

Incomplete poly-Bernoulli numbers associated with incomplete Stirling numbers

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Abstract. By using the associated and restricted Stirling numbers of the second kind, we give some generalizations of the poly-Bernoulli numbers. We also study their analytic and combinatorial properties. As an application, at the end of the paper we present a new infinite series representation of the Riemann zeta function via the Lambert W .

1. Introduction

Let $\mu \geq 1$ be an integer in the whole text. Our goal is to generalize the following relation for the poly-Bernoulli numbers $B_n^{(\mu)}$ ([Kan, Theorem 1]):

$$B_n^{(\mu)} = \sum_{k=0}^n (-1)^{n-k} \frac{k!}{(k+1)^\mu} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \quad (n \geq 0, \mu \geq 1),$$

where $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$ are the Stirling numbers of the second kind, determined by

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = \frac{1}{k!} \sum_{j=0}^k (-1)^i \binom{k}{j} (k-j)^n$$

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(see e.g., [Jor]). When $\mu = 1$, $B_n^{(1)}$ are the classical Bernoulli numbers, defined by the generating function

$$\frac{x}{1 - e^{-x}} = \sum_{n=0}^{\infty} B_n^{(1)} \frac{x^n}{n!}. \quad (1)$$

Notice that the classical Bernoulli numbers B_n are also defined by the generating function

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!},$$

satisfying $B_n^{(1)} = B_n$ ($n \neq 1$) with $B_1^{(1)} = 1/2 = -B_1$.

The generating function of the poly-Bernoulli numbers $B_n^{(\mu)}$ is given by

$$\frac{\text{Li}_{\mu}(1 - e^{-x})}{1 - e^{-x}} = \sum_{n=0}^{\infty} B_n^{(\mu)} \frac{x^n}{n!}, \quad (2)$$

where

$$\text{Li}_{\mu}(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^{\mu}}$$

is the μ -th polylogarithm function ([Kan, (1)]). The generating function of the poly-Bernoulli numbers can also be written in terms of iterated integrals ([Kan, (2)]):

$$e^x \cdot \underbrace{\frac{1}{e^x - 1} \int_0^x \frac{1}{e^x - 1} \int_0^x \cdots \frac{1}{e^x - 1} \int_0^x}_{\mu-1} \frac{x}{e^x - 1} \underbrace{dx dx \cdots dx}_{\mu-1} = \sum_{n=0}^{\infty} B_n^{(\mu)} \frac{x^n}{n!}. \quad (3)$$

Several generalizations of the poly-Bernoulli numbers have been considered ([BayHam1], [BayHam2], [CopCan], [Jol], [Sas]). However, most kinds of generalizations are based upon the generating functions of (1) and/or (2). On the contrary, our generalizations are based upon the explicit formula in terms of the Stirling numbers. In [KomMezSza], a similar approach is used to generalize the Cauchy numbers c_n , defined by $x/\log(1+x) = \sum_{n=0}^{\infty} c_n x^n/n!$. In this paper, by using the associated and restricted Stirling numbers of the second kind, we give substantial generalizations of the poly-Bernoulli numbers. One of the main results is to generalize the formula in (2) as

$$\sum_{n=0}^{\infty} B_{n, \leq m}^{(\mu)} \frac{t^n}{n!} = \frac{\text{Li}_{\mu}(1 - E_m(-t))}{1 - E_m(-t)}$$

and

$$\sum_{n=0}^{\infty} B_{n, \geq m}^{(\mu)} \frac{t^n}{n!} = \frac{\text{Li}_{\mu}(E_{m-1}(-t) - e^{-t})}{E_{m-1}(-t) - e^{-t}},$$

where $E_m(t) = \sum_{k=0}^m \frac{t^k}{k!}$. See Theorem 1 below.

