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On p-hypercyclically embedded subgroups of finite groups

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Abstract. Let G be a finite group and p a prime. A normal subgroup E of G is said to be p-hypercyclically embedded in G if every p-chief factor of G below E is cyclic. We say that a subgroup H of G is generalized $S\Phi$-supplemented in G if G has a subnormal subgroup T such that $G = HT$ and $(H \cap T)H_{sG}/H_{sG} \leq \Phi(H/H_{sG})$, where H_{sG} is the subgroup of H generated by all those subgroups of H which are s-permutable in G. In this paper, some new characterizations of p-hypercyclically embeddability of normal subgroups of a finite group are obtained based on the assumption that some primary subgroups are generalized $S\Phi$-supplemented in G.

1. Introduction

Throughout this paper, all groups considered are finite. G always denotes a group, p denotes a prime, and $|G|_p$ denotes the order of a Sylow p-subgroup of G.

A normal subgroup E of G is said to be hypercyclically embedded (resp. p-hypercyclically embedded) in G if every chief factor (resp. p-chief factor) of G below E is cyclic. The hypercyclically embedded subgroups have a great influence on the structure of a group, and some important classes of groups can be characterized in terms of hypercyclically embedded subgroups. For example, if all subgroups of G of prime order or order 4 are hypercyclically embedded in G,

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then G is supersoluble (Huppert [12], Doerk [5], see also [23]). A group G is quasisupersoluble (i.e. for every non-cyclic chief factor H/K of G, every automorphism of H/K induced by an element of G is inner) if and only if it has a normal hypercyclically embedded subgroup E such that G/E is semisimple (see [10, Theorem C]). Some recent results in this topic can be found in, for example, [2], [9], [11], [22], [23], [24], [25].

Recall that a subgroup H of G is said to be s-permutable in G if H permutes with every Sylow subgroup of G. A subgroup H of G is said to be weakly s-permutable in G [21] if G has a subnormal subgroup T such that $G = HT$ and $H \cap T \leq H_{sG}$, where H_{sG} is the subgroup of H generated by all those subgroups of H which are s-permutable in G. A subgroup H of a group G is called $S\Phi$-supplemented [17] (or Φ-s-supplemented [16]) in G if there exists a subnormal subgroup T of G such that $G = HT$ and $H \cap T \leq \Phi(H)$, where $\Phi(H)$ is the Frattini subgroup of H. Note that H_{sG} is normal in H. We now introduce the following concept which is closely related to the above two concepts.

Definition 1.1. A subgroup H of G is said to be generalized $S\Phi$-supplemented in G if there exists a subnormal subgroup T of G such that $G = HT$ and $(H \cap T)H_{sG}/H_{sG} \leq \Phi(H/H_{sG})$.

It is easy to see that weakly s-permutable subgroups and $S\Phi$-supplemented subgroups of G are all generalized $S\Phi$-supplemented in G. But the following examples show that the converse does not hold in general.

Example 1.2. Let $G = Q_8 = \langle a, b \mid a^4 = 1,\ a^2 = b^2,\ b^{-1}ab = a^{-1} \rangle$ and $H = \langle b^2 \rangle$. Then, clearly, H is s-permutable in G and H has the unique supplement G in G. Hence H is generalized $S\Phi$-supplemented in G. But H is not $S\Phi$-supplemented in G because $\Phi(H) = 1$.

Example 1.3. Let $G = S_5$ be the symmetric group of degree 5 and $H = \langle (1234) \rangle$. Then $H_{sG} = H_G = 1$. Since $G = HA_5$ and $H \cap A_5 = \Phi(H) = \langle (13)(24) \rangle$, H is generalized $S\Phi$-supplemented in G, but H is not weakly s-permutable in G.

A class of groups $\mathfrak{F}$ is called a formation if it is closed under taking homomorphic images and subdirect products. The $\mathfrak{F}$-residual of G, denoted by $G^{\mathfrak{F}}$, is the smallest normal subgroup of G with quotient in $\mathfrak{F}$. Let $Z_{\mathfrak{F}}(G)$ (resp. $Z_{p\mathfrak{F}}(G)$) denote the $\mathfrak{F}$-hypercentre (resp. $p\mathfrak{F}$-hypercentre) of G, that is, the product of all normal subgroups H of G such that all chief factors (resp. p-chief factors) L/K of G below H is $\mathfrak{F}$-hypercentral (i.e. $L/K \rtimes G/C_G(L/K) \in \mathfrak{F}$ (see [8, Chap. 1])). Let $\mathfrak{U}$ denote the classes of all supersoluble groups. Then $Z_{\mathfrak{U}}(G)$ (resp. $Z_{p\mathfrak{U}}(G)$) is the product of all normal hypercyclically embedded (resp. p-hypercyclically embedded) subgroups of G. Moreover, the generalized Fitting subgroup $F^*(G)$ (resp. the generalized p-Fitting subgroup $F_p^*(G)$) of G is the maximal quasinilpotent subgroup (resp. the maximal p-quasinilpotent subgroup) of G (for details, see [14, Chap. X] and [15]).

