscr{C}, \mathscr{D})$ be a balanced pair in $\mathscr{A}$. If we define $\mathcal{F}(X, Y)$ as the subset of $\operatorname{Ext}^1_{\mathscr{A}}(X, Y)$ consisting of the equivalence classes of short exact sequences of condition $(2' + 3')$ in Remark 2.1, then the assignment $(X, Y) \longmapsto \mathcal{F}(X, Y)$ is the definition on objects of a subfunctor of $\operatorname{Ext}^1_{\mathscr{A}}(-,-)$, and $\{\mathcal{F}\text{-projectives}\} = \mathscr{C}$, $\{\mathcal{F}\text{-injectives}\} = \mathscr{D}$. The converse is easy if we put $(\mathscr{C}, \mathscr{D}) = (\{\mathcal{F}\text{-projectives}\}, \{\mathcal{F}\text{-injectives}\})$.

Next we show that there exists a one-to-one correspondence between (2) and (3). Let $\mathcal{F}$ be a subfunctor of $\operatorname{Ext}^1_{\mathscr{A}}(-,-)$ such that $\mathscr{A}$ has enough $\mathcal{F}$-projectives and enough $\mathcal{F}$-injectives. Put $\mathcal{E} = \{\mathcal{F}\text{-sequences}\}$, and

$\{\mathcal{E}\text{-projectives}\} = \{\mathcal{F}\text{-projectives}\}$, $\{\mathcal{E}\text{-injectives}\} = \{\mathcal{F}\text{-injectives}\}$. We claim that $(\mathscr{A}, \mathcal{E})$ is an exact category with enough projective and enough injective objects.

Clearly, $\mathcal{E}$ contains $\{0 \to A \xrightarrow{1_A} A \to 0 \to 0\}$ and $\{0 \to 0 \to A \xrightarrow{1_A} A \to 0\}$ for any $A \in \mathscr{A}$.

For [E1], let

$$0 \to A_1 \xrightarrow{f} A_2 \to L \to 0 \quad \text{and} \quad 0 \to A_2 \xrightarrow{g} A_3 \to K \to 0$$

be $\mathcal{F}$-sequences in $\mathscr{A}$. We will prove

$$0 \to A_1 \xrightarrow{gf} A_3 \to \operatorname{Coker} gf \to 0$$

is an $\mathcal{F}$-sequence. In fact, for any $\mathcal{F}$-injective object I of $\mathscr{A}$ and $\alpha : A_1 \to I$, by [1, Proposition 1.5] there exists $\beta : A_2 \to I$ such that $\alpha = \beta f$. Consider the following commutative diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
0 \arrow[r] & A_1 \arrow[r, "f"] \arrow[d, equal] & A_2 \arrow[r] \arrow[d, "g"] & L \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_1 \arrow[r, "gf"] \arrow[d, "\alpha"] & A_3 \arrow[r] \arrow[d] \arrow[dl, dashed, "\theta"] & \operatorname{Coker} gf \arrow[r] & 0 \\
 & I & K \arrow[d] & & \\
 & & 0. & &
\end{tikzcd}$$

Then there exists $\theta : A_3 \to I$ such that $\beta = \theta g$, that is, $\alpha = \beta f = \theta(gf)$. So

$$0 \to A_1 \xrightarrow{gf} A_3 \to \operatorname{Coker} gf \to 0$$

is an $\mathcal{F}$-sequence by [1, Proposition 1.5].

For [E2], take an $\mathcal{F}$-sequence

$$0 \to L \xrightarrow{i} M \to N \to 0.$$

For any morphism $f : L \to M'$, consider the following pushout diagram:

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "i"] \arrow[d, "f"'] & M \arrow[r] \arrow[d] & N \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & M' \arrow[r, "i'"] & X \arrow[r] & N \arrow[r] & 0. \end{tikzcd}$$

By [1, Proposition 1.5], for any $\mathcal{F}$-injective object I of $\mathscr{A}$ and $s : M' \to I$, there exists $t : M \to I$ such that $sf = ti$. It follows from the universal property of pushouts, there exists $h : X \to I$ such that $s = hi'$. So

$$0 \to M' \xrightarrow{i'} X \to N \to 0$$

is an $\mathcal{F}$-sequence by [1, Proposition 1.5]. Dually [E1^{op}] and [E2^{op}] follow.