2. Incomplete Stirling numbers of the second kind

In place of the classical Stirling numbers of the second kind $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$ we substitute the restricted Stirling numbers and the associated Stirling numbers

$$\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}_{\leq m} \quad \text{and} \quad \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}_{\geq m},$$

respectively. Some combinatorial and modular properties of these numbers can be found in [Mez], and other properties can be found in the cited papers of [Mez]. The generating functions of these numbers are given by

$$\sum_{n=k}^{mk} \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}_{\leq m} \frac{x^n}{n!} = \frac{1}{k!} (E_m(x) - 1)^k \quad (4)$$

and

$$\sum_{n=mk}^{\infty} \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}_{\geq m} \frac{x^n}{n!} = \frac{1}{k!} (e^x - E_{m-1}(x))^k \quad (5)$$

respectively, where

$$E_m(t) = \sum_{k=0}^m \frac{t^k}{k!}$$

is the m th partial sum of the exponential function sum. These give the number of the k -partitions of an n -element set, such that each block contains at most or at least m elements, respectively. Since the generating function of $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$ is given by

$$\sum_{n=k}^{\infty} \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} \frac{x^n}{n!} = \frac{(e^x - 1)^k}{k!}$$

(see e.g., [Jor]), by $E_{\infty}(x) = e^x$ and $E_0(x) = 1$, we have

$$\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}_{\leq \infty} = \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}_{\geq 1} = \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}.$$

These give the number of the k -partitions of an n -element set, such that each block contains at most or at least m elements, respectively. Notice that these numbers

where $m = 2$ have been considered by several authors (e.g., [Com], [How], [Rio], [Zha]).

It is well-known that the Stirling numbers of the second kind satisfy the recurrence relation:

$$\left\{ \begin{matrix} n+1 \\ k \end{matrix} \right\} = k \left\{ \begin{matrix} n \\ k \end{matrix} \right\} + \left\{ \begin{matrix} n \\ k-1 \end{matrix} \right\} \quad (6)$$

for $k > 0$, with the initial conditions

$$\left\{ \begin{matrix} 0 \\ 0 \end{matrix} \right\} = 1 \quad \text{and} \quad \left\{ \begin{matrix} n \\ 0 \end{matrix} \right\} = \left\{ \begin{matrix} 0 \\ n \end{matrix} \right\} = 0$$

for $n > 0$. The restricted and associated Stirling numbers of the second kind satisfy the similar relations. It is easy to see the initial conditions

$$\begin{aligned} \left\{ \begin{matrix} 0 \\ 0 \end{matrix} \right\}_{\leq m} &= 1 \quad \text{and} \quad \left\{ \begin{matrix} n \\ 0 \end{matrix} \right\}_{\leq m} = \left\{ \begin{matrix} 0 \\ n \end{matrix} \right\}_{\leq m} = 0, \\ \left\{ \begin{matrix} 0 \\ 0 \end{matrix} \right\}_{\geq m} &= 1 \quad \text{and} \quad \left\{ \begin{matrix} n \\ 0 \end{matrix} \right\}_{\geq m} = \left\{ \begin{matrix} 0 \\ n \end{matrix} \right\}_{\geq m} = 0 \end{aligned}$$

for $n > 0$.

Proposition 1. For $k > 0$ we have

$$\left\{ \begin{matrix} n+1 \\ k \end{matrix} \right\}_{\leq m} = \sum_{i=0}^{m-1} \binom{n}{i} \left\{ \begin{matrix} n-i \\ k-1 \end{matrix} \right\}_{\leq m} \quad (7)$$

$$= k \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{\leq m} + \left\{ \begin{matrix} n \\ k-1 \end{matrix} \right\}_{\leq m} - \binom{n}{m} \left\{ \begin{matrix} n-m \\ k-1 \end{matrix} \right\}_{\leq m}, \quad (8)$$

$$\left\{ \begin{matrix} n+1 \\ k \end{matrix} \right\}_{\geq m} = \sum_{i=m-1}^n \binom{n}{i} \left\{ \begin{matrix} n-i \\ k-1 \end{matrix} \right\}_{\geq m} \quad (9)$$

$$= k \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{\geq m} + \binom{n}{m-1} \left\{ \begin{matrix} n-m+1 \\ k-1 \end{matrix} \right\}_{\geq m}. \quad (10)$$

