

Approximately Jensen-convex functions

By NOÉMI NAGY (Miskolc)

Abstract. In this paper we show that if a function satisfies the Jensen-inequality (or the inequality describing $\mathbb{Q}$ -convexity) with an appropriate error term, then the function is Jensen-convex (without error) as well.

First we consider a function f , which is defined on an open interval I of $\mathbb{R}$. We prove that if $f : I \rightarrow \mathbb{R}$ satisfies the inequality

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} + \psi(|x-y|)$$

for every $x, y \in I$, where $\lim_{t \rightarrow 0+} \frac{\psi(t)}{t^2} = 0$, then f is Jensen-convex.

We also prove that if a real function f , which is defined on an $\mathbb{F}$ -algebraically open and $\mathbb{F}$ -convex subset D of a vector space X over $\mathbb{F}$ (where $\mathbb{F}$ is a subfield of $\mathbb{R}$), satisfies the inequality

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) + c[\lambda(1-\lambda)|x-y|]^p$$

for every $x, y \in D$ and $\lambda \in [0, 1] \cap \mathbb{F}$, with a fixed non-negative real number c and a fixed exponent $p > 1$, then it has to be $\mathbb{F}$ -convex, i.e., f satisfies the above inequality with $c = 0$ as well. Considering $\mathbb{F} = \mathbb{Q}$, we obtain another characterization of Jensen-convex functions.

Mathematics Subject Classification: 26A51, 39B62.

Key words and phrases: functional inequality, Jensen-convexity, approximate convexity. This research was (partially) carried out in the framework of the Center of Excellence of Mechatronics and Logistics at the University of Miskolc.

1. Introduction

The first paper devoted to approximately convex functions is due to HYERS and ULAM [6]. Their result motivated further investigations of approximate convexity (see, for instance, [5], [8], [9], [11], [12], [13], [17] and the references therein).

ROLEWICZ [14], [15], [16] investigated continuous real functions f satisfying the functional inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + Ct(1-t)\alpha(|x-y|) \quad (1)$$

for every $x, y \in \mathbb{R}$, $t \in [0, 1]$, with a non-negative constant C and a non-decreasing function $\alpha : [0, +\infty[\rightarrow [0, +\infty[$ fulfilling $\lim_{t \rightarrow 0+} \alpha(t)/t = 0$. In particular, he proved that under the additional assumption

$$\lim_{t \rightarrow 0+} \frac{\alpha(t)}{t^2} = 0,$$

every continuous solution f of inequality (1) is convex. Motivated by this result we deal with an analogue of (1) for Jensen-convexity (i.e., when $t = 1/2$) on an open interval without any regularity assumption.

In [1], BOROS and the present author considered the inequality

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) + c[\lambda(1-\lambda)|x-y|]^p. \quad (2)$$

It was supposed that the function f was defined on a convex, open subset D of a linear normed space, c was a fixed non-negative real number, $p > 1$ was a fixed exponent, and the inequality (2) was satisfied by every $x, y \in D$ and $\lambda \in [0, 1]$. The properties of $\mathbb{F}$ -differentiability and $\mathbb{F}$ -convexity and their connection were described by BOROS and PÁLES in [3]. Based on these results we can show for any function f , which satisfies (2) under the additional restriction $\lambda \in \mathbb{F}$, that f is $\mathbb{F}$ -convex.

