

## GK-dimension of $2 \times 2$ generic Lie matrices

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**Abstract.** Recently Machado and Koshlukov have computed the Gelfand–Kirillov dimension of the relatively free algebra  $F_m = F_m(\text{var}(sl_2(K)))$  of rank  $m$  in the variety of algebras generated by the three-dimensional simple Lie algebra  $sl_2(K)$  over an infinite field  $K$  of characteristic different from 2. They have shown that  $\text{GKdim}(F_m) = 3(m-1)$ . The algebra  $F_m$  is isomorphic to the Lie algebra generated by  $m$  generic  $2 \times 2$  matrices. Now we give a new proof for  $\text{GKdim}(F_m)$  using classical results of Procesi and Razmyslov combined with the observation that the commutator ideal of  $F_m$  is a module of the center of the associative algebra generated by  $m$  generic traceless  $2 \times 2$  matrices.

### 1. Introduction

Let  $R$  be a (not necessarily associative) algebra generated by  $m$  elements  $r_1, \dots, r_m$  over a field  $K$  and let  $V_n$  be the vector subspace of  $R$  spanned by all products  $r_{i_1} \dots r_{i_k}$ ,  $k \leq n$ . The growth function of  $R$  with respect to the given system of generators is

$$g_R(n) = \dim(V_n), \quad n \geq 0.$$

The Gelfand–Kirillov dimension of  $R$  is defined as

$$\text{GKdim}(R) = \limsup_{n \rightarrow \infty} \log_n(g_R(n)).$$

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It does not depend on the choice of the generators of  $R$ . See the book [9] for a background on GKdim. If the algebra  $R$  is graded,

$$R = \bigoplus_{n \geq 0} R^{(n)},$$

where  $R^{(n)}$  is the homogeneous component of degree  $n$  of  $R$ , then the Hilbert series of  $R$  is the formal power series

$$H(R, t) = \sum_{n \geq 0} \dim(R^{(n)}) t^n.$$

If  $R$  is generated by its homogeneous elements of first degree, then its growth function is

$$g_R(n) = \sum_{l=0}^n \dim(R^{(l)}).$$

In the general case, if  $R$  is a graded algebra generated by a finite system of (homogeneous) elements of arbitrary degree, its Gelfand–Kirillov dimension can be expressed using again its Hilbert series as

$$\text{GKdim}(R) = \limsup_{n \rightarrow \infty} \log_n \left( \sum_{l=0}^n \dim(R^{(l)}) \right).$$

When studying varieties of  $K$ -algebras  $\mathfrak{V}$ , all information for the  $m$ -generated algebras in  $\mathfrak{V}$  is carried by the relatively free algebra  $F_m(\mathfrak{V})$  of rank  $m$  in  $\mathfrak{V}$ . When the base field  $K$  is of characteristic 0, a lot is known for the Gelfand–Kirillov dimension of relatively free associative algebras, see the book [9], the survey article [4], or the paper [11]. In particular,  $\text{GKdim}(F_m(\mathfrak{V}))$  is an integer for all proper varieties of associative algebras. Almost nothing is known for relatively free Lie algebras. Using the bases of free nilpotent-by-abelian Lie algebras given by SHMELKIN [17], it is easy to see that

$$\text{GKdim}(F_m(\mathfrak{N}_c\mathfrak{A})) = \text{GKdim}(L_m/(L'_m)^{c+1}) = mc,$$

where  $m > 1$  and  $L_m$  is the free  $m$ -generated Lie algebra. Together with free nilpotent Lie algebras where the Gelfand–Kirillov dimension is equal to 0, these are the only free polynilpotent Lie algebras of finite Gelfand–Kirillov dimension, see PETROGRADSKY [12].

