

## On groups with small verbal conjugacy classes

By PAVEL SHUMYATSKY (Brasilia) and AGENOR FREITAS DE ANDRADE (Luziânia)

**Abstract.** Given a group  $G$  and a word  $w$ , we denote by  $G_w$  the set of all  $w$ -values in  $G$  and by  $w(G)$  the corresponding verbal subgroup. The main result of the paper is the following theorem. Let  $n$  be a positive integer and let  $w$  be either the lower central word  $\gamma_n$  or the derived word  $\delta_n$ . Let  $G$  be a group in which for any element  $g \in G$  there exist finitely many Chernikov subgroups whose union contains  $g^{G_w}$ . Then  $\langle g^{w(G)} \rangle$  is Chernikov for all  $g \in G$ .

### 1. Introduction

Let  $w$  be a word in  $n$  variables, and let  $G$  be a group. The verbal subgroup  $w(G)$  of  $G$  determined by  $w$  is the subgroup generated by the set  $G_w$  consisting of all values  $w(g_1, \dots, g_n)$ , where  $g_1, \dots, g_n$  are elements of  $G$ . A word  $w$  is said to be concise if whenever  $G_w$  is finite for a group  $G$ , it always follows that  $w(G)$  is finite. P. Hall asked whether every word is concise, but it was later proved that this problem has a negative solution in its general form (see [5, p. 439]). On the other hand, many important words are known to be concise. For instance, TURNER-SMITH [9] showed that the lower central words  $\gamma_n$  and the derived words  $\delta_n$  are concise; here the words  $\gamma_n$  and  $\delta_n$  are defined by the formulae  $\gamma_1 = \delta_0 = x$ ,  $\gamma_n = [\gamma_{n-1}, \gamma_1]$  and  $\delta_n = [\delta_{n-1}, \delta_{n-1}]$ . The corresponding verbal subgroups for these words are the familiar  $n$ th term of the lower central series of  $G$  denoted by  $\gamma_n(G)$  and the  $n$ th derived group of  $G$  denoted by  $G^{(n)}$ .

---

*Mathematics Subject Classification:* 20E45, 20F24.

*Key words and phrases:* words, verbal subgroups, conjugacy classes.

The work of the first author was supported by CNPq-Brazil. The work of the second author was supported by CAPES/FAPEG-Brazil.

There are several natural ways to look at Hall's question from a different angle. The circle of problems arising in this context can be characterized as follows.

Given a word  $w$  and a group  $G$ , assume that certain restrictions are imposed on the set  $G_w$ . How does this influence the properties of the verbal subgroup  $w(G)$ ?

If  $X$  and  $Y$  are non-empty subsets of a group  $G$ , we will write  $X^Y$  to denote the set  $\{y^{-1}xy \mid x \in X, y \in Y\}$ . In [2] groups  $G$  with the property that  $x^{G_w}$  is finite for all  $x \in G$  were called  $FC(w)$ -groups. Recall that  $FC$ -groups are precisely groups with finite conjugacy classes. The main result of [2] tells us that if  $w$  is a concise word, then a group  $G$  is an  $FC(w)$ -group if and only if  $x^{w(G)}$  is finite for all  $x \in G$ . In particular, it follows that if  $w$  is a concise word and  $G$  is an  $FC(w)$ -group, then the verbal subgroup  $w(G)$  is  $FC$ . Later it was shown in [1] that there exists a function  $f = f(m, w)$  such that if, under the hypothesis of the above theorem,  $x^{G_w}$  has at most  $m$  elements for all  $x \in G$ , then  $x^{w(G)}$  has at most  $f$  elements for all  $x \in G$ . In relation with the above results, the following question was considered in [4].

Given a concise word  $w$  and a group  $G$ , assume that for all  $x \in G$  the subgroup  $\langle x^{G_w} \rangle$  satisfies a certain finiteness condition. Is it true that a similar condition is also satisfied by  $\langle x^{w(G)} \rangle$  for all  $x \in G$ ?

Here and throughout the paper  $\langle M \rangle$  denotes the subgroup generated by the set  $M$ . The following theorem is the main result of [4].