In the present paper, we will give a new characterization of p-hypercyclically embedded subgroups of G by using the generalized $S\Phi$-supplemented subgroups. Our main result is the following.

Theorem 1.4. *Let E and X be normal subgroups of G such that $F_p^*(E) \leq X \leq E$ and P a Sylow p-subgroup of X. If P has a subgroup D such that $1 < |D| < |P|$, and all subgroups H of P with $|H| = |D|$ and all cyclic subgroups of P of order 4 (when P is a non-abelian 2-group and $|D| = 2$) are generalized $S\Phi$-supplemented in G, then $E \leq Z_{p\mathfrak{U}}(G)$.*

The following example illustrates that the converse of Theorem 1.4 does not hold.

Example 1.5. Let $G = \langle a, b \mid a^5 = 1, b^4 = 1, b^{-1}ab = a^2\rangle$ and $H = \langle b^2\rangle$. Then clearly, G is 2-supersoluble, $H_{sG} = 1$ and $\Phi(H) = 1$. If H is generalized $S\Phi$-supplemented in G, then there exists a subnormal subgroup T of G such that $G = HT$ and $H \cap T = 1$. This implies that $\langle b\rangle = H(\langle b\rangle \cap T)$, and so $H \leq \langle b\rangle \leq T$. This contradiction shows that H is not generalized $S\Phi$-supplemented in G.

The proof of Theorem 1.4 consists of a large number of steps. The following propositions are the main stages of it.

Proposition 1.6. *Let P be a normal p-subgroup of G. If P has a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with $|H| = |D|$ and all cyclic subgroups of P of order 4 (when P is a non-abelian 2-group and $|D| = 2$) are generalized $S\Phi$-supplemented in G, then $P \leq Z_{\mathfrak{U}}(G)$.*

Proposition 1.7. *Let E be a normal subgroup of G and P a Sylow p-subgroup of E. If every cyclic subgroup of P of order p or 4 (when P is a non-abelian 2-group) is generalized $S\Phi$-supplemented in G, then $E \leq Z_{p\mathfrak{U}}(G)$.*

Proposition 1.8. *Let E be a normal subgroup of G and P a Sylow p-subgroup of E. If every maximal subgroup of P is generalized $S\Phi$-supplemented in G, then either $E \leq Z_{p\mathfrak{U}}(G)$ or $|E|_p = p$.*

Note that Propositions 1.6–1.8 are independently interesting since they cover main results of many papers among which one can find recent publications (for example, [17], [16], [19]). We prove Theorem 1.4 and Propositions 1.6–1.8 in Section 3. Some applications of these results will be discussed in Section 4.

All unexplained notation and terminology are standard, as in [6], [7], [8].

2. Preliminaries

Lemma 2.1 (see [8, Chap. 1, Lemma 5.34]). *Let $H \leq G$, $K \leq G$ and $N \trianglelefteq G$.*

(1) *If H is s-permutable in G, then H is subnormal in G.*

(2) *If H is s-permutable in G, then HN/N is s-permutable in G/N.*

(3) *If H is a p-group, then H is s-permutable in G if and only if $O^p(G) \leq N_G(H)$.*

(4) *If H is s-permutable in G, then $H \cap K$ is s-permutable in K.*

Lemma 2.2 (see [21, Lemma 2.8] or [8, Chap. 3, Lemma 3.6]). *Let $H \leq K \leq G$. Then:*

(1) *H_{sG} is an s-permutable subgroup of G;*

(2) *$H_{sG} \leq H_{sK}$;*

(3) *If $H \trianglelefteq G$, then $(K/H)_{s(G/H)} = K_{sG}/H$.*

Lemma 2.3. *Let $H \leq K \leq G$ and $N \trianglelefteq G$. Suppose that H is generalized $S\Phi$-supplemented in G. Then:*

(1) *H is generalized $S\Phi$-supplemented in K.*

(2) *If either $N \leq H$ or $(|H|, |N|) = 1$, then HN/N is generalized $S\Phi$-supplemented in G/N.*

Proof. By the hypothesis, G has a subnormal subgroup T such that $G = HT$ and$(H \cap T)H_{sG}/H_{sG} \leq \Phi(H/H_{sG})$. Let $V/H_{sG} = \Phi(H/H_{sG})$.

(1) By Dedekind's identity, $K = H(T \cap K)$. Then by Lemma 2.2(2), $H_{sG} \leq H_{sK}$, and so $(H \cap T)H_{sK}/H_{sK} \leq VH_{sK}/H_{sK} \leq \Phi(H/H_{sK})$. Hence H is generalized $S\Phi$-supplemented in K.