2. Rolewicz theorem for approximate Jensen-convexity

For the proof of our first theorem we also have to define the difference operator Δ_h^2 by the following recursion. If I is an open interval and $f : I \rightarrow \mathbb{R}$,

let

$$\begin{aligned}\Delta_h^1 f(x) &= f(x+h) - f(x) \quad (x \in I, h \in \mathbb{R} : x+h \in I), \\ \Delta_h^2 f(x) &= \Delta_h^1 \Delta_h^1 f(x) \\ &= f(x+2h) - 2f(x+h) + f(x) \quad (x \in I, h \in \mathbb{R} : x+2h \in I),\end{aligned}$$

and we define the second order lower Dinghas interval derivative of $f : I \rightarrow \mathbb{R}$ at $\xi \in I$ as

$$\underline{D}^2 f(\xi) := \liminf_{(x,h) \rightarrow (\xi,0), x \leq \xi \leq x+2h} \frac{\Delta_h^2 f(x)}{h^2}.$$

The following statement is a particular case of a result proved by GILÁNYI and PÁLES [4, Corollary 1] (see also the details in the paragraphs preceding Proposition 2 in [2]):

Proposition 2.1. *A function $f : I \rightarrow \mathbb{R}$ is Jensen-convex if and only if $\underline{D}^2 f(\xi) \geq 0$ for every $\xi \in I$.*

The following result claims that approximate Jensen-convexity implies Jensen-convexity if the error function ψ is sufficiently small in the vicinity of zero.

Theorem 2.2. *Let $I \subset \mathbb{R}$ be an open interval, d_I be the length of the interval I , and $J_I = [0, d_I[$. Let the function $\psi : J_I \rightarrow [0, +\infty[$ satisfy*

$$\lim_{t \rightarrow 0+} \frac{\psi(t)}{t^2} = 0.$$

If a function $f : I \rightarrow \mathbb{R}$ satisfies

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} + \psi(|x-y|)$$

for all $x, y \in I$, then f is Jensen-convex.

PROOF. Let us consider $x \in I$ and a positive real number h such that $y = x+2h \in I$. Then $\frac{x+y}{2} = x+h$, $|x-y| = 2h$, and we have

$$f(x+2h) - 2f(x+h) + f(x) \geq -2\psi(2h). \quad (3)$$

Dividing by h^2 , inequality (3) can be rewritten as

$$\frac{\Delta_h^2 f(x)}{h^2} \geq -2 \frac{\psi(2h)}{h^2}. \quad (4)$$

Now let $\xi \in I$ be arbitrary and let us take the $\liminf$ on both sides of (4) as h tends to 0 and x tends to ξ such that $x \leq \xi \leq x+2h$. We obtain that

$$\underline{D}^2 f(\xi) \geq 0.$$

Applying Proposition 2.1, we get that the function f is Jensen-convex. $\square$

A similar result was established in [10, Theorem 5 and Corollary 6]. In fact, [10, Theorem 5] seems to be more general. However, its proof is more complicated as well. The short proof of our theorem allows generalizations for higher order Jensen-convexity as it appears in [2, Section 3].

3. Approximate convexity with respect to a subfield

Throughout this section, let $\mathbb{F}$ be a subfield of $\mathbb{R}$, X be a vector space over $\mathbb{F}$ and $\mathbb{F}_+ = \mathbb{F} \cap]0, +\infty[$.

In [3] BOROS and PÁLES defined the notions of $\mathbb{F}$ -algebraically openness and $\mathbb{F}$ -convexity:

Definition 3.1. A subset D of the space X is called $\mathbb{F}$ -algebraically open if, for every $x \in D$ and $u \in X$, there exists $\delta > 0$ such that $x + ru \in D$ whenever $r \in \mathbb{F} \cap]-\delta, \delta[$.

We say that D is $\mathbb{F}$ -convex if $rx + (1 - r)y \in D$ for every $x, y \in D$ and $r \in [0, 1] \cap \mathbb{F}$.

Let D be an $\mathbb{F}$ -algebraically open and $\mathbb{F}$ -convex subset of X , $c \geq 0$, and $p > 1$. We use specific differences and difference ratios in order to reformulate the assumption that a function $f : D \rightarrow \mathbb{R}$ fulfils inequality (2) for every $x, y \in D$ and $\lambda \in [0, 1] \cap \mathbb{F}$. Our first observation is that the inequality (2) is obviously satisfied if $\lambda = 0$, $\lambda = 1$, or $x = y$. It is therefore sufficient to investigate functions $f : D \rightarrow \mathbb{R}$ that fulfil inequality (2) for every $x, y \in D$ and $\lambda \in \mathbb{F} \cap]0, 1[$ such that $x \neq y$.