**Theorem 1.1.** *Let  $n$  be a positive integer and let  $w$  be either the word  $\gamma_n$  or the word  $\delta_n$ . Suppose that  $G$  is a group in which  $\langle g^{G_w} \rangle$  is Chernikov for all  $g \in G$ . Then  $\langle g^{w(G)} \rangle$  is Chernikov for all  $g \in G$  as well.*

Recall that a group  $G$  is Chernikov if it has a subgroup of finite index that is a direct product of finitely many groups of type  $C_{p^\infty}$  for various primes  $p$  (quasicyclic  $p$ -groups, or Prüfer  $p$ -groups). By a deep result obtained independently by SHUNKOV [8], and KEGEL and WEHRFRITZ [3] Chernikov groups are precisely the locally finite groups satisfying the minimal condition on subgroups, that is, any non-empty set of subgroups possesses a minimal subgroup.

The purpose of the present paper is to strengthen Theorem 1.1 in the following way.

**Theorem 1.2.** *Let  $n$  be a positive integer and let  $w$  be either the word  $\gamma_n$  or the word  $\delta_n$ . Let  $G$  be a group in which for any element  $g \in G$  there exist finitely many Chernikov subgroups whose union contains  $g^{G_w}$ . Then  $\langle g^{w(G)} \rangle$  is Chernikov for all  $g \in G$ .*

A proof of Theorem 1.2 in the case where  $w = \gamma_n$  can be obtained from the case  $w = \delta_n$  by simply replacing everywhere in the proof the term “ $\delta_n$ -commutators” by “ $\gamma_n$ -commutators”. That is why we do not provide an explicit proof for the case  $w = \gamma_k$  concentrating instead on proving Theorem 1.2 in the case  $w = \delta_n$ .

The hypothesis in Theorem 1.2 is reminiscent of the situation considered in [7] where it was proved that if the set of  $\delta_n$ -commutators in a group  $G$  is contained in a union of finitely many Chernikov subgroups, then  $G^{(n)}$  is Chernikov. As a by-product of the proof of Theorem 1.2 we obtain a considerably stronger result – Corollary 2.11 in the next section says that for any word  $w$  if the set of  $w$ -values in a group  $G$  is contained in a union of finitely many Chernikov subgroups, then  $w(G)$  is Chernikov.

## 2. Preliminaries

Let  $G$  be a group acted on by a group  $A$ . As usual,  $[G, A]$  denotes the subgroup generated by all elements of the form  $x^{-1}x^a$ , where  $x \in G, a \in A$ . It is well-known that  $[G, A]$  is a normal subgroup of  $G$ . If  $B$  is a normal subset of  $A$  such that  $A = \langle B \rangle$ , then  $[G, A] = \langle [G, b]; b \in B \rangle$ . In particular, if  $A$  is cyclic, then  $[G, A] = [G, a]$ , where  $a$  is a generator of  $A$ .

The minimal subgroup of finite index of a Chernikov group  $T$  is called the radicable part of  $T$ . Throughout the article we denote this subgroup by  $T^0$ . In general a group  $T$  is called radicable if the equation  $x^n = a$  has a solution in  $T$  for every positive integer  $n$  and every  $a \in T$ . It is well-known that a periodic abelian radicable group is a direct product of quasicyclic  $p$ -subgroups. Suppose the radicable part of a Chernikov group  $T$  has index  $i$  and is a direct product of precisely  $j$  groups of type  $C_{p^\infty}$  (for various primes  $p$ ). The ordered pair  $(j, i)$  is called the size of  $T$ . The set of all pairs  $(j, i)$  is endowed with the lexicographic order. It is easy to check that if  $H$  is a proper subgroup of  $T$ , the size of  $H$  is necessarily strictly less than that of  $T$ . Also, if  $N$  is an infinite normal subgroup of  $T$ , the size of  $T/N$  is necessarily strictly less than that of  $T$ .

The following lemma is well-known (see for example [6, Part 1, Lemma 3.13]).

**Lemma 2.1.** *Suppose that  $R$  is a radicable abelian normal subgroup of the group  $G$  and suppose that  $H$  is a subgroup of  $G$  such that  $[R, \underbrace{H, \dots, H}_r] = 1$  for some natural number  $r$ . If  $H/H'$  is periodic, then  $[R, H] = 1$ .*

The next few lemmas can be easily deduced from the above. The interested reader can find their proofs for example in [4].

