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On the x-coordinates of Pell equations which are rep-digits

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Abstract. For a positive integer d which is not a square, we show that there is at most one value of the positive integer x participating in the Pell equation $x^2 - dy^2 = 1$ which is a rep-digit, that is all its base 10 digits are equal, with a few exceptions in the pairs (d, x) which we determine.

1. Introduction

Let $d > 1$ be a positive integer which is not a perfect square. It is well-known that the Pell equation

$$x^2 - dy^2 = 1 \tag{1}$$

has infinitely many positive integer solutions (x, y). Furthermore, putting (x_1, y_1) for the smallest solution with $x > 1$, all solutions are of the form (x_n, y_n) for some positive integer n where

$$x_n + \sqrt{d}y_n = (x_1 + \sqrt{d}y_1)^n.$$

There are many papers in the literature which solve Diophantine equations involving members of the sequences $\{x_n\}_{n\geq 1}$ or $\{y_n\}_{n\geq 1}$ being squares, or perfect powers of larger exponents of some other integers, etc. (see, for example, [3], [5]). In this paper, we study a new problem of this kind which we now describe.

Let $g \geq 2$ be an integer. A natural number N is called a *base g rep-digit* if all of its base g-digits are equal; that is, if

$$N = a\left(\frac{g^m - 1}{g - 1}\right), \quad \text{for some} \quad m \geq 1 \text{ and } a \in \{1, 2, \ldots, g - 1\}.$$

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When $g = 10$, we omit the base and simply say that N is a rep-digit. Diophantine equations involving rep-digits were also considered in several papers which found all rep-digits which are perfect powers, or Fibonacci numbers, or generalized Fibonacci numbers, and so on (see [1], [2], [4], [7], [8], [9], [10], [12] for a sample of such results). In this paper, we study when x_n can be a rep-digit, which reduces to the Diophantine equation

$$x_n = a\left(\frac{10^m - 1}{9}\right), \qquad m \geq 1 \text{ and } a \in \{1, \ldots, 9\}. \tag{2}$$

Of course, for every integer $x \geq 2$ there is a unique square-free integer $d \geq 2$ such that

$$x^2 - dy^2 = 1.$$

Namely d is the product of all prime factors of $x^2 - 1$ which appear at odd exponents in its factorization. In particular, taking $x = a(10^m - 1)/9$, we get that any rep-digit is the x-coordinate of the Pell equation corresponding to some specific square-free integer d. Here we study the square-free integers d such that the sequence $\{x_n\}_{n\geq 1}$ contains at least two rep-digits. Our result is the following.

Theorem. *Let $d \geq 2$ be square-free. The Diophantine equation*

$$x_n = a\left(\frac{10^m - 1}{9}\right), \quad m \geq 1 \text{ and } a \in \{1, \ldots, 9\} \tag{3}$$

has at most one positive integer solution n with the following exceptions:

(i) $d = 2$, $n \in \{1, 3\}$;

(ii) $d = 3$, $n \in \{1, 2\}$.

Our proof proceeds in two cases according to whether n is even or odd. If n is even, we reduce the problem to the study of integer points on twelve elliptic curves, getting only the solution $(d, n) = (3, 2)$ for which $x_2 = 7$ is a rep-digit. When n is odd, the proof is more difficult and it uses lower bounds for linear forms in complex and p-adic logarithms as well as some computations to lower the bounds to values up to which we can search for the potential solutions. The tools we need from the literature will be mentioned as needed.

2. The case n even

Write $n = 2n_1$. Since

$$x_n = x_{2n_1} = 2x_{n_1}^2 - 1,$$

it suffices to solve the equation

$$2x^2 - 1 = a\left(\frac{10^m - 1}{9}\right), \quad m \geq 1 \text{ and } a \in \{1, \ldots, 9\}.$$

Since the left-hand side above is odd, it follows that a is odd. If $a = 9$, we get $2x^2 = 10^m$, which has no integer solutions (x, m). So, $a \in \{1, 3, 5, 7\}$. We write $m = 3m_0 + r$ where $r \in \{0, 1, 2\}$ and proceed according to the value of r.

When $r = 0$, we get

$$2x^2 - 1 = \frac{a}{9}\left(y^3 - 1\right), \quad \text{with} \quad y = 10^{m_0}. \tag{4}$$

Multiplying both sides of equation (4) above by $72a^2$ and simplifying give the elliptic curves

$$Y^2 = X^3 + A_0, \tag{5}$$

where $X := 2ay$, $Y := 12ax$, and $A_0 := 8a^2(9 - a)$, with $a \in \{1, 3, 5, 7\}$. With MAGMA we find all integer points (X, Y) on the four curves above. None yields a convenient solution to our original problem.

When $r = 1$, we get

$$2x^2 - 1 = \frac{a}{9}\left(10y^3 - 1\right), \quad \text{with} \quad y = 10^{m_0}. \tag{6}$$

We multiply equation (6) by $7200a^2$ and then simplify the result to obtain the elliptic curves

$$Y^2 = X^3 + A_1, \tag{7}$$

where $X := 20ay$, $Y := 120ax$, and $A_1 := 800a^2(9 - a)$. With MAGMA we find all integer points (X, Y) on the four elliptic curves above. The only convenient solution is $(X, Y) = (140, 1680)$ when $a = 7$ which leads to $(d, n) = (3, 2)$ for which $x_2 = 7$.

When $r = 2$, we get

$$2x^2 - 1 = \frac{a}{9}\left(100y^3 - 1\right), \quad \text{with} \quad y = 10^{m_0}. \tag{8}$$

We multiply both sides of equation (8) by $8 \cdot 9 \cdot 10^4 \cdot a^2$ and get

$$Y^2 = X^3 + A_2, \tag{9}$$

where $X := 200ay$, $Y := 1200ax$, and $A_2 := 8 \cdot 10^4 \cdot a^2(9 - a)$. With MAGMA we find all integer points (X, Y) on the four elliptic curves above. None of these yields any convenient solution to our original problem.