Remark. The fourth relation (10) appeared in a different form in [How]. Since

$$\sum_{i=1}^{n-k+1} \binom{n}{i} \left\{ \begin{matrix} n-i \\ k-1 \end{matrix} \right\} = k \left\{ \begin{matrix} n \\ k \end{matrix} \right\},$$

the relation (7) and the relation (9) are both reduced to the relation (6), if $m \geq n - k + 2$ and if $m = 1$, respectively. It is trivial to see that the relations (8) and (10) are also reduced to the original relation (6), if $m > n$ and if $m = 1$, respectively.

PROOF OF PROPOSITION 1. The combinatorial proofs of the previous theorem are given as follows. We shall give combinatorial proofs. First, identity (7). To construct a partition with k blocks on $n + 1$ element we can do the following. The last element in its block can have i elements by side, where $i = 0, 1, \dots, m - 1$. We have to choose these i elements from n . This can be done in $\binom{n}{i}$ ways. The rest of the elements go into $k - 1$ blocks in $\left\{ \begin{smallmatrix} n-i \\ k-1 \end{smallmatrix} \right\}_{\leq m}$ ways. Summing over the possible values of i we are done.

The proof of (8). The above construction can be described in another way: the last element we put into a singleton and the other n elements must form a partition with $k - 1$ blocks: $\left\{ \begin{smallmatrix} n \\ k-1 \end{smallmatrix} \right\}_{\leq m}$ possibilities. Or we put this element into one existing block after constructing a partition of n elements into k blocks. This offers us $k \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}_{\leq m}$ possibilities, but we must subtract the possibilities when we exceed the block size limit m . This happens if we put the last element into a block of m elements. There are $\binom{n}{m} \left\{ \begin{smallmatrix} n-m \\ k-1 \end{smallmatrix} \right\}_{\leq m}$ such partitions in total. The proof is done.

The proof of (9) and (10) is similar. $\square$

Note that the classical Stirling numbers of the second kind $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$ satisfy the identities:

$$\begin{aligned} \left\{ \begin{smallmatrix} n \\ 1 \end{smallmatrix} \right\} &= \left\{ \begin{smallmatrix} n \\ n \end{smallmatrix} \right\} = 1, & \left\{ \begin{smallmatrix} n \\ n-1 \end{smallmatrix} \right\} &= \binom{n}{2}, \\ \left\{ \begin{smallmatrix} n \\ n-2 \end{smallmatrix} \right\} &= \frac{3n-5}{4} \binom{n}{3}, & \left\{ \begin{smallmatrix} n \\ n-3 \end{smallmatrix} \right\} &= \binom{n}{4} \binom{n-2}{2}, \\ \left\{ \begin{smallmatrix} n \\ 2 \end{smallmatrix} \right\} &= 2^{n-1} - 1, & \left\{ \begin{smallmatrix} n \\ 3 \end{smallmatrix} \right\} &= \frac{3^{n-1}}{2} - 2^{n-1} + \frac{1}{2}, \\ \left\{ \begin{smallmatrix} n \\ 4 \end{smallmatrix} \right\} &= \frac{4^{n-1}}{6} - \frac{3^{n-1}}{2} + 2^{n-2} - \frac{1}{6}. \end{aligned}$$

By the definition (4) or Proposition 1 (7), we list several basic properties about the restricted Stirling numbers of the second kind. Some basic properties about the associated Stirling numbers of the second kind can be found in [Com], [How], [Mez], [Zha].