(2) Clearly, $G/N = (HN/N)(TN/N)$ and $H_{sG}N/N \leq (HN/N)_{sG} = (HN)_{sG}/N$ by Lemma 2.2(3). Also, by Lemma 2.1(4), $(HN)_{sG} = ((HN)_{sG} \cap H)N \leq H_{sG}N$. This implies that $(HN)_{sG} = H_{sG}N$. Since either $N \leq H$ or $(|H|, |N|) = 1$, $HN \cap TN = (H \cap T)N$, and so $(HN \cap TN)(HN)_{sG}/(HN)_{sG} \leq VN/H_{sG}N \leq \Phi(HN/H_{sG}N)$. This shows that HN/N is generalized $S\Phi$-supplemented in G/N. $\square$

Let P be a p-group. If P is not a non-abelian 2-group, then we use $\Omega(P)$ to denote the subgroup $\Omega_1(P)$. Otherwise, $\Omega(P) = \Omega_2(P)$.

Lemma 2.4 (see [4, Lemma 2.12]). *Let P be a normal p-subgroup of G and C a Thompson critical subgroup of P* (see [7, p. 186]). *If $P/\Phi(P) \leq Z_{\mathfrak{U}}(G/\Phi(P))$ or $C \leq Z_{\mathfrak{U}}(G)$ or $\Omega(P) \leq Z_{\mathfrak{U}}(G)$, then $P \leq Z_{\mathfrak{U}}(G)$.*

Lemma 2.5 (see [3, Lemma 2.10]). *Let C be a Thompson critical subgroup of a nontrivial p-group P.*

(1) *If p is odd, then the exponent of $\Omega_1(C)$ is p.*

(2) *If P is an abelian 2-group, then the exponent of $\Omega_1(C)$ is 2.*

(3) *If $p = 2$, then the exponent of $\Omega_2(C)$ is at most 4.*

Lemma 2.6 (see [1, Theorem 2.1.6]). *Let G be a p-supersoluble group. Then the derived subgroup G' of G is p-nilpotent. In particular, if $O_{p'}(G) = 1$, then G has a unique Sylow p-subgroup.*

Lemma 2.7 (see [25, Lemma 2.13]). *Let $\mathfrak{F}$ be a formation and E a normal subgroup of G. Then $E \le Z_{p\mathfrak{F}}(G)$ if and only if $F_p^*(E) \le Z_{p\mathfrak{F}}(G)$.*

Lemma 2.8 (see [20, Lemma 2.6]). *Let V be an s-permutable subgroup of G of order 4.*

(1) *If $V = A \times B$, where $|A| = |B| = 2$ and A is s-permutable in G, then B is s-permutable in G.*

(2) *If $V = \langle x \rangle$ is cyclic, then $\langle x^2 \rangle$ is s-permutable in G.*

Lemma 2.9 (see [22, Theorem B]). *Let $\mathfrak{F}$ be any formation and E a normal subgroup of G. If $F^*(E) \le Z_{\mathfrak{F}}(G)$, then $E \le Z_{\mathfrak{F}}(G)$.*

3. Proof of main results

For a p-subgroup H of G, we know that $\Phi(H/H_{sG}) = \Phi(H)H_{sG}/H_{sG}$ (see [13, Chap. 3, Theorem 3.14(c)]). Therefore, if H is a generalized $S\Phi$-supplemented p-subgroup of G, then there exists a subnormal subgroup T of G such that $G = HT$ and $H \cap T \le \Phi(H)H_{sG}$.

Proof of Proposition 1.6. Suppose that the assertion is false and let (G, P) be a counterexample for which $|G| + |P|$ is minimal. Then:

(1) $|D| > p$.

If $|D| = p$, then by the hypothesis, every cyclic subgroup of P of order p or 4 (when P is a non-abelian 2-group) is generalized $S\Phi$-supplemented in G. Let P/R be a chief factor of G. Clearly, (G, R) satisfies the hypothesis of the proposition. The choice of (G, P) implies that $R \le Z_{\mathfrak{U}}(G)$. If $|P/R| = p$, then $P \le Z_{\mathfrak{U}}(G)$, a contradiction. Hence $|P/R| > p$. Suppose that $L \trianglelefteq G$ and $L < P$. Then, similarly as above, we have that $L \le Z_{\mathfrak{U}}(G)$. If $L \nleq R$, then $P = RL \le Z_{\mathfrak{U}}(G)$, a contradiction. Hence $L \le R$. This shows that G has a unique normal subgroup R

such that P/R is a chief factor of G. Let C be a Thompson critical subgroup of P. Note that C is characteristic in P (see [7, Chap. 5, Theorem 3.11]). If $\Omega(C) < P$, then $\Omega(C) \leq R \leq Z_{\mathfrak{U}}(G)$. It follows from Lemma 2.4 that $P \leq Z_{\mathfrak{U}}(G)$, which is impossible. Hence $P = C = \Omega(C)$. Then by Lemma 2.5, the exponent of P is p or 4 (when P is a non-abelian 2-group).