Recently MACHADO and KOSHLUKOV [11] have computed the Gelfand–Kirillov dimension of the relatively free algebra  $F_m = F_m(\text{var}(sl_2(K)))$  of rank

$m > 2$  in the variety of algebras generated by the three-dimensional simple Lie algebra  $sl_2(K)$  over an infinite field  $K$  of characteristic different from 2. They have shown that  $\text{GKdim}(F_m) = 3(m - 1)$ . Their proof is based on a careful analysis of the explicit expression of the Hilbert series of  $F_m$  obtained by DRENSKY [3]. The case  $m = 2$  was handled before by BAHTURIN [2] who showed that  $\text{GKdim}(F_2) = 3$ . The algebra  $F_m$  is isomorphic to the Lie algebra generated by  $m$  generic traceless  $2 \times 2$  matrices. The purpose of our paper is to give a new proof for  $\text{GKdim}(F_m)$  using classical results of PROCESI [13], [14] on Gelfand–Kirillov dimension of the algebra of generic matrices and RAZMYSLOV [16] on the weak polynomial identities of matrices, combined with the observation that the commutator ideal of  $F_m$  is a module over the center of the associative algebra generated by  $m$  generic traceless  $2 \times 2$  matrices. We believe that the present approach is more adequate for generalizations for other finite dimensional simple Lie algebras than the approach in [11].

## 2. The proof

The following statement and its corollary are folklore known. We include the proof for self-completeness of the exposition and also because we were not able to find an explicit reference.

**Lemma 1.** *Let  $R$  be a finitely generated graded algebra with Hilbert series of the form*

$$H(R, t) = h(t) \prod_{i=1}^s \frac{1}{(1 - t^{d_i})},$$

where  $h(t) \in \mathbb{C}[t]$  is a polynomial and the  $d_i$ 's are positive integers. Then the Gelfand–Kirillov dimension of  $R$  is equal to the multiplicity of 1 as a pole of  $H(R, t)$ .

PROOF. It is sufficient to consider the case when  $R$  is not finite dimensional and hence its Hilbert series has a nontrivial denominator. Let  $d$  be the least common multiple of the degrees  $d_i$ . Then

$$\begin{aligned} H(R, t) &= \sum_{n \geq 0} a_n t^n = f(t) + \sum_{p=1}^k \sum_{q=0}^{d-1} \frac{\alpha_{pq}}{(1 - \omega_q t)^p} \\ &= f(t) + \sum_{n \geq 0} \left( \sum_{p=1}^k \binom{n+p-1}{p-1} \sum_{q=0}^{d-1} \alpha_{pq} \omega_q^n \right) t^n, \end{aligned}$$

where  $f(t) \in \mathbb{C}[t]$ ,  $\alpha_{pq} \in \mathbb{C}$ ,  $\omega_0 = 1, \omega_1, \dots, \omega_{d-1}$  are the  $d$ -th roots of 1, and at least one of the coefficients  $\alpha_{kq}$  is different from zero. Since  $\omega_q^d = 1$ , the sequences

$$\beta_{pn} = \sum_{q=0}^{d-1} \alpha_{pq} \omega_q^n, \quad p = 1, \dots, k,$$

are periodic with period  $d$  and for  $n$  large enough the coefficients  $a_n$  of the Hilbert series  $H(R, t)$  are bounded by polynomials of degree  $k-1$  in  $n$ . Hence the sequence

$$\sum_{l=0}^n a_l = \sum_{l=0}^n \dim(R^{(l)})$$

needed for the definition of the Gelfand–Kirillov dimension of  $R$  is bounded by a polynomial of degree  $k$  in  $n$  and

$$\text{GKdim}(R) \leq k.$$

The asymptotics of the coefficients  $a_n$  of

$$H(R, t) = f(t) + \sum_{n \geq 0} \left( \sum_{p=1}^k \binom{n+p-1}{p-1} \beta_{pn} \right) t^n,$$

is determined by  $\beta_{kn}$ . Since  $a_n$  are positive integers, we derive that the periodic sequence  $\beta_{kn}$ ,  $n = 0, 1, 2, \dots$ , consists of nonnegative reals and at least one of them is positive. Since  $\omega_q^d = 1$ , if  $\omega_q \neq 1$ , then  $1 + \omega_q + \omega_q^2 + \dots + \omega_q^{d-1} = 0$ . Hence

$$0 < \sum_{l=0}^{d-1} \beta_{k, dn+l} = \sum_{l=0}^{d-1} \sum_{q=0}^{d-1} \alpha_{kq} \omega_q^{dn+l} = \sum_{q=0}^{d-1} \alpha_{kq} \sum_{l=0}^{d-1} \omega_q^l = d\alpha_{k0}.$$