This analysis shows that the only instance in which x_n is a rep-digit for some even n is when $n = 2$ for which $d = 3$. In the last computational section, we will find all indices n such that x_n is a rep-digit for all non-square $d \in \{1, 2, \ldots, 100\}$, which includes the case $d = 3$. From now on until the last computational section we assume that $d > 100$ and that there exist two odd $n_1 \neq n_2$ for which both x_{n_1} and x_{n_2} are rep-digits.

3. On the greatest common divisor of two rep-digits

Suppose that $n_1 \neq n_2$ are odd and

$$x_{n_1} = a_1 \left(\frac{10^{m_1} - 1}{9} \right), \quad x_{n_2} = a_2 \left(\frac{10^{m_2} - 1}{9} \right), \quad \text{where} \quad a_1, a_2 \in \{1, \ldots, 9\}.$$

Let $n_3 := \gcd(n_1, n_2)$. Since n_1 and n_2 are odd, from known properties of solutions to Pell equations, we get that $x_{n_3} = \gcd(x_{n_1}, x_{n_2})$. We put $a_3 := \gcd(a_1, a_2)$, $a_1' := a_1/a_3$, $a_2' := a_2/a_3$. We also put $m_3 := \gcd(m_1, m_2)$ and use the fact that $\gcd(10^{m_1} - 1, 10^{m_2} - 1) = 10^{m_3} - 1$. We then get that

$$\begin{aligned} x_n &= \gcd(x_{n_1}, x_{n_2}) = \gcd\left(a_1 \left(\frac{10^{m_1} - 1}{9} \right), a_2 \left(\frac{10^{m_2} - 1}{9} \right) \right) \\ &= a_3 \left(\frac{10^{m_3} - 1}{9} \right) \gcd\left(a_1' \left(\frac{10^{m_1} - 1}{10^{m_3} - 1} \right), a_2' \left(\frac{10^{m_2} - 1}{10^{m_3} - 1} \right) \right) \\ &:= a_3 \ell \left(\frac{10^{m_3} - 1}{9} \right). \end{aligned} \tag{10}$$

The quantities inside the greatest common divisor ℓ in the right-hand side of (10) above have the property that a_1', a_2' are coprime, that $(10^{m_1} - 1)/(10^{m_3} - 1)$ and $(10^{m_2} - 1)/(10^{m_3} - 1)$ are also coprime and each one of these two last numbers is coprime to 10. A quick analysis shows then that $\ell \in \{1, 3, 7, 9, 21, 63\}$. Furthermore, if $\ell > 1$, then $a_3 \in \{1, 2, 3\}$ and if $\ell > 3$ then $a_3 = 1$. Summarizing, we get that

$$x_{n_3} = a_3 \left(\frac{10^{m_3} - 1}{9} \right) \quad \text{with some} \quad a_3 \in \{1, 2, 3, \ldots, 9, 21, 63\}.$$

Since $n_1 \neq n_2$, we may assume that $n_1 < n_2$, and then $n_3 < n_2$ and n_3 is a proper divisor of n_2. Putting $n := n_2/n_3$ $d := x_{n_3}^2 - 1$, $m := m_3$, $\ell := m_2/m_3$, our problem is now seen to be a subproblem of the following slightly more general one:

A somewhat more general problem. *Find all instances in which*

$$x_1 = a\left(\frac{10^m - 1}{9}\right) \quad \textit{with} \quad a \in \{1, 2, \ldots, 9, 21, 63\} \quad \textit{and}$$
$$x_n = b\left(\frac{10^{m\ell} - 1}{9}\right) \quad \textit{with} \quad b \in \{1, 2, \ldots, 9\}, \tag{11}$$

where $n > 1$ is odd and ℓ, m are positive integers.

4. Bounds among ℓ, m, n

We assume that $m \geq 20$. The calculations for $m \leq 19$ appear in the last section. We put

$$\alpha := x_1 + \sqrt{x_1^2 - 1} = x_1 + y_1\sqrt{d}.$$

Then

$$x_n = \frac{1}{2}(\alpha^n + \alpha^{-n}).$$

Thus, from the first relation (11), we have

$$\frac{1}{2}(\alpha + \alpha^{-1}) = x_1 = a\frac{10^m - 1}{9} \quad \text{with} \quad a \in \{1, 2, \ldots, 9, 21, 63\}.$$

We get

$$\alpha > \frac{1}{2}(\alpha + \alpha^{-1}) = a\frac{10^m - 1}{9} > 10^{m-1}, \tag{12}$$

giving

$$m - 1 < \frac{\log \alpha}{\log 10}.$$

On the other hand,

$$\frac{\alpha}{2} < \frac{1}{2}(\alpha + \alpha^{-1}) = x_1 < 7 \cdot 10^m, \quad \text{or} \quad \alpha < 14 \cdot 10^m, \tag{13}$$

so that

$$\frac{\log \alpha}{\log 10} < m + \frac{\log 14}{\log 10} < m + 2.$$

Hence, we have that

$$m - 1 < \frac{\log \alpha}{\log 10} < m + 2. \tag{14}$$

We now exploit the second relation (11). We have

$$\alpha^n > x_n = b\left(\frac{10^{m\ell}-1}{9}\right) > 9^{\ell-1}\left(\frac{10^m-1}{9}\right)^{\ell} > 9^{\ell-1}\left(\frac{\alpha}{140}\right)^{\ell} > \left(\frac{9\alpha}{140}\right)^{\ell-1} > \alpha^{\frac{\ell-1}{2}}, \tag{15}$$