**Lemma 2.2.** *In a periodic nilpotent group  $G$  every radicable abelian subgroup  $Q$  is central.*

**Lemma 2.3.** *Let  $A$  be a periodic group acting on a periodic radicable abelian group  $G$ . Then  $[G, A, A] = [G, A]$ .*

**Lemma 2.4.** *Let  $A$  be a finite group acting on a periodic radicable abelian group  $G$ . Then  $[G, A]$  is radicable.*

**Lemma 2.5.** *Let  $A$  be a radicable group acting on a Chernikov group  $B$ . Then  $[B, A, A] = 1$ .*

**Lemma 2.6.** *Let  $G$  be a Chernikov group for which there exists a positive integer  $m$  such that  $G$  can be generated by elements of order dividing  $m$ . If  $G^0 \leq Z(G)$ , then  $G$  is finite.*

PROOF. Essentially, this is Lemma 2.7 in [4]. □

**Lemma 2.7.** *Let  $G$  be a group,  $y$  an element of  $G$ , and  $x$  is a  $\delta_n$ -commutator for some  $n \geq 0$ . Then  $[y, x, x]$  is a  $\delta_{n+1}$ -commutator.*

PROOF. This follows from the fact that  $[y, x, x]$  can be written as  $[x^{-y}, x]^x$ . □

**Lemma 2.8.** *Let  $G$  be a group generated by an element  $g$  and an abelian radicable subgroup  $S$ . Suppose that  $G$  has finitely many Chernikov subgroups whose union contains  $g^S$ . Then the subgroup  $\langle g^S \rangle$  is Chernikov.*

PROOF. Suppose that the lemma is false and the subgroup  $\langle g^S \rangle$  is not Chernikov. Let  $C_1, \dots, C_k$  be finitely many Chernikov subgroups such that  $g^S \subseteq \cup C_i$ . Without loss of generality we assume that the subgroups  $C_1, \dots, C_k$  are chosen in such a way that the sum of the sizes of  $C_1, \dots, C_k$  is as small as possible. In that case, of course, each subgroup  $C_i$  is generated by  $C_i \cap g^S$ . Remark that  $\langle g^G \rangle = \langle g^S \rangle$  and therefore the subgroup  $\langle g^S \rangle$  is normal. If all subgroups  $C_1, \dots, C_k$  are finite, then so is the set  $g^S$ . In that case the index  $[S : C_S(g)]$  is finite. Being radicable,  $S$  does not have proper subgroups of finite index and so we deduce that  $g^S = g$  and  $\langle g^S \rangle = \langle g \rangle$ . Since  $g$  is contained in a Chernikov subgroup,  $g$  must be of finite order and so  $\langle g \rangle$  is finite. Therefore, at least one of the subgroups  $C_1, \dots, C_k$  is infinite. Without loss of generality assume that  $C_1$  is infinite. Among all infinite subgroups of  $C_1$  that can be generated by elements

of  $g^S$  we choose a minimal one, say  $K$ . Let  $Y = K \cap g^S$  and so  $K = \langle Y \rangle$ . If  $x$  is an arbitrary element in  $S$ , the set  $Y^x$  has infinite intersection with at least one of the subgroups  $C_i$ . Suppose that  $C_j \cap Y^x$  is infinite and set  $L = \langle C_j \cap Y^x \rangle$ . It is clear that  $L^{x^{-1}}$  is an infinite subgroup of  $K$  generated by a subset of  $Y$ . Because of minimality of  $K$  we conclude that  $L = K^x$ . Thus, for any  $x \in S$  there exists  $j$  such that  $K^x \leq C_j$ . Choose  $a \in K^0$ . It follows that for any  $x \in S$  there exists  $j$  such that  $a^x \leq C_j^0$ . Since a radicable Chernikov group has only finitely many elements of any given order, we deduce that the class  $a^S$  is finite. Taking into account that  $S$  has no proper subgroups of finite index and that  $a$  was taken in  $K^0$  arbitrarily we now deduce that  $[K^0, S] = 1$ . Since  $Y$  normalizes  $K^0$  and since  $G = \langle S, Y \rangle$ , it follows that  $K^0$  is normal in  $G$ . The size of the image of  $C_1$  in  $G/K^0$  is strictly less than that of  $C_1$  and therefore, by induction,  $\langle g^S \rangle / K^0$  is Chernikov. Since also  $K^0$  is Chernikov, so is  $\langle g^S \rangle$ . The proof is complete.  $\square$