Therefore  $\alpha_{k0} > 0$ . We consider the partial sum  $p_{dn} = a_0 + a_1 + \dots + a_{dn}$  of the coefficients of the Hilbert series  $H(R, t)$ . Its asymptotics is determined by

$$\begin{aligned} \tilde{p}_{dn} &= \sum_{c=0}^{dn} \binom{c+k-1}{k-1} \beta_{kc} \approx \frac{1}{(k-1)!} \sum_{c=0}^{dn} c^{k-1} \beta_{kc} \\ &\approx \frac{1}{(k-1)!} \sum_{e=0}^n (ed)^{k-1} \sum_{l=0}^{d-1} \beta_{k, ed+l} = \frac{d\alpha_{k0}}{(k-1)!} \sum_{e=0}^n (ed)^{k-1} \end{aligned}$$

and this is a polynomial of degree  $k$  in  $n$ . Hence

$$\begin{aligned} \text{GKdim}(R) &= \limsup_{n \rightarrow \infty} \log_n \left( \sum_{l=0}^n a_l \right) \geq \limsup_{n \rightarrow \infty} \log_n \left( \sum_{c=0}^{dn} a_c \right) \\ &= \limsup_{n \rightarrow \infty} \log_{dn} (p_{dn}) = \limsup_{n \rightarrow \infty} \log_{dn} (\tilde{p}_{dn}) = k \end{aligned}$$

which, together with the opposite inequality  $\text{GKdim}(R) \leq k$ , completes the proof.  $\square$

**Corollary 2.** *Let  $R$  be a finitely generated graded algebra and let  $C$  be a finitely generated graded subalgebra of the center of  $R$  such that  $R$  is a finitely generated  $C$ -module. Then the Gelfand–Kirillov dimension of  $R$  is equal to the multiplicity of 1 as a pole of  $H(R, t)$ .*

PROOF. By the Hilbert–Serre theorem (see e.g., [1]), the Hilbert series of any finitely generated graded module  $M$  over a finitely generated graded commutative algebra  $C$  is of the form

$$H(M, t) = h(t) \prod_{i=1}^k \frac{1}{(1 - t^{d_i})}, \quad h(t) \in \mathbb{C}[t], d_i > 0.$$

Hence the proof follows immediately from Lemma 1.  $\square$

In the sequel we assume that the base field  $K$  is of characteristic 0. Let

$$\Omega_{km} = K[Y_{km}] = K[y_{pq}^{(i)} \mid p, q = 1, \dots, k, i = 1, \dots, m]$$

be the algebra of polynomials in  $k^2 m$  commuting variables and let

$$y_i = (y_{pq}^{(i)}), \quad i = 1, \dots, m,$$

be  $m$  generic  $k \times k$  matrices. We consider the following algebras:

$R_{km}$  – the generic matrix algebra. This is the subalgebra generated by  $y_1, \dots, y_m$  of the associative  $k \times k$  matrix algebra  $M_k(\Omega_{km})$  with entries from  $\Omega_{km}$ .

$C_{km}$  – the pure trace algebra. This is the subalgebra of  $\Omega_{km}$  generated by the traces of the products,  $\text{tr}(y_{i_1} \cdots y_{i_l})$ . We embed  $C_{km}$  in  $M_k(\Omega_{km})$  by  $f(Y_{km}) \rightarrow f(Y_{km})I_k$ , where  $I_k$  is the identity matrix.

$T_{km}$  – the mixed trace algebra. This is the subalgebra of  $M_k(\Omega_{km})$  generated by  $R_{km}$  and  $C_{km}$ .

For a background on generic matrices see e.g., [14] or [7]. Below we summarize the results we need.

**Proposition 3.** *Let  $k, m \geq 2$ . Then:*

- (i) *The mixed trace algebra  $T_{km}$  has no zero divisors;*
- (ii) *The pure trace algebra  $C_{km}$  coincides with the center of  $T_{km}$ . It is finitely generated and  $T_{km}$  is a finitely generated  $C_{km}$ -module;*
- (iii) (KIRILLOV [8], PROCESI [13])

$$\text{GKdim}(T_{km}) = \text{GKdim}(C_{km}) = \text{GKdim}(R_{km}) = k^2(m-1) + 1.$$

Further, we consider the generic traceless  $k \times k$  matrices

$$z_i = (z_{pq}^{(i)}) = y_i - \frac{1}{k} \text{tr}(y_i) I_k, \quad i = 1, \dots, m,$$

and the subalgebra  $W_{km}$  of  $T_{km}$  generated by  $z_1, \dots, z_m$ , the subalgebra  $C_{km}^{(0)}$  of  $C_{km}$  generated by the traces of the products,  $\text{tr}(z_{i_1} \cdots z_{i_l})$ , and the subalgebra  $T_{km}^{(0)}$  of  $T_{km}$  generated by  $W_{km}$  and  $C_{km}^{(0)}$ . Finally, let  $L_{km}$  be the Lie subalgebra of  $W_{km}$  generated by  $z_1, \dots, z_m$ .