where in the above chain of inequalities we used the right–most inequality (13) to infer that $10^{m-1} > \alpha/140$, as well as the fact that $m \geq 20$ together with (12) which implies that α is sufficiently large so that both inequalities $\alpha > 140$ and $9\alpha/140 > \alpha^{1/2}$ hold. Clearly, (15) shows that

$$\ell < 2n+1. \tag{16}$$

On the other hand

$$\frac{\alpha^n}{2} < \frac{1}{2}(\alpha^n + \alpha^{-n}) = x_n < 10^{m\ell} < (10\alpha)^{\ell},$$

where we used once again inequality (12). The last inequality above leads to

$$\alpha^n < (20\alpha)^{\ell} < (\alpha^{3/2})^{\ell}, \tag{17}$$

where we used the fact that $\alpha^{1/2} > 20$, which follows from (12) together with the fact that $m \geq 20$. Inequality (17) yields

$$\ell > 2n/3,$$

which together with (16) gives

$$2n/3 < \ell < 2n+1. \tag{18}$$

We will use (14) and (18) later.

5. Bounding m in terms of n

We recall that $x_n = P_n(x_1)$, where $P_n(X) \in \mathbb{Z}[X]$ is the nth Chebyshev polynomial given by

$$P_n(X) := \frac{1}{2}\big((X+\sqrt{X^2-1}\,)^n + (X-\sqrt{X^2-1}\,)^n\big).$$

Throughout this section, we work with divisibility relations among $\{3\}$-integers. That is, we work with algebraic numbers of the form $\delta/3^k$, for some algebraic

integer δ and some integer k. Given two $\{3\}$-integers δ and λ, we will say that $\delta \mid \lambda$ if λ/δ is a $\{3\}$-integer. Using (11), it follows that

$$-b/9 \equiv x_n \equiv P_n(x_1) \equiv P_n\left(-a/9\right) \pmod{10^m}. \tag{19}$$

We will exploit relation (19).

We treat first the case $a = 9$. Then $x_1 = 10^m - 1$, so $x_1 + 1 = 10^m$. We find the last two terms of $P_n(X)$ as a polynomial in $X + 1$. For this, we put $Y := X + 1$, $Q_n(Y) := P_n(Y - 1)$, so

$$Q_n(Y) = \frac{1}{2}\big(Y - 1 + \sqrt{(Y-1)^2 - 1}\,)^n + (Y - 1 - \sqrt{(Y-1)^2 - 1}\,)^n\big).$$

Taking $Y = 0$ in the above expression, we get

$$Q_n(0) = \frac{1}{2}((-1)^n + (-1)^n) = -1$$

because n is odd.

For the coefficient of Y, we take the derivative of $Q_n(Y)$ with respect to Y. We get

$$\begin{aligned}
Q_n(Y)' &= \frac{n}{2}\Bigg((Y - 1 + \sqrt{(Y-1)^2 - 1})^{n-1}\left(1 + \frac{(Y-1)}{\sqrt{(Y-1)^2 - 1}}\right) \\
&\quad + (Y - 1 - \sqrt{(Y-1)^2 - 1})^{n-1}\left(1 - \frac{(Y-1)}{\sqrt{(Y-1)^2 - 1}}\right)\Bigg) \\
&= \frac{n}{2}\Bigg(((Y - 1 + \sqrt{(Y-1)^2 - 1}\,)^{n-1} + (Y - 1 - \sqrt{(Y-1)^2 - 1}\,)^{n-1}) \\
&\quad + \frac{(Y-1)}{\sqrt{(Y-1)^2 - 1}}\Big((Y - 1 + \sqrt{(Y-1)^2 - 1}\,)^{n-1} \\
&\quad - (Y - 1 - \sqrt{(Y-1)^2 - 1}\,)^{n-1}\Big)\Bigg) \\
&= \frac{n}{2}\Bigg((Y - 1 + \sqrt{(Y-1)^2 - 1})^{n-1} + (Y - 1 - \sqrt{(Y-1)^2 - 1}\,)^{n-1}) \\
&\quad + 2(Y-1)\sum_{\substack{0 \le k \le n-1 \\ k \equiv 1 \pmod 2}} \binom{n-1}{k}(Y-1)^{n-1-k}\sqrt{(Y-1)^2 - 1}^{\,k-1}\Bigg).
\end{aligned}$$

For $Y = 0$ and using the fact that n is odd, we get that

$$Q_n'(0) = \frac{n}{2}\left((-1)^{n-1} + (-1)^{n-1} + 2(-1)\binom{n-1}{1}(-1)^{n-2}\right) = n^2.$$

Thus,

$$Q_n(X) = n^2(X+1) - 1 \pmod{(X+1)^2}.$$

Making $X := x_1$, we get that when $a = 9$, then

$$-b/9 \equiv x_n \equiv P_n(x_1) \equiv -1 + n^2 10^m \pmod{10^{2m}}.$$

If $b \neq 9$, we then get that $10^m \mid b - 9$, a contradiction because $m \geq 20$. If $b = 9$, we then get $10^m \mid n^2$, which is also a contradiction since n is odd. This takes care of the case $a = 9$.