Lemma 1. For $0 \leq n \leq k - 1$ or $n \geq mk + 1$, we have

$$\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}_{\leq m} = 0. \quad (11)$$

For $k \leq n \leq mk$, we have

$$\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}_{\leq m} = \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} \quad (k \leq n \leq m), \quad (12)$$

$$\left\{ \begin{matrix} n \\ n \end{matrix} \right\}_{\leq m} = 1 \quad (n \geq 0, m \geq 1), \quad (13)$$

$$\left\{ \begin{matrix} n \\ n-1 \end{matrix} \right\}_{\leq m} = \binom{n}{2} \quad (n \geq 2, m \geq 2), \quad (14)$$

$$\left\{ \begin{matrix} n \\ n-2 \end{matrix} \right\}_{\leq m} = \begin{cases} \frac{3n-5}{4} \binom{n}{3} & (n \geq 4, m \geq 3); \\ 3 \binom{n}{4} & (n \geq 4, m = 2), \end{cases} \quad (15)$$

$$\left\{ \begin{matrix} n \\ n-3 \end{matrix} \right\}_{\leq m} = \begin{cases} \binom{n}{4} \binom{n-2}{2} & (n \geq 4, m \geq 4); \\ 15 \binom{n}{6} + 10 \binom{n}{5} & (n \geq 4, m = 3); \\ 15 \binom{n}{6} & (n \geq 4, m = 2), \end{cases} \quad (16)$$

$$\left\{ \begin{matrix} n \\ 1 \end{matrix} \right\}_{\leq m} = 1 \quad (1 \leq n \leq m), \quad (17)$$

$$\left\{ \begin{matrix} n \\ 2 \end{matrix} \right\}_{\leq m} = 2^{n-1} - 1 \quad (2 \leq n \leq m+1). \quad (18)$$

PROOF. Some of the above special values are trivial. Some of them can be proven by analyzing the possible block structures.

We take (15) as a concrete example.

Let $m = 2$, and the number of blocks be $k = n - 2$. Then for the block structure we have the only one possibility

$$\underbrace{. | . | \cdots | . |}_{n-4} \cdots | \cdots$$

That is, there are $n-4$ singletons and two blocks of length 2. There are $\frac{1}{2} \binom{4}{2} \binom{n}{4} = 3 \binom{n}{4}$ such partitions: we have to choose those four elements going to the non singleton blocks in $\binom{n}{4}$ ways. Then we put two of four into the first block and the other two goes to the other: $\binom{4}{2} = 6$ cases. Finally, we have to divide by two because the order of the blocks does not matter. The last case of (15) follows.

If $m = 3$ then we have one more possible distribution of blocks sizes apart from the above:

$$\underbrace{. | . | \cdots | . |}_{n-5} \cdots$$

Into the last block we have $\binom{n}{3}$ possible option to put 3 elements. So if $m = 3$ and $k = n - 2$ then we have $\binom{n}{3} + 3 \binom{n}{4} = \frac{3n-5}{4} \binom{n}{3}$ cases in total.

The rest of the cases can be treated similarly. $\square$

3. Incomplete poly-Bernoulli numbers

3.1. Generating function and its integral representation. By using two types of incomplete Stirling numbers, define *restricted poly-Bernoulli numbers* $B_{n,\leq m}^{(\mu)}$ and *associated poly-Bernoulli numbers* $B_{n,\geq m}^{(\mu)}$ by

$$B_{n,\leq m}^{(\mu)} = \sum_{k=0}^n (-1)^{n-k} \frac{k!}{(k+1)^\mu} \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{\leq m} \quad (n \geq 0), \quad (19)$$

and

$$B_{n,\geq m}^{(\mu)} = \sum_{k=0}^n (-1)^{n-k} \frac{k!}{(k+1)^\mu} \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{\geq m} \quad (n \geq 0), \quad (20)$$

respectively. These numbers can be considered as generalizations of the usual poly-Bernoulli numbers $B_n^{(\mu)}$, since

$$B_{n,\leq \infty}^{(\mu)} = B_{n,\geq 1}^{(\mu)} = B_n^{(\mu)}.$$

We call these numbers as *incomplete poly-Bernoulli numbers*.