An idea from the proof of Lemma 2.8 can be used to significantly improve the result that if the set of  $\delta_n$ -commutators in a group  $G$  is contained in a union of finitely many Chernikov subgroups, then  $G^{(n)}$  is Chernikov [7]. We will now show that for any word  $w$  if the set of  $w$ -values in a group  $G$  is contained in a union of finitely many Chernikov subgroups, then  $w(G)$  is Chernikov. In fact we have the following rather general proposition.

**Proposition 2.9.** *Let  $X$  be a normal subset of a group  $G$  and suppose that  $G$  has Chernikov subgroups  $C_1, \dots, C_k$  whose union contains  $X$ . Then  $\langle X \rangle$  is Chernikov.*

Recall that a group having an ascending central series is called hypercentral. For the proof of Proposition 2.9 we will require the following well-known lemma whose proof can be easily deduced for example from [6, Part 2, Theorem 9.23 and Corollary 2, page 125].

**Lemma 2.10.** *Let  $G$  be a hypercentral group generated by its quasicyclic subgroups. Then  $G$  is abelian.*

**PROOF OF PROPOSITION 2.9.** Without loss of generality we assume that all subgroups  $C_i$  are generated by elements of  $X$ . Let  $C$  be the normal closure of the subgroups  $C_1^0, \dots, C_k^0$ . It is clear that  $C$  has no subgroups of finite index. If  $C = 1$ , then the set  $X$  is finite. Since the elements of  $X$  are contained in Chernikov subgroups, it follows that all elements of  $X$  have finite order. In that case  $\langle X \rangle$  is finite by Dietzmann's Lemma on elements of finite order having finitely many conjugates (see [6, Part 1, p. 45]). So we assume that  $C \neq 1$ . In particular, we assume that  $C_1^0 \neq 1$ . Let  $K$  be a minimal infinite subgroup of  $C_1$  generated by

elements of  $X$ . Because of minimality, for every  $x \in G$  there exists  $i$  such that  $K^x \leq C_i$ . Let  $a$  be an element of  $K^0$ . It follows that every conjugate  $a^x$  belong to  $C_i^0$  for some  $i$ . Since each subgroup  $C_i^0$  has only finitely many elements of any given order, we conclude that the conjugacy class  $a^G$  is finite. Since  $C$  has no subgroups of finite index,  $a \in Z(C)$ . Thus, we have shown that  $K^0 \leq Z(C)$ . Next, we can repeat the argument with  $G$  replaced by  $G/Z(C)$  and conclude that if  $C \neq Z(C)$ , then  $Z_2(C) \neq Z(C)$ . Thus, we see that  $C$  is hypercentral. Since  $C$  is generated by quasicyclic subgroups, Lemma 2.10 tells us that  $C$  is abelian. Recall that every conjugate  $(K^0)^x$  belongs to  $C_i^0$  for some  $i$ . Hence, the normal closure  $\langle (K^0)^G \rangle$  is Chernikov. Note that the sum of sizes of the images of  $C_1, \dots, C_k$  in the quotient  $G/\langle (K^0)^G \rangle$  is strictly smaller than that of  $C_1, \dots, C_k$ . Thus, by induction, the image of  $\langle X \rangle$  in  $G/\langle (K^0)^G \rangle$  is Chernikov. Therefore  $\langle X \rangle$  is Chernikov, as desired.  $\square$

The following corollary is now straightforward.

**Corollary 2.11.** *Let  $w$  be a group-word and  $G$  a group in which the set of  $w$ -values is contained in a union of finitely many Chernikov subgroups. Then  $w(G)$  is Chernikov.*

### 3. Proof of Theorem 1.2

We will now assume the hypothesis of Theorem 1.2 with  $w = \delta_n$ . Thus,  $n$  is a positive integer and  $G$  is a group in which for any element  $g \in G$  there exist finitely many Chernikov subgroups whose union contains  $g^{G_w}$ . We denote by  $X$  the set of all  $\delta_n$ -commutators in  $G$  and by  $H$  the  $n$ th derived group of  $G$ . In other words,  $H = \langle X \rangle$ . Our goal is to prove that  $\langle g^H \rangle$  is Chernikov for all  $g \in G$ .