Suppose now that $a \neq 9$. We put

$$\beta := -a/9 + \sqrt{(-a/9)^2 - 1},$$

and study these numbers for $a \in \{1, 2, \ldots, 8, 21, 63\}$. Then β is one of the numbers

$$\frac{-1+4\sqrt{-5}}{9}, \quad \frac{-2+\sqrt{-77}}{9}, \quad \frac{-1+2\sqrt{-2}}{3}, \quad \frac{-4+\sqrt{-65}}{9}, \quad \frac{-5+2\sqrt{-14}}{9},$$
$$\frac{-2+\sqrt{-5}}{3}, \quad \frac{-7+4\sqrt{-2}}{9}, \quad \frac{-8+\sqrt{-17}}{9}, \quad \frac{-7+2\sqrt{10}}{3}, \quad -7+4\sqrt{3}.$$

Clearly, β is a $\{3\}$-integer. The numbers from the above list are multiplicatively independent any two (in fact, they live in distinct quadratic fields) except for

$$\frac{-1+4\sqrt{-5}}{9} = \left(\frac{2+\sqrt{-5}}{3}\right)^2 \quad \text{and} \quad \left(\frac{1+2\sqrt{-2}}{3}\right)^2 = \frac{-7+4\sqrt{-2}}{9}.$$

The divisibility relation (19) that we exploit becomes

$$10^m \mid P_n(-a/9) + b/9. \tag{20}$$

Observe that

$$P_n(-a/9) + b/9 = \frac{1}{2}(\beta^n + \beta^{-n}) + b/9 = \frac{\beta^{-n}}{2}\left((\beta^n)^2 + (2b/9)(\beta)^n + 1\right)$$
$$= \frac{\beta^{-n}}{2}(\beta^n - \gamma)(\beta^n - \gamma^{-1}), \tag{21}$$

where

$$\gamma := -b/9 + \sqrt{(-b/9)^2 - 1}.$$

Since β is a $\{3\}$-unit (that is, β^{-1} is also a $\{3\}$-integer), divisibility relation (20) and relation (21) yield

$$10^m \mid (\beta^n - \gamma)(\beta^n - \gamma^{-1}). \tag{22}$$

We distinguish four cases.

Case 1. $b = 9$. Then $\gamma = -1$, and (22) implies

$$10^m \mid (\beta^n + 1)^2. \tag{23}$$

Let π be a prime ideal dividing 2 in $\mathbb{K} = \mathbb{Q}(\beta)$. Computing the norm of $\beta + 1$ from $\mathbb{K}$ to $\mathbb{Q}$, we get

$$N_{\mathbb{K}/\mathbb{Q}}(\beta + 1) = (\beta + 1)(\beta^{-1} + 1) = (-a/9 + 1)^2 - ((a/9)^2 - 1) = \frac{2(9 - a)}{9},$$

which is a rational number of the form r/s with odd s and even r. Hence,

$$\pi | 2 | (\beta + 1)(\beta^{-1} + 1).$$

Since also $\beta + \beta^{-1} = -2a/9 \equiv 0 \pmod{\pi}$, it follows that $\beta \equiv \beta^{-1} \pmod{\pi}$. In particular, $\pi \mid (\beta + 1)^2$, therefore $\pi \mid \beta + 1$. Note that

$$\beta^n + 1 = (\beta + 1)\left(\frac{\beta^n + 1}{\beta + 1}\right), \tag{24}$$

and

$$\frac{\beta^n + 1}{\beta + 1} = \beta^{n-1} + \cdots + 1 \equiv \underbrace{1 + \cdots + 1}_{n \text{ times}} \equiv n \equiv 1 \pmod{\pi}, \tag{25}$$

because n is odd. It now follows that the divisibility relation (23) gives $2^m \mid (\beta + 1)^2$. Taking norms we get

$$2^{2m} \mid N_{\mathbb{K}/\mathbb{Q}}(\beta + 1)^2 = \left(\frac{2(9 - a)}{9}\right)^2, \tag{26}$$

which leads to $m < 10$, a contradiction. Thus, the case $b = 9$ is not possible.

Case 2. $b = a$. Then $\gamma = \beta$, and we get

$$10^m \mid \frac{\beta^{-n}}{2}(\beta^n - \beta)(\beta^n - \beta^{-1}) = \frac{1}{2}(\beta^{(n+1)/2} - \beta^{-(n+1)/2})(\beta^{(n-1)/2} - \beta^{-(n-1)/2}). \tag{27}$$

For a positive integer k put

$$v_k = \frac{\beta^k - \beta^{-k}}{\beta - \beta^{-1}}.$$

The sequence $\{v_k\}_{k\geq 1}$ is a binary recurrent sequence of $\{3\}$-units. We are interested in the exponent of 2 in the factorization of v_k. Since $v_1 = 1$, $v_2 = -2a/9$ and

$$v_{k+2} = (-2a/9)v_{k+1} - v_k, \quad \text{for all} \quad k \geq 1,$$

one proves easily that v_k has an even numerator if and only if k is even. Further, writing an even k as $k = 2^u k_0$ with a positive integer u and an odd integer k_0, we get that

$$v_k = v_{2^u k_0} = (\beta + \beta^{-1})(\beta^2 + \beta^{-2}) \cdots (\beta^{2^{u-1}} + \beta^{-2^{u-1}}) \left(\frac{\beta^{2^u k_0} - \beta^{-2^u k_0}}{\beta^{2^u} - \beta^{-2^u}} \right).$$

The arguments from Case 1 (see (24) and (25)) show that the right–most factor above is odd (that is, it is a rational number of the form r/s with odd r), and also that $2 \| \beta^{2^i} + \beta^{-2^i}$ for $i = 1, 2, \ldots$. Further, $\beta + \beta^{-1} = -2a/9$ and since $a \in \{1, 2, \ldots, 8, 21, 63\}$ it follows that the exponent of 2 in $\beta + \beta^{-1}$ is at most 4. Thus, putting $\nu_2(r)$ for the exponent of 2 in the factorization of the rational number r, we get that

$$\nu_2(v_{2^u k_0}) \leq 3 + u = 3 + \log k / \log 2.$$

Now, we return to divisibility (27) which can be rewritten as

$$10^m \mid \frac{1}{2}(\beta - \beta^{-1})^2 v_{(n+1)/2} v_{(n-1)/2}.$$

Observing that $(\beta - \beta^{-1})^2 = \frac{4}{81}(a^2 - 9^2)$, that $\nu_2(a^2 - 9^2) \leq 5$ for our possibilities for a, and that one of $(n+1)/2$ and $(n-1)/2$ is odd, we get that

$$m \leq 1 + \nu_2\left((a^2 - 9^2) v_{(n+1)/2} v_{(n-1)/2}\right) \leq 1 + \nu_2(a^2 - 9^2) + \nu_2(v_{(n+1)/2}) + \nu_2(v_{(n-1)/2}) \leq 9 + \log((n+1)/2)/\log 2. \quad (28)$$