**Proposition 4.** *Let  $k, m \geq 2$ . Then*

- (i) (PROCESI [15])

$$T_{km} \cong K[\text{tr}(y_1), \dots, \text{tr}(y_m)] \otimes_K T_{km}^{(0)},$$

$$C_{km} \cong K[\text{tr}(y_1), \dots, \text{tr}(y_m)] \otimes_K C_{km}^{(0)};$$

- (ii) (RAZMYSLOV [16])

$$W_{km} \cong K\langle x_1, \dots, x_m \rangle / \text{Id}(M_k(K), sl_k(K))$$

where  $\text{Id}(M_k(K), sl_k(K))$  is the ideal of all weak polynomial identities in  $m$  variables for the pair  $(M_k(K), sl_k(K))$ , i.e., the polynomials in the free associative algebra  $K\langle x_1, \dots, x_m \rangle$  which vanish when evaluated on  $sl_k(K)$  considered as a subspace in  $M_k(K)$ .

- (iii) (RAZMYSLOV [16]) *The Lie algebra  $L_{km}$  is isomorphic to the relatively free algebra  $F_m(\text{var}(sl_k(K)))$  in the variety of Lie algebras generated by  $sl_k(K)$ .*

**Corollary 5.** *For  $k, m \geq 2$*

$$\text{GKdim}(T_{km}^{(0)}) = \text{GKdim}(C_{km}^{(0)}) = (k^2 - 1)(m - 1).$$

PROOF. The algebras  $T_{km}$  and  $C_{km}$  satisfy the conditions of Corollary 2. Hence the multiplicity of 1 as a pole of the Hilbert series of  $T_{km}$  and  $C_{km}$  is

equal to their Gelfand–Kirillov dimension  $k^2(m-1)+1$  (see Proposition 3 (iii)). Proposition 4 (i) gives that

$$H(T_{km}, t) = H(K[\text{tr}(y_1), \dots, \text{tr}(y_m)], t) H(T_{km}^{(0)}, t) = \frac{1}{(1-t)^m} H(T_{km}^{(0)}, t),$$

$$H(C_{km}, t) = \frac{1}{(1-t)^m} H(C_{km}^{(0)}, t).$$

Hence the multiplicity of 1 as a pole of  $H(T_{km}^{(0)}, t)$  and  $H(C_{km}^{(0)}, t)$  is equal to  $(k^2(m-1)+1) - m = (k^2-1)(m-1)$ . Both algebras  $T_{km}^{(0)}$  and  $C_{km}^{(0)}$  are finitely generated and graded. Hence the proof follows from Corollary 2.  $\square$

Now we shall summarize the information for  $2 \times 2$  generic matrices.

**Proposition 6.** *Let  $k = 2$  and  $m \geq 2$ . Then:*

(i) (SIBIRSKII [18]) *The trace polynomials*

$$\text{tr}(y_i), i = 1, \dots, m, \quad \text{tr}(y_i y_j), 1 \leq i \leq j \leq m,$$

$$\text{tr}(y_{i_1} y_{i_2} y_{i_3}), \quad 1 \leq i_1 < i_2 < i_3 \leq m,$$

*form a minimal system of generators of  $C_{2m}$ .*

(ii) (PROCESI [15]) *The algebras  $T_{2m}^{(0)}$  and  $W_{2m}$  coincide. The algebra  $C_{2m}^{(0)}$  is generated by*

$$\text{tr}(z_i z_j), 1 \leq i \leq j \leq m, \quad \text{tr}(z_{i_1} z_{i_2} z_{i_3}), 1 \leq i_1 < i_2 < i_3 \leq m,$$