Obviously, $P/R \cap Z(G_p/R) > 1$, where G_p is a Sylow p-subgroup of G. Suppose that $V/R \leq P/R \cap Z(G_p/R)$ and $|V/R| = p$. Let $x \in V \backslash R$ and $H = \langle x \rangle$. Then $V = HR$ and $|H| = p$ or 4. If $H = H_{sG}$, then by Lemma 2.2(1), H is s-permutable in G, and so $V/R = HR/R \trianglelefteq G/R$ by Lemma 2.1(2)(3). But since P/R is a chief factor of G, we have that $P = V$. It follows that $P/R = V/R$ is cyclic, and so $P \leq Z_{\mathfrak{U}}(G)$, a contradiction. Hence $H \neq H_{sG}$ and so $H_{sG} \leq \Phi(H)$. By the hypothesis, there exists a subnormal subgroup T of G such that $G = HT$ and $H \cap T \leq \Phi(H)$. In this case, $P \cap T < P$, and so $(P \cap T)^G = (P \cap T)^P < P$. This means that $(P \cap T)^G \leq R$, and so $P = H(P \cap T) = HR = V$, also a contradiction. Hence $|D| > p$.

(2) $|D| < |P|/p$.

Suppose that $p|D| = |P|$. By the hypothesis, every maximal subgroup of P is generalized $S\Phi$-supplemented in G. Let N be a minimal normal subgroup of G contained in P. Then by Lemma 2.3(2), $(G/N, P/N)$ satisfies the hypothesis of the proposition. The choice of (G, P) yields that $P/N \leq Z_{\mathfrak{U}}(G/N)$. If $|N| = p$, then $P \leq Z_{\mathfrak{U}}(G)$, which is impossible. Hence $|N| > p$. Suppose that G has another minimal normal subgroup L contained in P such that $N \neq L$. With a similar discussion as above, we have that $P/L \leq Z_{\mathfrak{U}}(G/L)$. It follows that $NL/L \leq Z_{\mathfrak{U}}(G/L)$, and so $|N| = p$, a contradiction. Thus G has a unique minimal normal subgroup N contained in P.

If $\Phi(P) = 1$, then P is elementary abelian. Let N_1 be a maximal subgroup of N such that N_1 is normal in some Sylow p-subgroup G_p of G, and let S be a complement of N in P. Then $P_1 = N_1S$ is a maximal subgroup of P. By [13, Chap. 3, Lemma 3.3], $\Phi(P_1) \leq \Phi(P) = 1$. Therefore, there exists a subnormal subgroup T of G such that $G = P_1T$ and $P_1 \cap T \leq (P_1)_{sG}$. Then $G = PT$ and $P = P_1(P \cap T)$. It is easy to see that $1 \neq P \cap T \trianglelefteq G$. Hence $N \leq P \cap T$, and so $P_1 \cap N \leq P_1 \cap T \leq (P_1)_{sG}$. It follows that $N_1 = P_1 \cap N = (P_1)_{sG} \cap N$ is s-permutable in G. By Lemma 2.1(3), $N_1 \trianglelefteq G$, and so $|N| = p$, a contradiction. Thus $\Phi(P) \neq 1$. Then $N \leq \Phi(P)$. Since $P/N \leq Z_{\mathfrak{U}}(G/N)$, $P/\Phi(P) \leq Z_{\mathfrak{U}}(G/\Phi(P))$. Applying Lemma 2.4, we obtain that $P \leq Z_{\mathfrak{U}}(G)$. The contradiction completes the proof of (2).

(3) *Final contradiction.*

We shall show that all subgroups H of P with $|H| = |D|$ are s-permutable in G. By the hypothesis, G has a subnormal subgroup T such that $G = HT$ and $H \cap T \leq \Phi(H)H_{sG}$. If $T < G$, then there exists a normal subgroup M of G such that $T \leq M$ and $|G : M| = p$. Since $|P : P \cap M| = |PM : M| = p$, $P \cap M$ is a maximal subgroup of P and so $|D| < |P \cap M|$ by (2). Clearly, $P \cap M \trianglelefteq G$. Then $(G, P \cap M)$ satisfies the hypothesis of the proposition. The choice of (G, P) yields that $P \cap M \leq Z_{\mathfrak{U}}(G)$. Consequently, $P \leq Z_{\mathfrak{U}}(G)$, which is impossible. Hence $T = G$. This implies that $H = H_{sG}$ is s-permutable in G by Lemma 2.2(1). Then by [24, Theorem], $P \leq Z_{\mathfrak{U}}(G)$. The final contradiction ends the proof. □

PROOF OF PROPOSITION 1.7. Suppose that the assertion is false and let (G, E) be a counterexample for which $|G| + |E|$ is minimal. We now proceed via the following steps.

(1) $O_{p'}(E) = 1$.

If $O_{p'}(E) \neq 1$, then by Lemma 2.3(2), $(G/O_{p'}(E), E/O_{p'}(E))$ satisfies the hypothesis of the proposition. The choice of (G, E) implies that $E/O_{p'}(E) \leq Z_{p\mathfrak{U}}(G/O_{p'}(E)) = Z_{p\mathfrak{U}}(G)/O_{p'}(E)$, and so $E \leq Z_{p\mathfrak{U}}(G)$, a contradiction. Hence $O_{p'}(E) = 1$.

(2) $E = G$.