Case 3. $a \neq b$ **but** β **and** γ **are multiplicatively dependent**. As we saw before, this happens only when $\mathbb{Q}(\beta) = \mathbb{Q}(\sqrt{-5})$ or $\mathbb{Q}(\sqrt{-2})$ and either $\gamma = \beta^2$ or $\beta = \gamma^2$. In this case, the divisibility (22) becomes one of

$$10^m \mid (\beta^{n+2} - 1)(\beta^{n-2} - 1) \quad \text{or} \quad 10^m \mid (\gamma^{2n+1} - 1)(\gamma^{2n-1} - 1).$$

The arguments from Case 1 show that either one of the above two relations yields that either $2^m \mid (\beta-1)^2$ or $2^m \mid (\gamma-1)^2$. By the argument used to prove (26), we get

$$2^{2m} \mid N_{\mathbb{K}/\mathbb{Q}}(\beta-1)^2 = \left(\frac{2(a+9)}{9}\right)^2,$$

or a similar relation with the pair (β, a) replaced by (γ, b), and none of them holds with some $m \geq 20$. So, this case cannot occur.

Case 4. β **and** γ **are multiplicatively independent**. Let $\mathbb{L} = \mathbb{Q}(\beta, \gamma)$, which is a field of degree $D = 4$. Let π be any prime ideal dividing 2 in $\mathbb{L}$. For an algebraic number $\delta \in \mathbb{L}$, we put $\nu_\pi(\delta)$ for the exponent of π in the factorization of the principal fractional ideal $\delta\mathcal{O}_\mathbb{L}$. Relation (22) gives

$$m \leq \nu_\pi(10^m) \leq \nu_\pi(\beta^n - \gamma) + \nu_\pi(\beta^n - \gamma^{-1}).$$

Clearly,

$$\min\{\nu_\pi(\beta^n-\gamma), \nu_\pi(\beta^n-\gamma^{-1})\} \leq \nu_\pi(\gamma-\gamma^{-1}). \tag{29}$$

Since

$$N_{\mathbb{L}/\mathbb{Q}}(\gamma-\gamma^{-1}) = \left(\frac{2\sqrt{b^2-9^2}}{9}\right)^4 = \frac{2^4(b-9)^2(b+9)^2}{9^4},$$

for $m \geq 20$, it follows that $\nu_2(N_{\mathbb{L}/\mathbb{Q}}(\gamma-\gamma^{-1})) \leq 14$. Putting e_π for the ramification of π, we get from (29) that

$$\min\{\nu_\pi(\beta^n-\gamma), \nu_\pi(\beta^n-\gamma^{-1})\} \leq \nu_\pi(\gamma-\gamma^{-1}) \leq e_\pi\nu_2\left(N_{\mathbb{L}/\mathbb{Q}}(\gamma-\gamma^{-1})\right) \leq 56.$$

Hence,

$$m \leq \max\{\nu_\pi(\beta^n-\gamma), \nu_\pi(\beta^n-\gamma^{-1})\} + 56.$$

Note also that $\nu_\pi(\gamma) = \nu_\pi(\gamma^{-1})$ because γ is a $\{3\}$-unit. Hence,

$$m \leq \max\{\nu_\pi(\beta^n\gamma^{\pm 1}-1)\} + 56. \tag{30}$$

To estimate the maximum above, we use a linear form in p-adic logarithms due to Kunrui Yu [13]. The statement is the following.

Theorem 5.1. *Let $\delta_1, \ldots, \delta_t$ be algebraic numbers in a field $\mathbb{L}$ of degree D and $b_1, \ldots, b_t$ be nonzero integers. Put*

$$\Lambda = \delta_1^{b_1} \cdots \delta_t^{b_t} - 1$$

and

$$B \geq \max\{|b_1|, \ldots, |b_t|, 3\}.$$

Let π be a prime ideal of $\mathbb{L}$ sitting above the rational prime p of ramification e_π and inertia f_π, respectively. Assume that

$$H_i \geq \max\{h(\delta_i), \log p\} \quad \text{for } i = 1, \ldots, t,$$

where $h(\delta)$ is the Weil height of δ. If $\Lambda \neq 0$, then

$$\nu_\pi(\Lambda) \leq 19(20\sqrt{t+1}D)^{2(t+1)} e_\pi^{t-1} \frac{p^{f_\pi}}{(f_\pi \log p)^2} \log(e^5 tD) H_1 \cdots H_t \log B. \tag{31}$$

For us, we take $t = 2$, $\delta_1 = \beta$, $\delta_2 = \gamma$, $b_1 = n$, $b_2 = \pm 1$. Thus, $B = n$. The degree D of the field $\mathbb{L}$ containing β and γ is 4. The form Λ is non-zero (because β and γ are multiplicatively independent). Hence, we get

$$\nu_\pi(\beta^n \gamma^{\pm 1} - 1) \leq 19(20\sqrt{3} \cdot 4)^6 e_\pi \frac{2^{f_\pi}}{(f_\pi \log 2)^2} \log(8e^5) H_1 H_2 \log n, \tag{32}$$

where

$$H_1 \geq \max\{h(\beta), \log 2\} \quad \text{and} \quad H_2 \geq \max\{h(\gamma), \log 2\},$$

and $h(\beta)$ and $h(\gamma)$ are the Weil heights of β and γ, respectively. Computing the heights of β, γ one gets that we can take $H_1 = H_2 = 1.5$. Since $e_\pi \leq 4$ and $f_\pi \leq 4$, a computation reveals that inequality (32) and (30) yield

$$m < 4 \times 10^{16} \log n. \tag{33}$$

Note that the upper bound (33) above is larger than (28) obtained at Case 2.

