

Continued fractional algebraic independence of sequences

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There are a lot of papers concerning algebraic independence ([3], [4]), the transcendence of continued fractions, ([2], [4], [5], [11], [12], [13], [15]) and the irrationality of infinite series ([6], [7], [8], [9], [10], [14], [16], [17], [18]), however there is no criterion describing the continued fractional algebraic independence of sequences. This paper deals with such a criterion.

Definition. Let $\{a_{in}\}_{n=1}^{\infty}$ ($i = 1, 2, \dots, k$) be sequences of positive real numbers. If for every sequence $\{c_n\}_{n=1}^{\infty}$ of positive integers, the continued fractions $[a_{i1}c_1, a_{i2}c_2, \dots]$ (where $i = 1, 2, \dots, k$) are algebraically independent, then the sequences $\{a_{in}\}_{n=1}^{\infty}$ are continued fractional algebraically independent.

Theorem (BUNDSCHUH [3]). *Let $\beta_1, \dots, \beta_k$ be given complex numbers, and let $g : \mathbb{N} \rightarrow \mathbb{R}_+$ satisfy $g(n) \rightarrow \infty$ as $n \rightarrow \infty$. Suppose that for each $\tau \in \{1, \dots, t\}$ there exists an infinite set $N_\tau \subset \mathbb{N}$ and τ sequences $\{\beta_{1n}\}_{n \in N_\tau}, \dots, \{\beta_{\tau n}\}_{n \in N_\tau}$ of algebraic numbers such that for each $n \in N_\tau$ the inequalities*

$$g(n) \sum_{\sigma=1}^{\tau-1} |\beta_\sigma - \beta_{\sigma n}| < |\beta_\tau - \beta_{\tau n}| \leq \\ \leq \exp(-g(n)) [\mathbb{Q}(\beta_{1n}, \dots, \beta_{\tau n}) : \mathbb{Q}] \sum_{\sigma=1}^{\tau} \frac{s(\beta_{\sigma n})}{\partial(\beta_{\sigma n})}$$

hold, where $\partial(\beta)$ and $H(\beta)$ denote the degree and the height respectively of an algebraic number β and $s(\beta) = \partial(\beta) + \log H(\beta)$. Then $\beta_1, \dots, \beta_t$ are algebraically independent.

Theorem 1. *Let $\{a_{in}\}_{n=1}^{\infty}$ ($i = 1, \dots, k$) be sequences of positive integers. If*

$$(1) \quad \limsup (\lg(\lg a_{1n}))/n = \infty$$

and

$$(2) \quad a_{i+1,n} 2^{2^n} > a_{in} > (1 + 1/n) (a_{i+1,n} + 1)$$

for $i = 1, \dots, k - 1$ hold, then $\{a_{in}\}_{n=1}^{\infty}$ are continued fractional algebraically independent sequences.

PROOF. It is sufficient to prove that the continued fractions $\alpha_i = [a_{i1}, a_{i2}, \dots]$ are algebraically independent. If $\{c_n\}_{n=1}^{\infty}$ denotes any sequence of positive integers, and $b_{ij} = c_j a_{ij}$, then the sequences $\{b_{in}\}_{n=1}^{\infty}$ ($i = 1, \dots, k$) satisfy (1) and (2). (1) also implies that there is a monotonically increasing sequence $\{H_n\}_{n=1}^{\infty}$ of positive real numbers H_n , with $\lim_{n \rightarrow \infty} H_n = \infty$, such that $\limsup_{n \rightarrow \infty} a_{1n}^{1/H_n} = \infty$. Let us put $S_n = a_{1n}^{1/H_n}$. Thus