Conversely, for any $X, Y \in \mathscr{A}$, we define $\mathcal{F}(X,Y)$ as the subset of $\mathrm{Ext}^1_{\mathscr{A}}(X,Y)$ consisting of the equivalence classes of admissible short exact sequences. Let $X', Y' \in \mathscr{A}$, $f : Y \to Y'$ and $g : X' \to X$. Then by [E2] and [E2^{op}], we get the following two commutative diagrams:

$$\begin{tikzcd} \mathcal{F}(X,Y) \arrow[r, "\text{inc.}"] \arrow[d, "\mathcal{F}(X,f)"] & \mathrm{Ext}^1_{\mathscr{A}}(X,Y) \arrow[d, "\mathrm{Ext}^1_{\mathscr{A}}(X,f)"] \\ \mathcal{F}(X,Y') \arrow[r, "\text{inc.}"] & \mathrm{Ext}^1_{\mathscr{A}}(X,Y') \end{tikzcd} \quad , \text{ and } \quad \begin{tikzcd} \mathcal{F}(X,Y) \arrow[r, "\text{inc.}"] \arrow[d, "\mathcal{F}(g,Y)"] & \mathrm{Ext}^1_{\mathscr{A}}(X,Y) \arrow[d, "\mathrm{Ext}^1_{\mathscr{A}}(g,Y)"] \\ \mathcal{F}(X',Y) \arrow[r, "\text{inc.}"] & \mathrm{Ext}^1_{\mathscr{A}}(X',Y). \end{tikzcd}$$

So $\mathcal{F}$ is a subfunctor of $\mathrm{Ext}^1_{\mathscr{A}}(-,-)$ such that $\mathscr{A}$ has enough $\mathcal{F}$-projectives and enough $\mathcal{F}$-injectives. □

3. $\mathcal{E}$-cotorsion pairs

In this section, a pair $(\mathscr{A}, \mathcal{E})$ means that $\mathscr{A}$ is an abelian category with small Ext groups and $\mathcal{E}$ is an exact structure on $\mathscr{A}$ such that $\mathscr{A}$ has enough $\mathcal{E}$-projectives and enough $\mathcal{E}$-injectives. We first give a strengthened version of the Wakamatsu lemma in the exact context, and then apply it to obtain complete hereditary $\mathcal{E}$-cotorsion pairs in $(\mathscr{A}, \mathcal{E})$. For any $M, N \in \mathscr{A}$ and $i \geq 1$, we use $\mathcal{E}\mathrm{xt}^i_{\mathscr{A}}(M,N)$ to denote the i-th cohomology group by taking $\mathcal{E}$-projective resolution of M or taking $\mathcal{E}$-injective coresolution of N.

Inspired by [20], we give the following

Definition 3.1. Let $(\mathscr{A}, \mathcal{E})$ be a pair as above.

(1) A pair $(\mathscr{X}, \mathscr{Y})$ of full subcategories of $\mathscr{A}$ is called an *$\mathcal{E}$-cotorsion pair* provided that $\mathscr{X} = {}^{\perp_*}\mathscr{Y}$ and $\mathscr{Y} = \mathscr{X}^{\perp_*}$, where

$$\mathscr{X}^{\perp_*} = \{N \in \mathscr{A} \mid \mathcal{E}\mathrm{xt}^1_{\mathscr{A}}(M, N) = 0 \text{ for any } M \in \mathscr{X}\}, \text{ and}$$

$$^{\perp_*}\mathscr{Y} = \{M \in \mathscr{A} \mid \mathcal{E}\mathrm{xt}^1_{\mathscr{A}}(M, N) = 0 \text{ for any } N \in \mathscr{Y}\}.$$

(2) An $\mathcal{E}$-cotorsion pair $(\mathscr{X}, \mathscr{Y})$ is called *complete* if for any $M \in \mathscr{A}$, there exist two conflations of the form:

$$0 \to Y \to X \to M \to 0, \quad \text{and} \quad 0 \to M \to Y' \to X' \to 0$$

with $X, X' \in \mathscr{X}$ and $Y, Y' \in \mathscr{Y}$.