One can deduce that these numbers have the generating functions.

Theorem 1. We have

$$\sum_{n=0}^{\infty} B_{n,\leq m}^{(\mu)} \frac{t^n}{n!} = \frac{\text{Li}_\mu(1 - E_m(-t))}{1 - E_m(-t)} \quad (21)$$

and

$$\sum_{n=0}^{\infty} B_{n,\geq m}^{(\mu)} \frac{t^n}{n!} = \frac{\text{Li}_\mu(E_{m-1}(-t) - e^{-t})}{E_{m-1}(-t) - e^{-t}}. \quad (22)$$

Remark. In the first formula $m \rightarrow \infty$ gives back the poly-Bernoulli numbers (2) since $E_\infty(-t) = e^{-t}$ and $\text{Li}_1(z) = -\log(1-z)$, while in the second we must take $m = 1$ since $E_0(-t) = 1$.

PROOF OF THEOREM 1. By the definition of (20) and using (4), we get

$$\begin{aligned} \sum_{n=0}^{\infty} B_{n,\leq m}^{(\mu)} \frac{t^n}{n!} &= \sum_{n=0}^{\infty} \sum_{k=0}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{\leq m} \frac{(-1)^{n-k} k!}{(k+1)^\mu} \frac{t^n}{n!} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k k!}{(k+1)^\mu} \sum_{n=k}^{\infty} \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{\leq m} \frac{(-t)^n}{n!} \end{aligned}$$

$$\begin{aligned} &= \sum_{k=0}^{\infty} \frac{(-1)^k k!}{(k+1)^\mu} \frac{1}{k!} \left((-t) + \frac{(-t)^2}{2!} + \cdots + \frac{(-t)^m}{m!} \right)^k \\ &= \sum_{k=0}^{\infty} \frac{(1 - E_m(-t))^k}{(k+1)^\mu} = \frac{\text{Li}_\mu(1 - E_m(-t))}{1 - E_m(-t)}. \end{aligned}$$

Similarly, by the definition of (19) and using (5), we get

$$\begin{aligned} \sum_{n=0}^{\infty} B_{n, \geq m}^{(\mu)} \frac{t^n}{n!} &= \sum_{n=0}^{\infty} \sum_{k=0}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{\leq m} \frac{(-1)^{n-k} k!}{(k+1)^{\mu}} \frac{t^n}{n!} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k k!}{(k+1)^{\mu}} \sum_{n=k}^{\infty} \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{\geq m} \frac{(-t)^n}{n!} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k k!}{(k+1)^{\mu}} \frac{1}{k!} \left(\frac{(-t)^m}{m!} + \frac{(-t)^{m+1}}{(m+1)!} + \cdots \right)^k \\ &= \sum_{k=0}^{\infty} \frac{(E_{m-1}(-t) - e^{-t})^k}{(k+1)^{\mu}} \\ &= \frac{\text{Li}_{\mu}(E_{m-1}(-t) - e^{-t})}{E_{m-1}(-t) - e^{-t}}. \end{aligned}$$

For $\mu \geq 1$, the generating functions can be written in the form of iterated integrals. We set $E_{-1}(-t) = 0$ for convenience.