For our convenience, let us substitute z in the place of x in (2). Clearly, if $y, z \in D$, $y \neq z$, and $\lambda \in \mathbb{F} \cap]0, 1[$, $x = \lambda z + (1 - \lambda)y$, $u = y - z$, $s = \lambda$, and $q = 1 - \lambda$, then $s, q \in \mathbb{F}_+$, $u \in X$, $z = x - qu$, and $y = x + su$. Conversely, if $x \in D$, $u \in X$, and $q, s \in \mathbb{F}_+$ such that $z = x - qu \in X$ and $y = x + su \in X$, then $\lambda = \frac{s}{q+s} \in \mathbb{F} \cap]0, 1[$ and $x = \lambda z + (1 - \lambda)y$. Applying these substitutions, we can formulate the following proposition:

Proposition 3.2. *Let $D \subset X$ be an $\mathbb{F}$ -algebraically open and $\mathbb{F}$ -convex set, $c \geq 0$, $p > 1$. A function $f : D \rightarrow \mathbb{R}$ fulfils inequality (2) for every $x, y \in D$ and $\lambda \in [0, 1] \cap \mathbb{F}$ if and only if f satisfies the inequality*

$$f(x) \leq \frac{s}{q+s}f(x-qu) + \frac{q}{q+s}f(x+su) + c \left[\frac{qs}{q+s} \right]^p |u|^p \quad (5)$$

for every $x \in D$, $s, q \in \mathbb{F}_+$, and $u \in X$ such that $x - qu, x + su \in D$.

The proof of the following lemma is simple calculation, so it is left to the reader. We assume that D , c , p and f satisfy the assumptions of the previous proposition.

Lemma 3.3. *Suppose that $x \in D$, $s, q \in \mathbb{F}_+$, and $u \in X$ such that $x - qu$, $x + su \in D$. Then the following two inequalities are equivalent to inequality (5):*

$$\frac{f(x) - f(x - qu)}{q} \leq \frac{f(x + su) - f(x)}{s} + c \left[\frac{qs}{q + s} \right]^{p-1} |u|^p, \quad (6)$$

$$\frac{f(x) - f(x - qu)}{q} \leq \frac{f(x + su) - f(x - qu)}{q + s} + c \left[\frac{s}{q + s} \right]^p q^{p-1} |u|^p. \quad (7)$$

If we substitute a in the place of $x - qu$ in (7), we get

$$\frac{f(a + qu) - f(a)}{q} \leq \frac{f(a + (q + s)u) - f(a)}{q + s} + c \left[\frac{s}{q + s} \right]^p q^{p-1} |u|^p. \quad (8)$$

We can therefore formulate the following statement.

Lemma 3.4. *Inequality (5) holds for all $x \in D$, $s, q \in \mathbb{F}_+$ and $u \in X$ with $x - qu$, x , $x + su \in D$ if and only if inequality (8) holds for all $a \in D$, $u \in X$, $q, s \in \mathbb{F}_+$ with $a + (q + s)u \in D$.*

With the aid of the above lemmas, we can establish the main result of this section.