(3) A subcategory $\mathscr{T}$ of $\mathscr{A}$ is called *closed under $\mathcal{E}$-extensions* if the end terms in a conflation are in $\mathscr{T}$ implies that the middle term is also in $\mathscr{T}$. The subcategory $\mathscr{T}$ is called *$\mathcal{E}$-resolving* if it contains all $\mathcal{E}$-projectives of $\mathscr{A}$, closed under $\mathcal{E}$-extensions, and for any conflation

$$0 \to X \to Y \to Z \to 0$$

with $Y, Z \in \mathscr{T}$, we have $X \in \mathscr{T}$. Dually, the notion of *$\mathcal{E}$-coresolving subcategories* is defined.

(4) An $\mathcal{E}$-cotorsion pair $(\mathscr{X}, \mathscr{Y})$ is called *hereditary* if $\mathscr{X}$ is $\mathcal{E}$-resolving.

Remark 3.1. (1) By [11, Lemma 2.2.10], an $\mathcal{E}$-cotorsion pair $(\mathscr{X}, \mathscr{Y})$ is hereditary if and only if $\mathscr{Y}$ is $\mathcal{E}$-coresolving.

(2) If $\mathcal{E}$ is the abelian structure, then all the notions in Definition 3.1 coincide with the classical ones in [5, p. 6].

(3) $\{\mathcal{E}\text{-projectives}\} = {}^{\perp_*}\mathscr{A}$ and $\{\mathcal{E}\text{-injectives}\} = \mathscr{A}^{\perp_*}$.

The following result is a strengthened version of the translation of the Wakamatsu lemma to the exact context.

Theorem 3.1. *Let $(\mathscr{A}, \mathcal{E})$ be a pair as above, $\mathscr{X}$ a full subcategory of $\mathscr{A}$ closed under $\mathcal{E}$-extensions and $A \in \mathscr{A}$.*

(1) *If $f\colon X \to A$ is an epimorphic $\mathscr{X}$-cover, then* $\operatorname{Ker} f \in \mathscr{X}^{\perp_*}$.

(2) *If $f\colon A \to X$ is a monomorphic $\mathscr{X}$-envelope, then* $\operatorname{Coker} f \in {}^{\perp_*}\mathscr{X}$.

Proof. We only prove (2), and (1) is dual to (2).

By assumption, there exists an exact sequence

$$0 \to A \xrightarrow{f} X \xrightarrow{g} \operatorname{Coker} f \to 0$$

in $\mathscr{A}$ with $f : A \to X$ a monomorphic $\mathscr{X}$-envelope. To prove $\operatorname{Coker} f \in {}^{\perp_*}\mathscr{X}$, it suffices to prove that for any $X' \in \mathscr{X}$, any conflation

$$0 \to X' \to M \xrightarrow{h} \operatorname{Coker} f \to 0$$

in $\mathscr{A}$ splits. Consider the pullback of g and h:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & X' \arrow[r, equal] \arrow[d] & X' \arrow[d] & \\
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & Y \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A \arrow[r, "f"] & X \arrow[r, "g"] \arrow[d] & \operatorname{Coker} f \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0. &
\end{tikzcd}$$

Then the middle column is a conflation. Since $X', X \in \mathscr{X}$ and $\mathscr{X}$ is closed under $\mathcal{E}$-extensions, we have $Y \in \mathscr{X}$. Because $f : A \to X$ is an $\mathscr{X}$-envelope of A, we obtain the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, equal] & X \arrow[r] \arrow[d] & \operatorname{Coker} f \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & Y \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A \arrow[r, "f"] & X \arrow[r] & \operatorname{Coker} f \arrow[r] & 0.
\end{tikzcd}$$

Since f is left minimal, we have that the composition

$$X \to Y \to X$$

is an isomorphism. This implies that the composition

$$\operatorname{Coker} f \to M \xrightarrow{h} \operatorname{Coker} f$$

is also an isomorphism. Therefore the conflation

$$0 \to X' \to M \xrightarrow{h} \operatorname{Coker} f \to 0$$

splits and the assertion follows. $\square$

Theorem 3.2. *Let $(\mathscr{A},\mathcal{E})$ be a pair as above and $(\mathscr{X},\mathscr{Y})$ a pair of full subcategories of $\mathscr{A}$. Then the following statements are equivalent.*