Theorem 2.

$$\frac{1}{1 - E_m(-t)} \cdot \underbrace{\int_0^t \frac{E_{m-1}(-t)}{1 - E_m(-t)} \int_0^t \cdots \int_0^t \frac{E_{m-1}(-t)}{1 - E_m(-t)} \int_0^t \frac{E_{m-1}(-t)}{1 - E_m(-t)}}_{\mu-1}$$
$$\times (-\log(E_m(-t))) \underbrace{dt \dots dt}_{\mu-1} = \sum_{n=0}^{\infty} B_{n,\leq m}^{(\mu)} \frac{x^n}{n!}, \quad (23)$$

$$\begin{aligned} & \frac{1}{E_{m-1}(-t) - e^{-t}} \cdot \underbrace{\int_0^t \frac{e^{-t} - E_{m-2}(-t)}{E_{m-1}(-t) - e^{-t}} \int_0^t \cdots \int_0^t \frac{e^{-t} - E_{m-2}(-t)}{E_{m-1}(-t) - e^{-t}} \int_0^t \frac{e^{-t} - E_{m-2}(-t)}{E_{m-1}(-t) - e^{-t}}}_{\mu-1} \\ & \times \left(-\log(1 + e^{-t} - E_{m-1}(-t)) \right) \underbrace{dt \cdots dt}_{\mu-1} = \sum_{n=0}^{\infty} B_{n, \geq m}^{(\mu)} \frac{x^n}{n!}. \end{aligned} \quad (24)$$

Remark. If $m \rightarrow \infty$ in (23), by $E_\infty(-t) = e^{-t}$, and if $m = 1$ in (24), by $E_0(-t) = 1$ and $E_{-1}(-t) = 0$, both of them are reduced to (3).

PROOF OF THEOREM 2. Since for $\mu \geq 1$

$$\frac{d}{dt} \text{Li}_\mu(1 - E_m(-t)) = \frac{E_{m-1}(-t)}{1 - E_m(-t)} \text{Li}_{\mu-1}(1 - E_m(-t)),$$

we have

$$\begin{aligned} \text{Li}_\mu(1 - E_m(-t)) &= \int_0^t \frac{E_{m-1}(-t)}{1 - E_m(-t)} \text{Li}_{\mu-1}(1 - E_m(-t)) dt \\ &= \int_0^t \frac{E_{m-1}(-t)}{1 - E_m(-t)} \int_0^t \frac{E_{m-1}(-t)}{1 - E_m(-t)} \text{Li}_{\mu-2}(1 - E_m(-t)) dt dt \\ &= \int_0^t \frac{E_{m-1}(-t)}{1 - E_m(-t)} \int_0^t \frac{E_{m-1}(-t)}{1 - E_m(-t)} \cdots \int_0^t \frac{E_{m-1}(-t)}{1 - E_m(-t)} \text{Li}_1(1 - E_m(-t)) \underbrace{dt \cdots dt}_{\mu-1} \\ &= \int_0^t \frac{E_{m-1}(-t)}{1 - E_m(-t)} \int_0^t \frac{E_{m-1}(-t)}{1 - E_m(-t)} \cdots \int_0^t \frac{E_{m-1}(-t)}{1 - E_m(-t)} (-\log(E_m(-t))) \underbrace{dt \cdots dt}_{\mu-1}. \end{aligned}$$

Therefore, we obtain (23). Similarly, by

$$\frac{d}{dt} \text{Li}_\mu(E_{m-1}(-t) - e^{-t}) = \frac{e^{-t} - E_{m-2}(-t)}{E_{m-1}(-t) - e^{-t}} \text{Li}_{\mu-1}(E_{m-1}(-t) - e^{-t}),$$

we obtain (24). $\square$

If $\mu = 1$ in Theorem 1 or in Theorem 2, the generating functions of the *restricted Bernoulli numbers* $B_{n, \leq m}^{(1)}$ and *associated Bernoulli numbers* $B_{n, \geq m}^{(1)}$ are given. Both functions below are reduced to the generating function (1) of the Bernoulli numbers $B_n^{(1)}$ if $m \rightarrow \infty$ and $m = 1$, respectively.