Let  $B$  be the subgroup of  $G$  generated by all subgroups of the form  $[T, x]$ , where  $T$  is an abelian radicable subgroup,  $x \in X$  and  $x$  normalizes  $T$ .

**Lemma 3.1.** *The subgroup  $B$  is abelian.*

PROOF. Let  $S = [T, x]$ , where  $T$  is an abelian radicable subgroup,  $x \in X$  and  $x$  normalizes  $T$ . By Lemma 2.4  $S$  is radicable. Lemma 2.3 shows that  $S = [T, x, x]$ . In view of Lemma 2.7 every element in  $[T, x, x]$  is a  $\delta_n$ -commutator. Thus,  $S$  is an abelian radicable subgroup contained in  $X$ . Choose an arbitrary element  $g \in G$ . By Lemma 2.8  $\langle g^S \rangle$  is Chernikov. It follows from Lemma 2.5 that  $[\langle g^S \rangle, S, S] = 1$ . In particular  $[g, S, S] = 1$  and so  $S$  commutes with  $S^g$ . This happens for every  $g \in G$  and therefore the normal subgroup  $\langle S^G \rangle$  is abelian. Lemma 2.2 shows that

in any group a product of normal abelian radicable periodic subgroups is abelian. Being a product of such subgroups,  $B$  is abelian.  $\square$

**Lemma 3.2.** *The quotient  $H/B$  is an  $FC$ -group.*

PROOF. Since every element of  $H$  is a product of finitely many elements from  $X$ , it is sufficient to show that under the additional hypothesis that  $B = 1$  the index  $[H : C_H(x)]$  is finite for every  $x \in X$ . Thus, we assume that  $B = 1$ . Suppose that the lemma is false and choose  $x \in X$  such that  $[H : C_H(x)]$  is infinite. Set  $Y = x^X$ . Let  $C_1, \dots, C_k$  be finitely many Chernikov subgroups such that  $Y \subseteq \cup C_i$ . Without loss of generality we assume that the subgroups  $C_1, \dots, C_k$  are chosen in such a way that the sum of the sizes of  $C_1, \dots, C_k$  is as small as possible. In that case, of course, each subgroup  $C_i$  is generated by  $C_i \cap Y$ . If the subgroups  $C_1, \dots, C_k$  were all finite, then in view of the main result of [2]  $[H : C_H(x)]$  would be finite. Thus, at least one of the subgroups  $C_1, \dots, C_k$  is infinite. Assume that  $C_1$  is infinite and let  $Y_1 = Y \cap C_1$ . For any  $y \in Y_1$  we have  $[C_1^0, y] \leq B$ . Since  $B = 1$  and  $C_1 = \langle Y_1 \rangle$ , it follows that  $C_1^0 \leq Z(C_1)$  whence, by Lemma 2.6,  $C_1$  is finite, a contradiction.  $\square$

**Lemma 3.3.** *For each  $g \in G$  the image of  $\langle g^H \rangle$  in  $G/B$  is Chernikov.*

PROOF. It follows from Lemma 3.2 that  $G$  is locally finite. Let us assume that  $B = 1$ . Then  $H$  is an  $FC$ -group and, since radicable groups have no proper subgroups of finite index, all radicable subgroups of  $H$  are contained in the center. Choose  $g \in G$  and let  $C_1, \dots, C_k$  be finitely many Chernikov subgroups such that  $g^X \subseteq \cup C_i$ . The subgroup  $J = \langle C_1^0, \dots, C_k^0 \rangle$  is Chernikov since it is generated by finitely many commuting Chernikov subgroups. Since  $g$  has finite order, it is clear that  $J_1 = \prod_i J^{g^i}$  is Chernikov, too. Set  $M = H\langle g \rangle$ . We remark that  $J_1$  is normal in  $M$ . The subgroups  $C_1, \dots, C_k$  all have finite images in  $M/J_1$  and therefore the image of the verbal conjugacy class  $g^X$  is finite. By [4, Lemma 2.9] the image of the conjugacy class  $g^H$  is finite as well. Since  $g$  is of finite order, by Dietzmann's lemma the image of  $\langle g^H \rangle$  in  $M/J_1$  is finite. Since  $J_1$  is Chernikov, the result follows.  $\square$