We record this as a lemma.

Lemma 5.1. *In* (11), *we have for $m \geq 20$ that*

$$m < 4 \times 10^{16} \log n. \tag{34}$$

6. Bounding all variables ℓ, m, n

The second equation (11) gives

$$\alpha^n + \alpha^{-n} = 2x_n = (2b/9)10^{m\ell} - (2b/9),$$

or

$$(2b/9)10^{m\ell} - \alpha^n = \alpha^{-n} + (2b/9).$$

This leads to

$$0 < (2b/9)10^{m\ell}\alpha^{-n} - 1 < \frac{3}{\alpha^n} < \frac{1}{\alpha^{n-1}}. \tag{35}$$

The left-hand side above is nonzero because α^n is irrational. We find a lower bound on it using a result of Matveev [11] which we now state.

Theorem 6.1. *In the notations of Theorem 5.1, assume additionally that* $\mathbb{L}$ *is real and*

$$H_i \geq \max\{Dh(\delta_i), |\log \delta_i|, 0.16\} \quad \text{for} \quad i = 1, \ldots, t.$$

If $\Lambda \neq 0$, *then*

$$\log |\Lambda| \geq -1.4 \cdot 30^{t+3} \cdot t^{4.5} \cdot D^2(1 + \log D)(1 + \log B)H_1 \cdots H_t.$$

We take $t = 3$, $\delta_1 = 2b/9$, $\delta_2 = 10$, $\delta_3 = \alpha$, $b_1 = 1$, $b_2 = m\ell$, $b_3 = -n$. Since $\ell \leq 2n$ by (18), it follows that we can take $B = 2mn$. We take $\mathbb{L} = \mathbb{Q}(\alpha)$ which is real of degree $D = 2$. We can clearly take $H_1 = 2\log 16$, $H_2 = 2\log 10$ and $H_3 = \log \alpha$. Then applying Theorem 6.1 and using (35), we get

$$(n-1)\log \alpha < 1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2(1 + \log 2)(1 + \log(2nm))(2\log 16)(2\log 10)\log \alpha,$$

which gives

$$n - 1 < 3 \times 10^{13}(1 + \log(2mn)), \tag{36}$$

and using (34), we get

$$n - 1 < 3 \times 10^{13}(1 + \log(8 \times 10^{16} n \log n)), \tag{37}$$

giving $n < 3 \times 10^{15}$. Inequalities (18) and (34) now give

$$\ell \leq 6 \times 10^{15} \quad \text{and} \quad m < 2 \times 10^{18}.$$

We summarize these calculations as follows.

Lemma 6.1. *In* (11) *we have for* $m \geq 20$ *that*

$$\ell \leq 6 \times 10^{15}, \quad m \leq 2 \times 10^{18}, \quad n \leq 3 \times 10^{15}.$$

7. The final calculations

7.1. Small cases. In this subsection, we deal with the cases deemed as "small" along the way. These are the cases for which either $d \leq 100$ (end of Section 2), or

$$x_1^2 - 1 = dy_1^2,$$

for some x_1 as in (11) for some $m \leq 37$. Put

$$\mathcal{D}_1 = \{2 \leq d \leq 100 : \mu^2(d) = 1\},$$

$$\mathcal{D}_2 := \{d : x_1^2 - 1 = dy_1^2, \text{ with } \mu^2(d) = 1, \text{ for some } x_1 \text{ as in (11) with } m \leq 37\},$$

and

$$\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2.$$

Here, $\mu(d)$ is the Möbius function of the positive integer d. The set $\mathcal{D}$ has 451 elements. The smallest one is 2 and the largest one is

$$306249999999999999999999999999999999938750000000000000000000000000000000003.$$

For each $d \in \mathcal{D}$, we let (x_1, y_1) be the minimal solution of the Pell equation

$$x^2 - dy^2 = 1,$$

and put as before $\alpha = x_1 + \sqrt{d}y_1$. We need to find $n \geq 2$ or deduce that it does not exist such that

$$x_n = \frac{b(10^M - 1)}{9}, \quad \text{for some} \quad b \in \{1, 2, \ldots, 9\}. \tag{38}$$

Thus, we get

$$\frac{1}{2}(\alpha^n + \alpha^{-n}) = \frac{b(10^M - 1)}{9}. \tag{39}$$

This can be regrouped as

$$\left|\alpha^n - (2b/9)10^M\right| < \alpha^{-n} + 2b/9 < 3. \tag{40}$$

The maximal value of α for $d \in \mathcal{D}_1$ corresponds to $d = 61$, with corresponding $x_1 = 1766319049$, but this gives a smaller value of α then the largest element corresponding to $d \in \mathcal{D}_2$, which is

$$x_1 = \frac{63(10^{37} - 1)}{9} = 7(10^{37} - 1),$$

with corresponding

$$\alpha = x_1 + \sqrt{x_1^2 - 1} < 14 \cdot 10^{37}.$$

So, estimate (40) gives

$$10^{M-1} < (2b/9)10^M < \alpha^n + 3 < (14 \cdot 10^{37})^n + 3 \leq 10^{39n},$$

implying

$$M \leq 39n. \tag{41}$$

On the other hand, since $\alpha \geq 2+\sqrt{3}$, we get, again from (40), that

$$(2+\sqrt{3}\,)^n \leq \alpha^n < (2b/9)10^M + 3 < 10^{M+1},$$

therefore

$$n \leq \frac{\log 10}{\log(2+\sqrt{3}\,)}(M+1) \leq 1.8(M+1). \tag{42}$$

So, from (41) and (42), we record that

$$\frac{M}{39} \leq n \leq 1.8(M+1). \tag{43}$$

Now from (40), and the fact that $\alpha \geq 2+\sqrt{3} > 3$, we get that

$$\left|\alpha^{-n}(2b/9)10^M - 1\right| < \frac{3}{\alpha^n} < \frac{1}{\alpha^{n-1}}. \tag{44}$$