Suppose that $E < G$. Then by Lemma 2.3(1), (E, E) satisfies the hypothesis of the proposition. The choice of (G, E) implies that E is p-supersoluble. By (1) and Lemma 2.6, we see that $P \trianglelefteq E$. Thus $P \trianglelefteq G$. Then by Proposition 1.6, we have $P \leq Z_{\mathfrak{U}}(G)$. Consequently, $E \leq Z_{p\mathfrak{U}}(G)$, which is absurd. Therefore, $E = G$.

(3) *$Z_{p\mathfrak{U}}(G)$ is the unique normal subgroup of G such that $G/Z_{p\mathfrak{U}}(G)$ is a chief factor of G, $G^{\mathfrak{U}} = G$ and $O_p(G) = Z(G) = Z_{\mathfrak{U}}(G)$ is the Sylow p-subgroup of $Z_{p\mathfrak{U}}(G)$.*

Let G/K be a chief factor of G. Obviously, (G, K) satisfies the hypothesis of the proposition. By the choice of the (G, E), $K \leq Z_{p\mathfrak{U}}(G)$, and so $K = Z_{p\mathfrak{U}}(G)$. This shows that $Z_{p\mathfrak{U}}(G)$ is the unique normal subgroup of G such that $G/Z_{p\mathfrak{U}}(G)$ is a chief factor of G. By Proposition 1.6, $O_p(G) \leq Z_{\mathfrak{U}}(G) \leq Z_{p\mathfrak{U}}(G)$. Then by (1), (2) and Lemma 2.6, $O_p(G)$ is the Sylow p-subgroup of $Z_{p\mathfrak{U}}(G)$. If $G^{\mathfrak{U}} < G$, then $G^{\mathfrak{U}} \leq Z_{p\mathfrak{U}}(G)$. So $G^{\mathfrak{U}} \cap O_p(G)$ is the Sylow p-subgroup of $G^{\mathfrak{U}}$. Let $P_1 = G^{\mathfrak{U}} \cap O_p(G)$. Note that $(G/P_1)/(G^{\mathfrak{U}}/P_1)$ is supersoluble and $G^{\mathfrak{U}}/P_1$ is a p'-group. Hence G/P_1 is p-supersoluble, and so G is p-supersoluble because $P_1 \leq Z_{\mathfrak{U}}(G)$, a contradiction. Thus $G^{\mathfrak{U}} = G$. It follows from [6, Chap. IV, Theorem 6.10] that

$Z_{\mathfrak{U}}(G) \leq Z(G)$. Since $O_{p'}(Z(G)) \leq O_{p'}(G) = 1$ by (1) and (2), $Z(G) \leq O_p(G)$. Therefore, $O_p(G) = Z(G) = Z_{\mathfrak{U}}(G)$.

(4) *Final contradiction.*

By (3), we have that $G' = G$. If P is abelian, then by (3) and [13, Chap. VI, Theorem 14.3], $Z(G)=1$. Hence by (3), $Z_{p\mathfrak{U}}(G)$ is a p'-group. Then by (1) and (2), $Z_{p\mathfrak{U}}(G) = 1$, and so G is simple by (3) again. Let x be an element of G of order p. Then by the hypothesis, G has a subnormal subgroup T such that $G = \langle x\rangle T$ and $\langle x\rangle \cap T \leq \langle x\rangle_{sG}$. In this case, clearly, $T = G$ and so $\langle x\rangle$ is s-permutable in G by Lemma 2.2(1). Then $\langle x\rangle$ is subnormal in G by Lemma 2.1(1). So $G = \langle x\rangle$, which is impossible. Thus P is non-abelian.

By [13, Chap. IV, Satz 5.5], we see that there exists a cyclic subgroup H of P of order p or 4 which is not contained in $Z(G)$. Then by the hypothesis, H is generalized $S\Phi$-supplemented in G. Thus G has a subnormal subgroup T such that $G = HT$ and $H \cap T \leq \Phi(H)H_{sG}$. If $T < G$, then G has a normal subgroup M such that $T \leq M$ and $|G : M| = p$. It is easy to see that (G, M) satisfies the hypothesis of the proposition. The choice of (G, E) implies that $M \leq Z_{p\mathfrak{U}}(G)$, and so $G \leq Z_{p\mathfrak{U}}(G)$, which is impossible. Hence $T = G$. Then $H = H_{sG}$ is s-permutable in G by Lemma 2.2(1). Since $H \nleq Z(G)$ and $Z(G)$ is the Sylow p-subgroup of $Z_{p\mathfrak{U}}(G)$ by (3), $H \nleq Z_{p\mathfrak{U}}(G)$. Hence by (3) and Lemma 2.1(3), we have that $G = (HZ_{p\mathfrak{U}}(G))^G = (HZ_{p\mathfrak{U}}(G))^P \leq PZ_{p\mathfrak{U}}(G)$. But since $G/Z_{p\mathfrak{U}}(G)$ is a chief factor of G, $|G/Z_{p\mathfrak{U}}(G)| = p$. This shows that G is p-supersoluble, a contradiction. This completes the proof. □

Proof of Proposition 1.8. Suppose that the assertion is false and let (G, E) be a counterexample for which $|G| + |E|$ is minimal. Then:

(1) $O_{p'}(E) = 1$ *and* $E = G$.