**Lemma 3.4.** *The subgroup  $[B, h]$  is Chernikov for every  $h \in H$ .*

PROOF. Suppose first that  $h \in X$ . Then, as we have remarked earlier,  $[B, h] \subseteq X$ . Let  $C_1, \dots, C_k$  be finitely many Chernikov subgroups such that  $h^{[B, h]} \subseteq \cup C_i$ . Then  $[B, h] = [B, h, h] \subseteq \cup (C_i \cap [B, h])$ . In view of Lemma 3.1, the subgroups  $C_i \cap [B, h]$  commute. Thus,  $[B, h]$  is contained in a union of commuting Chernikov subgroups and hence is Chernikov itself.

We now drop the assumption that  $h \in X$ . Since  $h \in H$ , we can write  $h$  as a product of several elements from  $X$ . Suppose that  $h = x_1 \cdots x_s$ , where  $x_i \in X$ . Then it is clear that  $[B, h] \leq \prod_i [B, x_i]$ . Since each  $[B, x_i]$  is Chernikov and all  $[B, x_i]$  commute, the result follows.  $\square$

**Lemma 3.5.** *Let  $A$  be a subgroup of  $H$  whose image in  $G/B$  is abelian and radicable. Then  $[B, A] = 1$ .*

PROOF. Let  $a \in A$ . Then, since  $B$  is abelian,  $A/B$  naturally acts on  $[B, a]$  and of course  $[B, a, A/B] = [B, a, A]$ . By Lemma 3.4 the subgroup  $[B, a]$  is Chernikov. According to Lemma 2.5  $[B, a, A, A] = 1$ . In particular  $[B, a, a, a] = 1$  and so Lemma 2.3 shows that  $[B, a] = 1$ . This happens for every  $a \in A$  and therefore  $[B, A] = 1$ .  $\square$

**Lemma 3.6.** *For every  $g \in G$  the subgroup  $[B, g]$  is Chernikov.*

PROOF. It was mentioned in the proof of Lemma 3.1 that if  $T$  is an abelian radicable subgroup,  $x \in X$  and  $x$  normalizes  $T$ , then  $[T, x]$  is an abelian radicable subgroup contained in  $X$ . Therefore  $B$  is the product of its subgroups  $S_1, S_2, \dots$  each of which is contained in  $X$ . Given  $g \in G$ , let  $C_1, \dots, C_k$  be finitely many Chernikov subgroups such that  $g^X \subseteq \cup C_i$  and  $B_i = C_i \cap B$  for  $i = 1, \dots, k$ . Denote by  $D$  the product of all subgroups of the form  $(B_i)^{g^j}$  for  $i \leq k$  and  $j = 0, 1, \dots$ . Since  $g$  has finite order,  $D$  is a product of finitely many commuting Chernikov subgroups and so is Chernikov itself. It is clear that  $D$  is normal in  $B\langle g \rangle$ .

Since each  $S_l$  is contained in  $X$ , it follows that  $g^{S_l} \subseteq \cup C_i$  for every  $l = 1, 2, \dots$ . We look at the image of the class  $g^{S_l}$  in the quotient  $B\langle g \rangle/D$  and conclude the image is finite since  $B$  has finite index in  $B\langle g \rangle$ . It follows that modulo  $D$  the element  $g$  centralizes a subgroup of finite index in  $S_l$ . Taking into account that  $S_l$  has no proper subgroups of finite index we conclude that  $[S_l, g] \leq D$ . This happens for every  $l = 1, 2, \dots$ . Because  $[B, g]$  is the product of subgroups of the form  $[S_l, g]$ , we have  $[B, g] \leq D$ .  $\square$