The left-hand side above is not 0. To find a lower bound for it, we apply again Matveev's Theorem 6.1 in the same way as we did it for estimate (35). The only difference now is that the exponent $m\ell$ was replaced by M and $M \leq 39n$, so instead of estimate (36), we get

$$n - 1 < 3 \times 10^{13}(1 + \log(39n)),$$

giving $n < 2 \times 10^{15}$ and then $M < 8 \times 10^{16}$.

We record what we have proved.

Lemma 7.1. *If $d \in \mathcal{D}$, then in* (38)*, we have $n < 2\times 10^{15}$ and $M < 8\times 10^{16}$.*

We now need to lower n. Clearly $n \geq 3$ since it is odd, so that $\alpha^{n-1} > 2$. We put

$$\Gamma = M\log 10 - n\log\alpha + \log(2b/9).$$

Relation (39) implies that $\Gamma > 0$. By (44), we have $0 < e^{\Gamma} - 1 < \alpha^{-(n-1)} < 1/2$, so $e^{\Gamma} \in (1/2, 3/2)$. Thus,

$$\Gamma < e^{\Gamma} - 1 < \frac{1}{\alpha^{n-1}}.$$

In particular,

$$0 < M\left(\frac{\log 10}{\log\alpha}\right) - n + \frac{\log(2b/9)}{\log\alpha} < \frac{A}{\alpha^n}, \tag{45}$$

where $A = \alpha/\log\alpha$. We now recall a result of Dujella and Pethő, the proof of Lemma 5 in [6].

Lemma 7.2. *Let M be a positive integer, let p/q be a convergent of the continued fraction of the irrational γ such that $q > 6M$, and let A, B, μ be some real numbers with $A > 0$ and $B > 1$. Let $\varepsilon := \|\mu q\| - M\|\gamma q\|$, where $\|\cdot\|$ denotes the distance from the nearest integer. If $\varepsilon > 0$, then there is no solution to the inequality*

$$0 < u\gamma - v + \mu < AB^{-w},$$

in positive integers $u, |v|$ and w with

$$u \le M \quad \text{and} \quad w \ge \frac{\log(Aq/\varepsilon)}{\log B}.$$

We applied Lemma 7.2 with $\gamma = \log 10/\log\alpha$, $\mu = \log(2b/9)/\log\alpha$, $B = \alpha$ for all $b \in \{1, \ldots, 9\}$ and each of the 451 α's corresponding to $d \in \mathcal{D}$ with the bounds on n and M given by Lemma 7.1. We got bounds for n and M that are at most 33 and the remaining calculations were done by brute force. We only obtained the small solutions indicated in the statement of Theorem 1. In fact, we have:

- $d = 2$, $(n, m, b, x) = (1, 1, 3, 3), (3, 2, 9, 99)$,
- $d = 3$, $(n, m, b, x) = (1, 1, 2, 2), (2, 1, 7, 7)$,
- $d = 62$, $(n, m, b, x) = (1, 1, 63, 63), (3, 6, 9, 999999)$.

Note that the case $d = 62$ above gives a convenient solution of (11) but not to our original problem since 63 is not a rep-digit. Thus, this example is not included in the statement of Theorem 1. So, from now on we are entitled to continue with the computational part when $d > 100$ and $m \ge 20$.

7.2. The large cases. Similarly to (35), the first equation (11) gives

$$0 < (2a/9)10^m\alpha^{-1} - 1 < \frac{3}{\alpha}. \tag{46}$$

Relations (35) and (46) lead via the inequality $e^z - 1 > z$ for positive real z to the inequalities

$$|m\ell \log 10 - n\log\alpha + \log(2b/9)| < \frac{3}{\alpha^n},$$

$$|m\log 10 - \log\alpha + \log(2a/9)| < \frac{3}{\alpha}.$$

Multiplying the second relation above by n, using the triangular inequality, the upper bound of Lemma 6.1 on n as well as inequality (12), we get

$$|m(\ell - n)\log 10 - n\log(2a/9) + \log(2b/9)| < \frac{3n+3}{\alpha} < \frac{10^{16}}{10^{m-1}} = \frac{1}{10^{m-17}}.$$

If $\ell = n$, we get

$$|n\log(2a/9) - \log(2b/9)| < \frac{1}{1000}, \tag{47}$$

and this has no solution $n \geq 3$, $a \in \{1, 2, \ldots, 8, 21, 63\}$, $b \in \{1, 2, \ldots, 9\}$ as we checked computationally. Thus, $\ell \neq n$. Then we get

$$\left| n\left(\frac{\log(2a/9)}{\log 10}\right) - m(\ell - n) - \frac{\log(2b/9)}{\log 10}\right| < \frac{1}{(\log 10)10^{m-17}} < \frac{0.5}{10^{m-17}}. \tag{48}$$