(1) *$\mathscr{X}$ is $\mathcal{E}$-resolving, $\mathscr{Y} = \mathscr{X}^{\perp_*}$ and each $A \in \mathscr{A}$ admits an epimorphic $\mathscr{X}$-precover $p\colon X \to A$ such that $\operatorname{Ker} p \in \mathscr{Y}$.*

(2) *$\mathscr{Y}$ is $\mathcal{E}$-coresolving, $\mathscr{X} = {}^{\perp_*}\mathscr{Y}$ and each $A \in \mathscr{A}$ admits a monomorphic $\mathscr{Y}$-preenvelope $j\colon A \to Y$ such that $\operatorname{Coker} j \in \mathscr{X}$.*

(3) *$(\mathscr{X},\mathscr{Y})$ is a complete hereditary $\mathcal{E}$-cotorsion pair.*

Proof. (2) $\Longrightarrow$ (1) First observe that $\mathscr{X}(= {}^{\perp_*}\mathscr{Y})$ is closed under direct summands, $\mathcal{E}$-extensions and contains all $\mathcal{E}$-projective objects of $\mathscr{A}$. For any conflation

$$0 \to A \to B \to C \to 0$$

in $\mathscr{A}$ with $B, C \in \mathscr{X}$, we have an exact sequence

$$0 = \mathcal{E}\mathrm{xt}^1_{\mathscr{A}}(B,Y) \to \mathcal{E}\mathrm{xt}^1_{\mathscr{A}}(A,Y) \to \mathcal{E}\mathrm{xt}^2_{\mathscr{A}}(C,Y)$$

for any $Y \in \mathscr{Y}$. Take a conflation

$$0 \to Y \to E \to L \to 0$$

in $\mathscr{A}$ with E $\mathcal{E}$-injective. We have $L \in \mathscr{Y}$ since $\mathscr{Y}$ is $\mathcal{E}$-coresolving. Then we have an exact sequence:

$$0 = \mathcal{E}\mathrm{xt}^1_{\mathscr{A}}(C,E) \to \mathcal{E}\mathrm{xt}^1_{\mathscr{A}}(C,L) \to \mathcal{E}\mathrm{xt}^2_{\mathscr{A}}(C,Y) \to \mathcal{E}\mathrm{xt}^2_{\mathscr{A}}(C,E) = 0.$$

Notice that $\mathcal{E}\mathrm{xt}^1_{\mathscr{A}}(C,L) = 0$, so $\mathcal{E}\mathrm{xt}^2_{\mathscr{A}}(C,Y) = 0$, and hence $\mathcal{E}\mathrm{xt}^1_{\mathscr{A}}(A,Y) = 0$, that is, $A \in \mathscr{X}$. Consequently we conclude that $\mathscr{X}$ is $\mathcal{E}$-resolving.

Next for each $A \in \mathscr{A}$, there exists a conflation

$$0 \to K \to P \to A \to 0$$

in $\mathscr{A}$ with P $\mathcal{E}$-projective. By (2) and Lemma 2.1, we have a conflation

$$0 \to K \xrightarrow{j} Y \to M \to 0$$

with $Y \in \mathscr{Y}$ and $M \in \mathscr{X}$. Consider the following pushout diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & K \arrow[r] \arrow[d, "j"] & P \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Y \arrow[r] \arrow[d] & X \arrow[r, "p"] \arrow[d] & A \arrow[r] & 0 \\
 & M \arrow[r, equal] \arrow[d] & M \arrow[d] & & \\
 & 0 & 0. & &
\end{tikzcd}$$

Since $\mathscr{X}$ is $\mathcal{E}$-resolving and $P \in \mathscr{X}$, it follows from the middle column in the above diagram that $X \in \mathscr{X}$. Obviously, the middle row

$$0 \to Y \to X \xrightarrow{p} A \to 0$$

is a conflation and $Y \in \mathscr{Y}$. Therefore we deduce that $p\colon X \to A$ is an epimorphic $\mathscr{X}$-precover, as required.