Theorem 3.5. *Let $D \subset X$ be an $\mathbb{F}$ -algebraically open and $\mathbb{F}$ -convex set, $c \geq 0$, $p > 1$ and $f : D \rightarrow \mathbb{R}$ such that f satisfies (2) for every $x, y \in D$ and $\lambda \in [0, 1] \cap \mathbb{F}$. Then f satisfies (2) with $c = 0$ as well, thus f is $\mathbb{F}$ -convex.*

PROOF. Let $x \in D$ and $u \in X$. We define the set $S_{\mathbb{F}}f(x, u)$ as

$$S_{\mathbb{F}}f(x, u) := \left\{ \frac{f(x + su) - f(x)}{s} : s \in \mathbb{F}_+ \text{ such that } x + su \in D \right\}$$

and we show that $S_{\mathbb{F}}f(x, u)$ is bounded from below. Let $s, q \in \mathbb{F}_+$ such that $x + su$, $x - qu \in D$. From inequality (6) we get that

$$\begin{aligned} \frac{f(x + su) - f(x)}{s} &\geq \frac{f(x) - f(x - qu)}{q} - c \left[\frac{qs}{q + s} \right]^{p-1} |u|^p \\ &\geq \frac{f(x) - f(x - qu)}{q} - cq^{p-1} |u|^p, \end{aligned}$$

which verifies the boundedness of $S_{\mathbb{F}}f(x, u)$ from below. Denote by $d_{\mathbb{F}}f(x, u)$ the infimum of $S_{\mathbb{F}}f(x, u)$, i.e., $d_{\mathbb{F}}f(x, u) := \inf S_{\mathbb{F}}f(x, u) \in \mathbb{R}$ and let $\varepsilon > 0$. Since

$$\lim_{d \rightarrow 0+} c|u|^p d^{p-1} = 0,$$

there exists $\delta > 0$ such that

$$c|u|^p \delta^{p-1} < \frac{\varepsilon}{2}.$$

Moreover, there exists $r \in \mathbb{F}_+$ such that $x + ru \in D$ and

$$\frac{f(x + ru) - f(x)}{r} < d_{\mathbb{F}}f(x, u) + \frac{\varepsilon}{2}.$$

Let $\bar{\delta} := \min\{\delta, r\} > 0$. If for $t \in \mathbb{F}$ we have that $0 < t < \bar{\delta}$, then $0 < t < r$ and writing t in the place of q , $r - t$ in the place of s and x in the place of a in inequality (8), we get

$$\begin{aligned} \frac{f(x + tu) - f(x)}{t} &\leq \frac{f(x + ru) - f(x)}{r} + c \left[\frac{r - t}{r} \right]^p t^{p-1} |u|^p \\ &\leq \frac{f(x + ru) - f(x)}{r} + c|u|^p t^{p-1} \\ &< d_{\mathbb{F}}f(x, u) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = d_{\mathbb{F}}f(x, u) + \varepsilon. \end{aligned}$$

Hence, we have

$$d_{\mathbb{F}}f(x, u) = \lim_{s \rightarrow 0, s \in \mathbb{F}_+} \frac{f(x + su) - f(x)}{s}.$$

Applying inequality (6) for $q, s \in \mathbb{F}_+$ fulfilling $x + su, x - qu \in D$, we get

$$\begin{aligned} -\frac{f(x + q(-u)) - f(x)}{q} &= \frac{f(x) - f(x - qu)}{q} \\ &\leq \frac{f(x + su) - f(x)}{s} + c \left[\frac{qs}{q + s} \right]^{p-1} |u|^p, \end{aligned}$$

consequently

$$-\lim_{q \rightarrow 0, q \in \mathbb{F}_+} \frac{f(x + q(-u)) - f(x)}{q} \leq \frac{f(x + su) - f(x)}{s} + cs^{p-1} |u|^p,$$

which yields

$$-d_{\mathbb{F}}f(x, -u) \leq \lim_{s \rightarrow 0, s \in \mathbb{F}_+} \left[\frac{f(x + su) - f(x)}{s} + cs^{p-1} |u|^p \right]$$

and thus

$$-d_{\mathbb{F}}f(x, -u) \leq d_{\mathbb{F}}f(x, u). \quad (9)$$

From inequality (9), for every $q, s \in \mathbb{F}_+$, $u \in X$ and $x \in D$, where $x - qu$, $x + su \in D$ we get the following :

$$\begin{aligned} \frac{f(x) - f(x - qu)}{q} &= -\frac{f(x + q(-u)) - f(x)}{q} \\ &\leq -d_{\mathbb{F}}f(x, -u) \leq d_{\mathbb{F}}f(x, u) \\ &\leq \frac{f(x + su) - f(x)}{s}. \end{aligned}$$