See steps (1) and (2) in the proof of Proposition 1.7.

(2) *Let* N *be a minimal normal subgroup of* G. *Then either* G/N *is* p-*supersoluble or* $|G/N|_p = p$.

Suppose that M/N is a maximal subgroup of PN/N. Then there exists a maximal subgroup P_1 of P such that $M = P_1N$ and $P \cap N = P_1 \cap N$. By the hypothesis, G has a subnormal subgroup T such that $G = P_1T$ and $P_1 \cap T \leq \Phi(P_1)(P_1)_{sG}$. Clearly, $(|N : P_1 \cap N|, |N : T \cap N|) = 1$. Hence $N = (P_1 \cap N)(T \cap N)$, and so $P_1N \cap TN = (P_1 \cap T)N$. By discussing similarly as in the proof of Lemma 2.3(2), $M/N = P_1N/N$ is generalized $S\Phi$-supplemented in G/N. This shows that $(G/N, G/N)$ satisfies the hypothesis of the proposition. The choice of (G, E) implies that either G/N is p-supersoluble or $|G/N|_p = p$. Hence (2) holds.

(3) *If* $PN < G$, *then* $N \leq O_p(G)$.

By Lemma 2.3(1), (PN, PN) satisfies the hypothesis of the proposition, and so the choice of (G, E) implies that either PN is p-supersoluble or $|PN|_p = p$. Then by (1), $N \leq O_p(G)$.

(4) N *is the unique minimal normal subgroup of* G.

Let N and L be two distinct minimal normal subgroups of G. By (2), we may discuss the following three possible cases.

(i) If G/N and G/L are all p-supersoluble, then G is p-supersoluble, a contradiction.

(ii) Without loss of generality, we may assume that G/N is p-supersoluble and $|G/L|_p = p$. Since LN/N is a minimal normal subgroup of G/N and $p||L|$ by (1), $|L| = |LN/N| = p$, and so $|P| = p^2$. Then by (1), $|N|_p = |P \cap N| = p$ and N is a non-abelian simple group. Let $N_1 = P \cap N$. Then $(N_1)_{sG} = 1$ by Lemma 2.1(1). By the hypothesis, N_1 is generalized $S\Phi$-supplemented in G. Thus G has a subnormal subgroup T such that $G = N_1T$ and $N_1 \cap T = 1$. Thus $T \trianglelefteq G$. It follows that either $N \cap T = 1$ or $N \leq T$. For the former case, we have $N = N \cap N_1T = N_1$, a contradiction. For the latter case, it follows that $N_1 = 1$, which is impossible.

(iii) Suppose that $|G/N|_p = p$ and $|G/L|_p = p$. Without loss of generality, we may assume that N and L are non-abelian simple groups. Then $P = (P \cap N)(P \cap L)$, and so $|P| = p^2$. Then with a similar discussion as above, we can derive a contradiction. Hence (4) holds.

(5) $N \nleq \Phi(P)$.

Suppose that $N \leq \Phi(P)$. Then $N \leq \Phi(G)$. By (2), either G/N is p-supersoluble or $|G/N|_p = p$. But the former case is clearly impossible. Hence we may assume that $|G/N|_p = p$. Then $|P/N| = p$. This implies that P is cyclic, and so $|N| = p$. Then $|P| = p^2$. We show that G/N is a non-abelian simple group. Let $A/N = O_{p'}(G/N)$. Then $A \cap P \leq N \leq \Phi(P)$, and so A is p-nilpotent by [13, Chap. IV, Satz 4.7]. It follows from (1) that $A = N$. Thus $O_{p'}(G/N) = 1$. Suppose that K/N is a chief factor of G. Then $|K/N|_p = p$, and so $P \leq K$. Obviously, (G, K) satisfies the hypothesis of the proposition. If $K < G$, the choice of (G, E) yields that $K \leq Z_{p\mathfrak{U}}(G)$. Thus G is p-supersoluble. This contradiction shows that $G = K$. Then G/N is a non-abelian simple group. Since $|N| = p$, $G/C_G(N)$ is abelian, and so $C_G(N) = G$. It follows that $N \leq Z(G)$, which contradicts [13, Chap. VI, Satz 14.3].

(6) $O_p(G) = 1$.