**Lemma 3.7.** *For every  $g \in G$  the subgroup  $[B, \langle g^H \rangle]$  is Chernikov.*

PROOF. Choose  $g \in G$  and set  $K = \langle g^H \rangle$  and  $C = C_K(B)$ . Then  $K/C$  naturally acts on  $B$  and  $[B, K] = [B, K/C]$ . By Lemma 3.3 the image of  $K$  in  $G/B$  is Chernikov. Let  $A$  be the subgroup of  $K \cap H$  whose image in  $G/B$  is the radicable part of the image of  $K \cap H$ . By Lemma 3.5 the subgroup  $A$  is contained in  $C$ . Obviously,  $K \cap H$  has finite index in  $K$  and therefore the index of  $A$  in  $K$  is finite. Thus,  $K/C$  is finite and so  $[B, K]$  is a product of finitely many subgroups of the form  $[B, u]$  for suitable elements  $u \in K$ . By Lemma 3.6 each of the subgroups  $[B, u]$  is Chernikov and the result follows.  $\square$



We are now ready to complete the proof of Theorem 1.2. Choose  $g \in G$  and set  $K = \langle g^H \rangle$ . By Lemma 3.7 the subgroup  $[B, K]$  is Chernikov. We remark that  $[B, K]$  is normal in  $HK$  and pass to the quotient  $\bar{V} = HK/[B, K]$ . The image of a subgroup  $T$  of  $HK$  in  $\bar{V}$  will be denoted by  $\bar{T}$ .

We have  $[\bar{B}, \bar{K}] = 1$ . It follows from Lemma 3.3 that  $\bar{K}/Z(\bar{K})$  is Chernikov. A theorem of Polovickii [6, Part 1, p. 129] now tells us that  $\bar{K}'$ , the derived group of  $\bar{K}$ , is Chernikov.

Therefore  $K'$  is Chernikov as well. The subgroup  $\langle g^X \rangle$  is generated by finitely many Chernikov subgroups and has Chernikov derived group  $\langle g^X \rangle'$ . We conclude that  $\langle g^X \rangle$  is Chernikov for all  $g \in G$ . The main theorem of [4] now tells us that  $\langle g^H \rangle$  is Chernikov for all  $g \in G$ . The proof is now complete.

## References

- [1] S. BRAZIL, A. KRASILNIKOV and P. SHUMYATSKY, Groups with bounded verbal conjugacy classes, *J. Group Theory* **9** (2006), 127–137.
- [2] S. FRANCIOSI, F. DE GIOVANNI and P. SHUMYATSKY, On group with finite verbal conjugacy classes, *Houston J. Math.* **28** (2002), 683–689.
- [3] O. H. KEGEL and B. A. F. WEHRFRITZ, Strong finiteness conditions in locally finite groups, *Math. Z.* **117** (1970), 309–324.
- [4] J. M. MUÑOZ-ESCOLANO and P. SHUMYATSKY, A finiteness condition for verbal conjugacy classes in a group, *Publ. Math. Debrecen* **82** (2013), 97–105.
- [5] A. YU. OLSHANSKII, Geometry of Defining Relations in Groups, *Kluwer Academic Publishers, Dordrecht*, 1991.
- [6] D. J. S. ROBINSON, Finiteness Conditions and Generalized Soluble Groups, *Springer, Berlin*, 1972.
- [7] J. R. ROGÉRIO and P. SHUMYATSKY, A finiteness condition for verbal subgroups, *J. Group Theory* **10** (2007), 811–815.
- [8] V. P. SHUNKOV, On the minimality problem for locally finite groups, *Algebra and Logic* **9** (1970), 137–151.
- [9] R. F. TURNER-SMITH, Finiteness conditions for verbal subgroups, *J. London Math. Soc.* **41** (1966), 166–176.

PAVEL SHUMYATSKY  
DEPARTMENT OF MATHEMATICS  
UNIVERSITY OF BRASÍLIA  
BRASÍLIA-DF, 70910-900  
BRAZIL  
E-mail: pavel@unb.br

AGENOR FREITAS DE ANDRADE  
DEPARTMENT OF ACADEMIC AREAS  
FEDERAL INSTITUTE OF GOIAS  
LUZIANIA-GO, 72811-580  
BRAZIL  
E-mail: agenor.andrade@ifg.edu.br

(Received July 20, 2015; revised December 3, 2015)