We have thus proved that f satisfies the inequality (5) with $c = 0$ (i.e., without error term) as well. Applying Proposition 3.2 also with $c = 0$, we obtain that f satisfies the inequality (2) with $c = 0$, as stated. $\square$

Remark 3.6. JENSEN [7] proved (see also [8]) that every Jensen-convex function is $\mathbb{Q}$ -convex. Hence, considering the case $\mathbb{F} = \mathbb{Q}$, our last theorem says that approximately Jensen-convex functions in the sense of (2), with $\lambda \in \mathbb{Q}$, are, in fact, Jensen-convex.

References

- [1] Z. BOROS and N. NAGY, Approximately convex functions, *Annales Univ. Sci. Budapest* **40** (2013), 143–150.
- [2] Z. BOROS and N. NAGY, Generalized Rolewicz theorem for convexity of higher order, *Math. Inequal. Appl.* **18** (2015), 1275–1281.
- [3] Z. BOROS and ZS. PÁLES, $\mathbb{Q}$ -subdifferential of Jensen-convex functions, *J. Math. Anal. Appl.* **321** (2006), 99–113.
- [4] A. GILÁNYI and ZS. PÁLES, On Dinghas-type derivatives and convex functions of higher order, *Real Anal. Exchange* **27** (2001/02), 485–494.
- [5] J. W. GREEN, Approximately convex functions, *Duke Math. J.* **19** (1952), 499–504.
- [6] D. H. HYERS and S. M. ULAM, Approximately convex functions, *Proc. Amer. Math. Soc.* **3** (1952), 821–828.
- [7] J. L. W. V. JENSEN, Sur les fonctions convexes et les inégalités entre les valeurs moyennes, *Acta Math.* **30** (1906), 175–193.
- [8] M. KUCZMA, An Introduction to the Theory of Functional Equations and Inequalities, 2nd Edition, *Birkhäuser Verlag, Basel*, 2009.
- [9] D. T. LUC, H. V. NGAI and M. THÉRA, Approximate convex functions, *J. Nonlinear Convex Anal.* **1** (2000), 155–176.
- [10] J. MAKÓ and ZS. PÁLES, Strengthening of strong and approximate convexity, *Acta Math. Hungar.* **132** (2011), 78–91.
- [11] J. MAKÓ and ZS. PÁLES, On φ -convexity, *Publ. Math. Debrecen* **80** (2012), 107–126.

- [12] C. T. NG and K. NIKODEM, On approximately convex functions, *Proc. Amer. Math. Soc.* **118** (1993), 103–108.
- [13] Zs. PÁLES, On approximately convex functions, *Proc. Amer. Math. Soc.* **131** (2003), 243–252.
- [14] S. ROLEWICZ, On γ -paraconvex multifunctions, *Math. Japon.* **24** (1979), 293–300.
- [15] S. ROLEWICZ, On $\alpha(\cdot)$ -paraconvex and strongly $\alpha(\cdot)$ -paraconvex multifunctions, *Control Cybernet.* **29** (2000), 367–377.
- [16] S. ROLEWICZ, Paraconvex analysis, *Control Cybernet.* **34** (2005), 951–965.
- [17] JA. TABOR and JÓ. TABOR, Generalized approximate midconvexity, *Control Cybernet.* **38** (2009), 655–669.

NOÉMI NAGY
DEPARTMENT OF APPLIED MATHEMATICS
UNIVERSITY OF MISKOLC
H-3515 MISKOLC, EGYETEMVÁROS U. 1.
HUNGARY

E-mail: matnagyn@uni-miskolc.hu

(Received March 26, 2015; revised January 26, 2016)