Corollary 1. *We have*

$$\sum_{n=0}^{\infty} B_{n, \leq m}^{(1)} \frac{t^n}{n!} = \frac{\log E_m(-t)}{E_m(-t) - 1}$$

and

$$\sum_{n=0}^{\infty} B_{n, \geq m}^{(1)} \frac{t^n}{n!} = \frac{\log(1 + e^{-t} - E_{m-1}(-t))}{e^{-t} - E_{m-1}(-t)}.$$

3.2. Basic divisibility for non-positive μ . In this short subsection we deduce a basic divisibility property for both the restricted and associated poly-Bernoulli numbers.

It is known [GraKnuPat] that

$$\left\{ \begin{matrix} p \\ k \end{matrix} \right\} \equiv 0 \pmod{p} \quad (1 < k < p)$$

for any prime p . The proof of this basic divisibility is the same for the restricted and associated Stirling numbers, so we can state that

$$\left\{ \begin{matrix} p \\ k \end{matrix} \right\}_{\leq m} \equiv 0 \pmod{p} \quad (k = 0, 1, \dots),$$

and

$$\left\{ \begin{matrix} p \\ k \end{matrix} \right\}_{\geq m} \equiv 0 \pmod{p} \quad (k \geq 2).$$

(Note that $\left\{ \begin{matrix} p \\ 1 \end{matrix} \right\}_{\geq m} = 1$.) These immediately lead to the next statement.

Theorem 3. *For any $\mu \leq 0$ we have that*

$$\begin{aligned} B_{p, \leq m}^{(\mu)} &\equiv 0 \pmod{p}, \\ B_{p, \geq m}^{(\mu)} &\equiv 2^{|\mu|} \pmod{p} \end{aligned}$$

hold for any prime p .

4. A new series representation for the Riemann zeta function

To present our result, we need to recall the definition of the Lambert W function. $W(a)$ is the solution of the equation

$$xe^x = a,$$

that is, $W(a)e^{W(a)} = a$. Since this equation, in general, has infinitely many solutions, the W function has infinitely many complex branches denoted by $W_k(a)$ where $k \in \mathbb{Z}$. What we prove is the following:

Theorem 4. *For any $\mu \in \mathbb{C}$ with $\Re(\mu) > 1$ we have that*

$$\zeta(\mu) = \sum_{n=0}^{\infty} B_{n, \geq 2}^{(\mu)} \frac{(W_k(-1))^n}{n!}$$

for $k = 0, -1$, where ζ is the Riemann zeta function.

PROOF. Let us recall the generating function of $B_{n,\geq m}^{(\mu)}$ in the particular case when $m = 2$:

$$\sum_{n=0}^{\infty} B_{n,\geq 2}^{(\mu)} \frac{(-t)^n}{n!} = \frac{\text{Li}_{\mu}(1+t-e^t)}{1+t-e^t}. \quad (25)$$

By a simple transformation it can be seen that the equation $1+t-e^t = 1$ is solvable in terms of the Lambert W function, and that the solution is $-W_k(-1)$ for any branch $k \in \mathbb{Z}$. However, (25) is valid only for t such that $|1+t-e^t| \leq 1$, at least when $\Re(\mu) > 1$. (This comes from the proof of Theorem 1.) Since the absolute value of $-W_k(-1)$ grows with k , the only two branches which belong to the convergence domain of (25) is $k = -1, 0$. Hence, substituting one of these in place of t we have that

$$\sum_{n=0}^{\infty} B_{n,\geq 2}^{(\mu)} \frac{(W_k(-1))^n}{n!} = \frac{\text{Li}_{\mu}(1-W_k(-1)-e^{-W_k(-1)})}{1-W_k(-1)-e^{-W_k(-1)}} = \frac{\text{Li}_{\mu}(1)}{1} = \zeta(\mu). \quad \square$$

Note that

$$W_0(-1) = \overline{W_{-1}(-1)} \approx -0.318132 + 1.33724i,$$

so all the terms in the incomplete Bernoulli sum are complex, but the sum itself always converges to the real number $\zeta(\mu)$.

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