Suppose that $O_p(G) \neq 1$. By (4), $N \leq O_p(G)$. If G/N is p-supersoluble, then $N \nleq \Phi(G)$. Therefore there exists a maximal subgroup M of G such that $G = NM$ and $N \cap M = 1$. Since $P = N(P \cap M)$, P has a maximal subgroup P_1 containing $P \cap M$ and $P = NP_1$. If $(P_1)_{sG} \neq 1$, then by (4), Lemma 2.1(3) and Lemma 2.2(1), $N \leq ((P_1)_{sG})^G = ((P_1)_{sG})^P \leq P_1$, a contradiction. Thus $(P_1)_{sG} = 1$. Then by the hypothesis, there exists a subnormal subgroup T of G such that $G = P_1T$ and $P_1 \cap T \leq \Phi(P_1)$. Note that $N \leq O^p(G) \leq T$ by (4). Thus $P_1 \cap N \leq \Phi(P_1)$. This induces that $P_1 = (P_1 \cap N)(P \cap M) = P \cap M$. Hence $P_1 \cap N = 1$, and so $|N| = p$, a contradiction. Now assume that $|G/N|_p = p$. Then $|P/N| = p$. By (5), P has a maximal subgroup P_2 such that $P = P_2N$. With a similar argument as above, we have that $(P_2)_{sG} = 1$. Therefore, by the hypothesis, there exists a subnormal subgroup T of G such that $G = P_2T$ and $P_2 \cap T \leq \Phi(P_2)$. Then clearly, $N \leq T$, and so $|G : T| = p$. This implies that $T \trianglelefteq G$ and T/N is a p'-group. Thus G/N is p-supersoluble. This case has been dealt with in the above. Hence we have (6).

(7) *Final contradiction.*

By (3) and (6), we have that $G = PN$. If $P \leq N$, then G is a non-abelian simple group. Let P_1 be a maximal subgroup of P. Then P_1 is generalized $S\Phi$-supplemented in G. It follows that $P_1 = (P_1)_{sG}$ is s-permutable in G by Lemma 2.2(1), and so $P_1 = 1$ by Lemma 2.1(1). Thus $|G|_p = |P| = p$. This contradiction shows that P has a maximal subgroup P_2 such that $P \cap N \leq P_2$. Then $(P_2)_{sG} = 1$ by (6), Lemma 2.1(1) and Lemma 2.2(1). Hence, by the hypothesis, G has a subnormal subgroup T such that $G = P_2T$ and $P_2 \cap T \leq \Phi(P_2) \leq \Phi(P)$. By [21, Lemma 2.5(7)], we have $O^p(G) \leq T$. Hence by (4), $N \leq O^p(G) \leq T$, and thereby $P \cap N = P_2 \cap N \leq \Phi(P)$. Then by [13, Chap. IV, Satz 4.7], N is p-nilpotent, and so N is a p-group by (1), which contradicts (6). The proof is thus completed. □

Proof of Theorem 1.4. Suppose that the result is false and let (G, E) be a counterexample for which $|G| + |E|$ is minimal. We now proceed via the following steps.

(1) $O_{p'}(E) = 1$ *and* $X = E = G$.

Suppose that $X < E$. Then clearly, $F_p^*(X) = F_p^*(E)$. Hence (G, X) satisfies the hypothesis of the theorem. The choice of (G, E) implies that $F_p^*(E) \leq X \leq Z_{p\mathfrak{U}}(G)$, and so $E \leq Z_{p\mathfrak{U}}(G)$ by Lemma 2.7. This contradiction shows that $X = E$. With a similar argument as in steps (1) and (2) in the proof of Proposition 1.7, we have that $O_{p'}(E) = 1$ and $E = G$.

(2) $p < |D| < |P|/p$.

It follows immediately from Propositions 1.7 and 1.8.

(3) *If* $H \leq P$ *and* $|H| = |D|$, *then* H *is* s*-permutable in* G.

By the hypothesis, G has a subnormal subgroup T such that $G = HT$ and $H \cap T \leq \Phi(H)H_{sG}$. If $T < G$, then there exists a normal subgroup M of G such that $T \leq M$ and $|G : M| = p$. Hence by (2), (G, M) satisfies the hypothesis of the theorem. The choice of (G, E) implies that $M \leq Z_{p\mathfrak{U}}(G)$, and so G is p-supersoluble, a contradiction. Thus $T = G$. It follows that $H = H_{sG}$ is s-permutable in G by Lemma 2.2(1).

(4) *Final contradiction.*

Let N be a minimal normal subgroup of G. Then by (1), $p \mid |N|$. If $N \nleq O_p(G)$, then we may take a subgroup H of P such that $|H| = |D|$ and $H \cap N \neq 1$. By (3) and Lemma 2.1(1), $H \cap N \leq O_p(N) = 1$, a contradiction. Hence $N \leq O_p(G)$. If $|N| > |D|$, then N has a subgroup H such that $H \trianglelefteq P$ and $|H| = |D|$. By (3) and Lemma 2.1(3), $H \trianglelefteq G$, a contradiction. Now assume that $|N| = |D|$. Then by (2), there exists a subgroup V of P such that $N < V < P$, $V \trianglelefteq P$ and $|V : N| = p$. If $\Phi(V) = N$, then V is cyclic, and so $|N| = p$, which contradicts (2). Thus $\Phi(V) < N$. It follows that N has a subgroup N_1 such that $\Phi(V) \leq N_1 < N$, $N_1 \trianglelefteq P$ and $|N : N_1| = p$. Then V has a subgroup H such that $|H| = |D|$ and $H \cap N = N_1$. By (3), N_1 is s-permutable in G, and so $N_1 \trianglelefteq G$ by Lemma 2.1(3). Thus $N_1 = 1$, which implies that $|N| = |D| = p$, which contradicts (2). Therefore, we have that $|N| < |D|$.