It remains to prove that $\mathscr{Y} = \mathscr{X}^{\perp_*}$. Observe that $\mathscr{Y} \subseteq ({}^{\perp_*}\mathscr{Y})^{\perp_*} = \mathscr{X}^{\perp_*}$. By (2) and Lemma 2.1, for any $A' \in \mathscr{X}^{\perp_*} = ({}^{\perp_*}\mathscr{Y})^{\perp_*}$, there exists a conflation

$$0 \to A' \to Y' \to X' \to 0 \tag{3.1}$$

with $Y' \in \mathscr{Y}$ and $X' \in \mathscr{X} = {}^{\perp_*}\mathscr{Y}$. So (3.1) splits. Thus $A' \in \mathscr{Y}$ since $\mathscr{Y}$ is closed under direct summands.

Dually, we get (1) $\Longrightarrow$ (2).

The implications (1)+(2) $\Longrightarrow$ (3) and (1) $\Longleftarrow$ (3) $\Longrightarrow$ (2) are clear. □

As direct consequences of Theorems 3.1 and 3.2, we have the following

Corollary 3.1. (1) *If $\mathscr{X}$ is an epi-covering $\mathcal{E}$-resolving subcategory of $\mathscr{A}$, then each $A \in \mathscr{A}$ admits a monomorphic $\mathscr{Y}$-preenvelope $j\colon A \to Y$ with* $\operatorname{Coker} j \in \mathscr{X}$, *where $\mathscr{Y} = \mathscr{X}^{\perp_*}$.*

(2) *If $\mathscr{Y}$ is a mono-enveloping $\mathcal{E}$-coresolving subcategory of $\mathscr{A}$, then each $A \in \mathscr{A}$ admits an epimorphic $\mathscr{X}$-precover $p\colon X \to A$ with* $\operatorname{Ker} p \in \mathscr{Y}$, *where $\mathscr{X} = {}^{\perp_*}\mathscr{Y}$.*

Corollary 3.2. *Let $(\mathscr{X},\mathscr{Y})$ be a complete hereditary (classical) cotorsion pair in $\mathscr{A}$. Then we have*

(1) *If $\mathscr{X}$ contains the $\mathcal{E}$-projective objects of $\mathscr{A}$, then $(\mathscr{X}, \mathscr{X}^{\perp_*})$ is a complete hereditary $\mathcal{E}$-cotorsion pair and each $A \in \mathscr{A}$ admits a monomorphic $\mathscr{X}^{\perp_*}$-preenvelope $j\colon A \to Y$ such that* $\operatorname{Coker} j \in \mathscr{X}$.

(2) *If $\mathscr{Y}$ contains the $\mathcal{E}$-injective objects of $\mathscr{A}$, then $({}^{\perp_*}\mathscr{Y}, \mathscr{Y})$ is a complete hereditary $\mathcal{E}$-cotorsion pair and each $A \in \mathscr{A}$ admits an epimorphic ${}^{\perp_*}\mathscr{Y}$-precover $p\colon X \to A$ such that* $\operatorname{Ker} p \in \mathscr{Y}$.

4. Applications

In this section, we will apply the results obtained in Section 3 to the module category.

Let R be an associative ring with identity and $\operatorname{mod} R$ the category of finitely presented right R-modules. Recall that a short exact sequence

$$\xi : 0 \to A \to B \to C \to 0$$

in $\operatorname{Mod} R$ is called *pure exact* if the induced sequence $\operatorname{Hom}_R(F, \xi)$ is exact for any $F \in \operatorname{mod} R$. In this case A is called a *pure submodule* of B and C is called a *pure quotient module* of B. In addition, modules that are projective (resp. injective) relative to pure exact sequences are called *pure projective* (resp. *pure injective*). The subcategory of $\operatorname{Mod} R$ consisting of pure projective (resp. pure injective) modules is denoted by $\mathcal{PP}$ (resp. $\mathcal{PI}$).