$$(3) \quad \limsup_{n \rightarrow \infty} S_n = \infty.$$

Then for infinitely many n we have

$$(4) \quad S_{n+1} > (1 + 1/n^2) \max_{1 \leq k \leq n} S_k.$$

If not, then for a fixed N and for every positive integer $n > N$

$$S_{n+1} < (1 + 1/n^2) \dots (1 + 1/N^2) \max_{1 \leq k \leq N} S_k.$$

This implies

$$S_{n+1} < K_1 \prod_{i=N}^{\infty} (1 + 1/i^2) = K_2,$$

a contradiction with (3). Thus (4) holds. Now for infinitely many n we

have

$$\begin{aligned}
 (5) \quad a_{1,n+1}^{1/(H_n-1)} &= S_{n+1}^{H_{n+1}^{n+1}/(H_n-1)} > \\
 &> (1 + 1/n^2)^{H_n^{n+1}/(H_n-1)} \cdot \max_{1 \leq k \leq n} S_k^{H_n^{n+1}/(H_n-1)} > \\
 &> (1 + 1/n^2)^{H_n^{n+1}/(H_n-1)} \cdot \max_{1 \leq k \leq n} S_k^{(H_n^{n+1}-1)/(H_n-1)} \geq \\
 &\geq (1 + 1/n^2)^{H_n^{n+1}/(H_n-1)} \prod_{i=1}^n \max_{1 \leq k \leq n} S_k^{H_i} \geq \\
 &\geq (1 + 1/n^2)^{H_n^{n+1}/(H_n-1)} \prod_{i=1}^n a_{1i}.
 \end{aligned}$$

Using Bundschuh's Theorem it is enough to prove that for infinitely many n and for every $j = 1, \dots, k$

$$(6) \quad g(n) \sum_{i=1}^{j-1} |\alpha_i - \alpha_{in}| < |\alpha_j - \alpha_{jn}| < H(\alpha_{1n})^{-g(n)}$$

hold, where $\alpha_{in} = [a_{i1}, \dots, a_{in}] = p_{in}/q_{in}$. It is well known that there is a constant $c = c(\alpha_1, \dots, \alpha_k)$ such that

$$(7) \quad \frac{c}{a_{i,n+1}q_{in}^2} < |\alpha_i - \alpha_{in}| < \frac{1}{a_{i,n+1}q_{in}^2}$$

(for the proof see e.g. [11] chapter 10) and

$$(8) \quad \prod_{i=1}^n a_{ji} < q_{jn} < \prod_{i=1}^n (a_{ji} + 1), \quad (j = 1, \dots, n)$$

which can be proved by mathematical induction and using

$$q_{j,n+1} = a_{n+1}q_{jn} + q_{j,n-1}.$$

(6) and (7) imply that it is sufficient to prove that for infinitely many n

$$(9) \quad g(n)ja_{j,n+1}q_{jn}^2 < c a_{j-1,n+1}q_{j-1,n}^2$$

and

$$(10) \quad q_{1n}^{g(n)} < a_{j,n+1}q_{jn}^2$$

hold. (2) and (8) imply that

$$\lim_{n \rightarrow \infty} ca_{j-1,n+1}q_{j-1,n}^2 (a_{j,n+1}q_{jn}^2)^{-1} = \infty.$$

Then we can put

$$(11) \quad g(n) = \min(H_n, ca_{j-1,n+1}q_{j-1,n}^2(a_{j,n+1}q_{jn}^2)^{-1})$$

and this immediately implies (9). (5) implies

$$a_{1,n+1} > (1 + 1/n^2)^{H_n^{n+1}} \left(\prod_{i=1}^n a_{1i} \right)^{H_n-1}.$$

Thus

$$\begin{aligned} \prod_{i=1}^{n+1} a_{1i} &> (1 + 1/n^2)^{H_n^{n+1}} \left(\prod_{i=1}^n a_{1i} \right)^{H_n} = \\ &= (1 + 1/n^2)^{H_n^{n+1}} 2^{-nH_n} \left(\prod_{i=1}^n 2a_{1i} \right)^{H_n} \geq \\ &\geq (1 + 1/n^2)^{H_n^{n+1}} 2^{-nH_n} \left(\prod_{i=1}^n (a_{1i} + 1) \right)^{H_n}. \end{aligned}$$

Using (8) we obtain

$$\prod_{i=1}^{n+1} a_{1i} \geq (1 + 1/n^2)^{H_n^{n+1}} 2^{-nH_n} \cdot q_{1n}^{H_n}.$$

This, (2) and (8) imply

$$\begin{aligned} (12) \quad &(1 + 1/n^2)^{H_n^{n+1}} 2^{-nH_n} \cdot q_{1n}^{H_n} \leq \prod_{i=1}^{n+1} a_{ji} 2^{(j-1)2^i} \leq \\ &\leq 2^{(j-1)2^{n+2}} \cdot \prod_{i=1}^{n+1} a_{ji} \leq 2^{(j-1)2^{n+2}} a_{j,n+1} q_{jn}^2 \end{aligned}$$

Now we have

$$\lim_{n \rightarrow \infty} 2^{(j-1)2^{n+2}} \cdot 2^{nH_n} \cdot (1 + 1/n^2)^{-H_n^{n+1}} = 0.$$

This, (11) and (12) imply (10) and the proof is finished.

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