*which belong to  $W_{2m}$ .*

(iii) (DRENSKY [5]) *The algebra  $C_{2m}^{(0)}$  is generated by*

$$z_i^2, i = 1, \dots, m, \quad z_i z_j + z_j z_i, 1 \leq i \leq j \leq m,$$

$$s_3(z_{i_1}, z_{i_2}, z_{i_3}), \quad 1 \leq i_1 < i_2 < i_3 \leq m,$$

*where*

$$s_3(x_1, x_2, x_3) = \sum_{\sigma \in S_3} \text{sign}(\sigma) x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)}$$

*is the standard polynomial of degree 3.*

PROOF. We shall present the proof of (ii) and (iii) as a consequence of (i). Clearly  $C_{2m}^{(0)}$  is generated by  $\text{tr}(z_i z_j)$ ,  $1 \leq i \leq j \leq m$ , and  $\text{tr}(z_{i_1} z_{i_2} z_{i_3})$ ,  $1 \leq i_1 < i_2 < i_3 \leq m$ . Now the proof of (ii) and (iii) follows immediately from the equalities in  $T_{2m}^{(0)}$

$$\text{tr}(z_1^2) = 2z_1^2, \quad \text{tr}(z_1 z_2) = z_1 z_2 + z_2 z_1, \quad \text{tr}(z_1 z_2 z_3) = \frac{1}{3} s_3(z_1, z_2, z_3)$$

which may be checked by direct verification.  $\square$

**Lemma 7.** *The commutator ideal  $L'_{2m}$  is a  $C_{2m}^{(0)}$ -module.*

PROOF. The following equalities which can be verified directly hold in  $W_{2m}$ :

$$\begin{aligned} [z_1, z_2]z_3^2 &= \frac{1}{4}([z_1, z_2, z_3, z_3] - [[z_1, z_3], [z_2, z_3]]), \\ [z_1, z_2](z_3z_4 + z_4z_3) &= \frac{1}{4}([z_1, z_2, z_3, z_4] + [z_1, z_2, z_4, z_3] \\ &\quad - [[z_1, z_3], [z_2, z_4]] - [[z_1, z_4], [z_2, z_3]]), \\ z_4s_3(z_1, z_2, z_3) &= \frac{3}{8} \sum_{\sigma \in S_3} \text{sign}(\sigma)[z_4, z_{\sigma(1)}, z_{\sigma(2)}, z_{\sigma(3)}]. \end{aligned}$$

The elements of the commutator ideal are linear combinations of (left normed) commutators  $u_i = [z_{i_1}, z_{i_2}, \dots, z_{i_n}]$ . If  $v$  is a generator of  $C_{2m}^{(0)}$ , then

$$u_i v = [z_{i_1}, z_{i_2}, \dots, z_{i_n}]v = [[z_{i_1}, z_{i_2}]v, \dots, z_{i_n}]$$

and the above equalities guarantee that  $u_i v$  is a linear combination of commutators, i.e., belongs to  $L'_{2m}$  again. Hence  $L'_{2m}C_{2m}^{(0)} \subset L'_{2m}$ .  $\square$

*Remark 8.* It is known that  $W_{2m}$  is a  $C_{2m}^{(0)}$ -module generated by 1,  $z_i$ ,  $i = 1, \dots, m$ , and  $[z_i, z_j]$ ,  $1 \leq i < j \leq m$ . Using the equality

$$[z_1, z_2, z_3] = 2(z_1(z_2z_3 + z_3z_2) - z_2(z_1z_3 + z_3z_1)),$$

as in the proof of Lemma 7 we can show that  $L'_{2m}$  is a  $C_{2m}^{(0)}$ -module generated by all commutators  $[z_i, z_j]$  and  $[z_{i_1}, z_{i_2}, z_{i_3}]$ . For  $m = 2$ , the commutator ideal  $L'_{22}$  is a free  $C_{22}^{(0)}$ -module generated by  $[z_1, z_2]$ ,  $[z_1, z_2, z_1]$ ,  $[z_1, z_2, z_2]$ , see [6].

The proof of the following theorem established in [11] is the main result of our paper.