Lemma 4.1 ([18, Corollary 3.5(c)])**.** *Let $\mathscr{F}$ be a subcategory of* $\operatorname{Mod} R$ *closed under pure submodules. Then $\mathscr{F}$ is preenveloping if and only if $\mathscr{F}$ is closed under direct products.*

Lemma 4.2 ([12, Theorem 2.5])**.** *Let $\mathscr{F}$ be a subcategory of* $\operatorname{Mod} R$ *closed under pure quotient modules. Then the following statements are equivalent.*

(1) *$\mathscr{F}$ is closed under direct sums.*

(2) *$\mathscr{F}$ is precovering.*

(3) *$\mathscr{F}$ is covering.*

Lemma 4.3. *Let $(\mathscr{A}, \mathcal{E})$ be a pair as in Section 3 and $A \in \mathscr{A}$. Then the following statements are equivalent.*

(1) $\operatorname{Ext}^1_{\mathscr{A}}(-, A)$ *vanishes on all $\mathcal{E}$-projective objects of $\mathscr{A}$.*

(2) *Any short exact sequence*

$$0 \to A \to Z \to V \to 0$$

is a conflation.

PROOF. (2) $\Longrightarrow$ (1) is trivial.

(1) $\Longrightarrow$ (2) For any $\mathcal{E}$-projective object P of $\mathscr{A}$ and $f : P \to V$, consider the pullback diagram:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, equal] & N \arrow[r, "i"] \arrow[d, "\alpha"] & P \arrow[r] \arrow[d, "f"] & 0 \\ 0 \arrow[r] & A \arrow[r] & Z \arrow[r, "g"] & V \arrow[r] & 0. \end{tikzcd}$$

By (1), there exists $j : P \to N$ such that $ij = 1_P$. Then we have $f = fij = g\alpha j$. So by Lemma 2.1, the short exact sequence

$$0 \to A \to Z \to V \to 0$$

is a conflation. $\square$

Recall from [3] that a subcategory of $\operatorname{Mod} R$ is called *definable* if it is closed under arbitrary direct products, direct limits, and pure submodules.

Theorem 4.4. *Let $\mathcal{E}$ be an exact structure on $\operatorname{Mod} R$ with enough $\mathcal{E}$-projectives and enough $\mathcal{E}$-injectives such that $\mathcal{E}$ contains all pure short exact sequences. If $\mathscr{X} \subseteq \mathcal{PP}$ and $\mathscr{Y} \subseteq \mathcal{PI}$ are two subcategories, then we have the following*

(1) *$\mathscr{X}^{\perp_*}$ is preenveloping. If moreover, each $X \in \mathscr{X}$ admits an $\mathcal{E}$-presention*

$$C_2 \to C_1 \to C_0 \to X \to 0,$$

where the C_i are $\mathcal{E}$-projective in $\operatorname{Mod} R$ and finitely presented, then $\mathscr{X}^{\perp_*}$ is also covering.

(2) *${}^{\perp_*}\mathscr{Y}$ is covering and closed under pure quotients.*

(3) *If $\mathscr{Y}$ is closed under taking $\mathcal{E}$-cosyzygies, then we have*
   (i) *$({}^{\perp_*}\mathscr{Y}, ({}^{\perp_*}\mathscr{Y})^{\perp_*})$ is a complete hereditary $\mathcal{E}$-cotorsion pair.*
   (ii) *If $\operatorname{Ext}^1_R(-, Y)$ vanishes on $\mathcal{E}$-projective objects of $\operatorname{Mod} R$ for any $Y \in \mathscr{Y}$, then $({}^{\perp_1}\mathscr{Y}, ({}^{\perp_1}\mathscr{Y})^{\perp_*})$ is a complete hereditary $\mathcal{E}$-cotorsion pair.*

PROOF. (1) First observe that $\mathscr{X}^{\perp_*}$ is closed under direct products. Let

$$0 \to A \to B \to C \to 0$$

be a pure exact sequence in $\operatorname{Mod} R$ with $B \in \mathscr{X}^{\perp_*}$. Then it is a conflation by assumption. For any $X \in \mathscr{X}$, we have the following exact sequence:

$$\operatorname{Hom}_R(X, B) \xrightarrow{f} \operatorname{Hom}_R(X, C) \to \mathcal{E}\mathrm{xt}^1_{\mathscr{A}}(X, A) \to \mathcal{E}\mathrm{xt}^1_{\mathscr{A}}(X, B) = 0.$$

Notice that f is epic, so $\mathcal{E}\mathrm{xt}^1_{\mathscr{A}}(X, A) = 0$ and $A \in \mathscr{X}^{\perp_*}$. Thus $\mathscr{X}^{\perp_*}$ is closed under pure submodules. It follows from Lemma 4.1 that $\mathscr{X}^{\perp_*}$ is preenveloping.