The above inequality is of the type

$$|u\gamma - v + \mu| < \frac{A}{B^w},$$

where

$$u := n, \quad \gamma := \frac{\log(2a/9)}{\log 10}, \quad v := m(\ell - n), \quad \mu = -\frac{\log(2b/9)}{\log 10},$$
$$A := 0.5, \quad B := 10,$$

and $w := m - 17$. We first suppose $a \neq b$, where recall

$$a \in \{1, 2, \ldots, 9, 21, 63\} \quad \text{and} \quad b \in \{1, 2, \ldots, 9\}.$$

This we treat with Lemma 7.2 by distinguishing two cases according to the sign of the expression on the left-hand side of (48). We took $M := 3 \times 10^{15}$, which is acceptable by Lemma 7.2, in order to get a small bound for m. A program was written in PARI/GP running with 200 digits. For the computations, if the first convergent such that $q > 6M$ does not satisfy the condition $\varepsilon > 0$, then we use the next convergent until the condition is satisfied. Fortunately, the bound was reduced at the first round. It took about 3 minutes to run our program and in all cases, we obtained $m \leq 36$ if $a \leq 9$, $m \leq 34$ if $a = 21$, and $m \leq 37$ if $a = 63$. But this case has been covered at Subsection 7.1.

In the case $a = b$, we cannot use Lemma 7.2 because then (48) becomes

$$\left|(n-1)\frac{\log(2a/9)}{\log 10} - m(\ell - n)\right| < \frac{0.5}{10^{m-17}},$$

and now the left-hand side is "homogeneous". That is, the parameter μ in Lemma 7.2 becomes 0 and ε is always negative. So, we use Legendre's criterion. Namely, let $a = b \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Note that $n > 1$. Then we get

$$\left|\frac{\log(2a/9)}{\log 10} - \frac{m(\ell - n)}{n-1}\right| < \frac{0.5}{(n-1)10^{m-17}}. \tag{49}$$

Recall that $m \geq 33$. Then $10^{m-17} > 3\times 10^{15} \geq n-1$, so that the right-hand side is $< 1/(2(n-1)^2)$, so $m(\ell-n)/(n-1) = p/q$ is a convergent of $\log(2a/9)/\log 10$ with denominator $q < 3 \times 10^{15}$. We computed all the convergents $p_t/q_t = [a_0, \ldots, a_t]$ with t maximal such that $q_t < 3\times 10^{15}$ and recorded $A = \max\{a_i : i = 0, \ldots, t+1\}$. We obtained the following values of A:

$$a = 1,\ A = 99;\ a = 2,\ A = 89;\ a = 3,\ A = 44;\ a = 4,\ A = 18;\ a = 5,\ A = 29;$$
$$a = 6,\ A = 254;\ a = 7,\ A = 459;\ a = 8,\ A = 509;\ a = 9,\ A = 42.$$

By the theory of continued fractions, we have

$$\left|\frac{\log(2a/9)}{\log 10} - \frac{p}{q}\right| > \frac{1}{(A+2)q^2}. \tag{50}$$

Combining (49) and (50) we get

$$\frac{1}{(A+2)q^2} < \frac{1}{2(n-1)10^{m-17}}.$$

This gives

$$10^{m-17} < \frac{(A+2)q^2}{2(n-1)} \leq \frac{(A+2)(n-1)}{2} \leq 1.5 \times 10^{15}(A+2).$$

Thus, with the above values of A we get $m \leq 34$. So, again we are in the situation treated at Subsection 7.1.

This completes the proof of Theorem 1.

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References

[1] S. Díaz Alvarado and F. Luca, Fibonacci numbers which are sums of two repdigits, Proceedings of the XIVth International Conference on Fibonacci numbers and their applications, Sociedad Matematica Mexicana, Aportaciones Matemáticas, Investigación **20**, 2011, 97–108.

[2] J. Bravo and F. Luca, On a conjecture concerning k-generalized Fibonacci numbers with only one distinct digit, *Publ. Math. Debrecen* **82** (2013), 623–639.

[3] Y. Bugeaud, M. Mignotte and S. Siksek, Classical and modular approaches to exponential Diophantine equations I. Fibonacci and Lucas perfect powers, *Ann. of Math.* **163** (2006), 969–1018.

[4] Y. Bugeaud and M. Mignotte, On integers with identical digits, *Mathematika* **46** (1999), 411–417.

[5] Y. Bugeaud and P. Mihailescu, On the Nagell–Ljunggren equation $(x^n-1)/(x-1) = y^q$, *Math. Scand.* **101** (2007), 177–183.

[6] A. Dujella and A. Pethő, A generalization of a theorem of Baker and Davenport, *Quart. J. Math. Oxford Ser.* **49** (1998), 291–306.

[7] F. Luca, Fibonacci and Lucas numbers with only one distinct digit, *Port. Math.* **57** (2000), 243–254.

[8] F. Luca, Repdigits which are sums of at most three Fibonacci number, *Math. Comm.* **17** (2012), 1–11.

[9] D. Marques and A. Togbé, On terms of linear recurrence sequences with only one distinct block of digits, *Colloq. Math.* **124** (2011), 145–155.

[10] D. Marques and A. Togbé, On repdigits as product of consecutive Fibonacci numbe rs, *Rend. Istit. Mat. Univ. Trieste,* **44** (2012), 393–397.

[11] E. M. Matveev, An explicit lower bound for a homogeneous rational linear form in logarithms of algebraic numbers, II, *Izv. Ross. Akad. Nauk Ser. Mat.* **64** (2000), 125–180, English translation in *Izv. Math.* **64** (2000), 1217–1269.

[12] R. Obláth, Une propriété des puissances parfaites, *Mathesis* **65** (1956), 356–364.

[13] K. Yu, p-adic logarithmic forms and group varieties II, *Acta Arith.* **89** (1999), 337–378.

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