**Theorem 9.** *Let  $K$  be a field of characteristic 0 and let  $L_{2m}$  be the Lie algebra generated by  $m$  generic traceless  $2 \times 2$  matrices,  $m \geq 2$ . Then*

$$\text{GKdim}(L_{2m}) = \text{GKdim}(F_m(\text{var}(sl_2(K))) = 3(m-1).$$

PROOF. Let

$$H(C_{2m}^{(0)}, t) = \sum_{n \geq 0} c_n t^n, \quad H(L_{2m}, t) = \sum_{n \geq 1} l_n t^n, \quad H(W_{2m}, t) = \sum_{n \geq 1} w_n t^n$$



be the Hilbert series of  $C_{2m}^{(0)}$ ,  $L_{2m}$ , and  $W_{2m}$ , respectively. Since the algebra  $L_{2m}$  is finitely generated, its Gelfand–Kirillov dimension is

$$\mathrm{GKdim}(L_{2m}) = \limsup_{n \rightarrow \infty} \log_n \left( \sum_{k=1}^n l_k \right).$$

The algebra  $W_{2m}$  has no zero divisors and hence  $[z_1, z_2]C_{2m}^{(0)} \subset L'_{2m} \subset L_{2m}$  is a free  $C_{2m}^{(0)}$ -module. Therefore

$$\sum_{k=0}^{n-2} c_k \leq \sum_{k=1}^n l_k \leq \sum_{k=0}^n w_k,$$

which implies that

$$3(m-1) = \mathrm{GKdim}(C_{2m}^{(0)}) \leq \mathrm{GKdim}(L_{2m}) \leq \mathrm{GKdim}(W_{2m}) = 3(m-1). \quad \square$$

*Remark 10.* As in [11], the formula for the Gelfand–Kirillov dimension of  $F_m(\mathrm{var}(sl_2(K)))$  obtained in characteristic 0 holds also for any infinite field  $K$  of characteristic different from 2.

*Remark 11.* In characteristic 2, the algebra  $sl_2(K)$  is nilpotent of class 2 and hence  $F_m(\mathrm{var}(sl_2(K)))$  is isomorphic to the free nilpotent of class 2 Lie algebra  $F_m(\mathfrak{N}_2)$  which is finite dimensional. Therefore  $\mathrm{GKdim}(F_m(\mathrm{var}(sl_2(K)))) = 0$ . When  $K$  is an infinite field of characteristic 2, a much more interesting object is the relatively free algebra  $F_m(\mathrm{var}(M_2(K)^{(-)}))$  of the variety generated by the  $2 \times 2$  matrix algebra  $M_2(K)$  considered as a Lie algebra. VAUGHAN-LEE [19] showed that the algebra  $M_2(K)^{(-)}$  does not have a finite basis of its polynomial identities. (It is easy to see that the four-dimensional Lie algebra constructed in [19] is isomorphic to  $M_2(K)^{(-)}$ .) The algebra  $M_2(K)^{(-)}$  satisfies the center-by-metabelian polynomial identity

$$[[[x_1, x_2], [x_3, x_4]], x_5] = 0.$$

It is well known that the free center-by-metabelian Lie algebra  $F_m([\mathfrak{A}^2, \mathfrak{E}])$  over any field  $K$  is spanned by

$$[x_{i_1}, x_{i_2}, x_{i_3}, \dots, x_{i_n}], \quad [[x_{i_1}, x_{i_2}, x_{i_3}, \dots, x_{i_n}], [x_{i_{n+1}}, x_{i_{n+2}}]],$$

where  $i_1 > i_2 \leq i_3 \leq \dots \leq i_n$  and the commutators are left normed, e.g.,  $[x_1, x_2, x_3] = [[x_1, x_2], x_3]$ . (A basis of  $F_m([\mathfrak{A}^2, \mathfrak{E}])$  is given by KUZMIN [10].) Since

the commutators  $[x_{i_1}, x_{i_2}, x_{i_3}, \dots, x_{i_n}]$  form a basis of the free metabelian Lie algebra  $F_m(\mathfrak{A}^2)$  and are linearly independent in  $F_m(\text{var}(M_2(K)^{(-)}))$ , we obtain immediately that

$$\text{GKdim}(F_m(\text{var}(M_2(K)^{(-)}))) = m, \quad m > 1.$$

In characteristic 2 there is another three-dimensional simple Lie algebra which is an analogue of the Lie algebra of the three-dimensional real vector space with the vector multiplication. It is interesting to see whether this algebra has a finite basis of its polynomial identities (probably not) and, when the field is infinite, to compute the Gelfand–Kirillov dimension of the corresponding relatively free algebras.

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