It is easy to see that $\mathscr{X}^{\perp_*}$ is closed under direct limits by assumption; in particular $\mathscr{X}^{\perp_*}$ is closed under direct sums. Then $\mathscr{X}^{\perp_*}$ is definable. So $\mathscr{X}^{\perp_*}$ is closed under pure quotient modules by [3, Proposition 4.3(3)], and hence $\mathscr{X}^{\perp_*}$ is covering by Lemma 4.2.

(2) Obviously, ${}^{\perp_*}\mathscr{Y}$ is closed under direct sums. By Lemma 4.2, it suffices to show that ${}^{\perp_*}\mathscr{Y}$ is closed under pure quotient modules. Let

$$0 \to A \to B \to C \to 0$$

be a pure exact sequence in $\operatorname{Mod} R$ with $B \in {}^{\perp_*}\mathscr{Y}$. Then it is a conflation by assumption. By using an argument similar to that in the proof of (1), we get $C \in {}^{\perp_*}\mathscr{Y}$.

(3) (i) Because ${}^{\perp_*}\mathscr{Y}$ contains all projective modules, we have that ${}^{\perp_*}\mathscr{Y}$ is epimorphic covering by (2). We claim that ${}^{\perp_*}\mathscr{Y}$ is $\mathcal{E}$-resolving. This will complete the proof of (i) by Theorems 3.1 and 3.2.

Clearly, ${}^{\perp_*}\mathscr{Y}$ contains all $\mathcal{E}$-projective objects of $\operatorname{Mod} R$ and ${}^{\perp_*}\mathscr{Y}$ is closed under direct summands and $\mathcal{E}$-extensions. Now take a conflation

$$0 \to M \to N \to L \to 0$$

in $\operatorname{Mod} R$ with $N, L \in {}^{\perp_*}\mathscr{Y}$. By assumption, we have $\operatorname{Ker} \mathcal{E}\mathrm{xt}^1_{\mathscr{A}}(-, Y) \subseteq \operatorname{Ker} \mathcal{E}\mathrm{xt}^2_{\mathscr{A}}(-, Y)$ for any $Y \in \mathscr{Y}$. Now by the dimension shifting it yields that $M \in {}^{\perp_*}\mathscr{Y}$. So ${}^{\perp_*}\mathscr{Y}$ is $\mathcal{E}$-resolving. The claim follows.

(ii) By Lemma 4.3, we have ${}^{\perp_1}\mathscr{Y} = {}^{\perp_*}\mathscr{Y}$. So, as a particular case of (i), the assertion follows. $\square$

Recall that a classical cotorsion pair $(\mathscr{X}, \mathscr{Y})$ in $\operatorname{Mod} R$ is *perfect* if $\mathscr{X}$ is covering and $\mathscr{Y}$ is enveloping. As a consequence of Theorem 4.4, we have the following

Corollary 4.1. *If $\mathscr{X} \subseteq \mathcal{PP}$ and $\mathscr{Y} \subseteq \mathcal{PI}$ are two subcategories in* $\operatorname{Mod} R$*, then we have*

(1) *$\mathscr{X}^{\perp_1}$ is preenveloping. If moreover, R is a right coherent ring and $\mathscr{X}$ is a subcategory of* $\operatorname{mod} R$*, then $\mathscr{X}^{\perp_1}$ is also covering.*

(2) *${}^{\perp_1}\mathscr{Y}$ is covering and closed under pure quotients.*

(3) *([11, Theorem 3.2.9]) $({}^{\perp_1}\mathscr{Y}, ({}^{\perp_1}\mathscr{Y})^{\perp_1})$ is a perfect hereditary cotorsion pair.*

Proof. The assertions (1) and (2) follow directly from Theorem 4.4, and the assertion (3) follows from Theorem 4.4 and [7, Theorem 7.2.6]. □

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