If $p > 2$ or $p = 2$ and P/N is abelian or $p = 2$ and $|D| > 2|N|$, then by Lemma 2.3(2), we see that $(G/N, G/N)$ satisfies the hypothesis of the theorem. Now assume that $p = 2$, P/N is non-abelian and $|D| = 2|N|$. Then P is non-abelian. By (3) and Lemma 2.1(2), all subgroups of P/N of order 2 are s-permutable in G/N. Let L/N be a cyclic subgroup of order 4 of P/N. If $N \leq \Phi(L)$, then L is cyclic, and so $|D| = 2|N| = 4$. By (3), all subgroups of P of order 4 are s-permutable in G. For any subgroup K of P of order 2 with $K \neq N$, NK is s-permutable in G. Thus by Lemma 2.8, K is s-permutable in G. Now by Proposition 1.7, we have that G is p-supersoluble, a contradiction. Hence we may assume that $N \nleq \Phi(L)$. Then there exists a maximal subgroup L_1 of L such that $L = L_1N$. Since $|L_1| = |D|$, $L/N = L_1N/N$ is s-permutable in G/N by (3) and Lemma 2.1(2). This shows that $(G/N, G/N)$ also satisfies the hypothesis of the theorem. Hence, by the choice of (G, E), G/N is p-supersoluble. Then clearly, N is the unique normal subgroup of G and $N \nleq \Phi(G)$. It follows that G has a maximal subgroup M such that $G = N \rtimes M$. Since $O_p(G) \cap M = 1$,

$N = O_p(G)$, and so $|N| \geq |D|$ by (3) and Lemma 2.1(1). The final contradiction completes the proof. $\square$

4. Further applications

By Theorem 1.4, we can prove the following corollaries.

Corollary 4.1. *Let E be a normal subgroup of G and P a Sylow p-subgroup of E, where $(|E|, p-1) = 1$. If P has a subgroup D such that $1 < |D| < |P|$, and all subgroups H of P with $|H| = |D|$ and all cyclic subgroups of P of order 4 (when P is a non-abelian 2-group and $|D| = 2$) are generalized $S\Phi$-supplemented in G, then E is p-nilpotent.*

Proof. By Theorem 1.4, $E \leq Z_{p\mathfrak{U}}(G)$, and so E is p-supersoluble. Since $(|E|, p-1) = 1$, we see that E is p-nilpotent. $\square$

Corollary 4.2. *Let E and X be normal subgroups of G such that $F^*(E) \leq X \leq E$. If for any non-cyclic Sylow subgroup P of X, P has a subgroup D such that $1 < |D| < |P|$, and all subgroups H of P with $|H| = |D|$ and all cyclic subgroups of P of order 4 (when P is a non-abelian 2-group and $|D| = 2$) are generalized $S\Phi$-supplemented in G, then $E \leq Z_{\mathfrak{U}}(G)$.*

Proof. By Lemma 2.3(2) and Corollary 4.1, we have that X has a Sylow tower of supersoluble type. If P is cyclic, then clearly, $X \leq Z_{p\mathfrak{U}}(G)$. Now assume that P is non-cyclic. Then by Theorem 1.4, $X \leq Z_{p\mathfrak{U}}(G)$ also holds. Therefore, $F^*(E) \leq X \leq Z_{\mathfrak{U}}(G)$, and so $E \leq Z_{\mathfrak{U}}(G)$ by Lemma 2.9. $\square$

Corollary 4.3. *Let E be a normal subgroup of G such that G/E is p-nilpotent and P a Sylow p-subgroup of E such that $N_G(P)$ is p-nilpotent. If P has a subgroup D such that $1 < |D| < |P|$, and all subgroups H of P with $|H| = |D|$ and all cyclic subgroups of P of order 4 (when P is a non-abelian 2-group and $|D| = 2$) are generalized $S\Phi$-supplemented in G, then G is p-nilpotent.*

Proof. Suppose that the result is false and let (G, E) be a counterexample for which $|G| + |E|$ is minimal. Assume that $O_{p'}(E) \neq 1$. Since

$$N_{G/O_{p'}(E)}(PO_{p'}(E)/O_{p'}(E)) = N_G(P)O_{p'}(E)/O_{p'}(E), \quad (G/O_{p'}(E), E/O_{p'}(E))$$

satisfies the hypothesis of the corollary by Lemma 2.3(2). The choice of (G, E) implies that $G/O_{p'}(E)$ is p-nilpotent, and so G is p-nilpotent, a contradiction.

Hence $O_{p'}(E) = 1$. Note that by Theorem 1.4, E is p-supersoluble. Then by Lemma 2.6, $P \trianglelefteq G$. Hence $G = N_G(P)$ is p-nilpotent, a contradiction. $\square$

Note that Corollaries 4.1–4.3 generalize many known results, for example, [16, Theorems 3.1, 3.6, 3.11, 4.1, 4.3 and 4.4], [17, Theorems 3.1–3.5], [18, Theorems 1.2 and 1.3], [19, Theorems 3.1 and 3.2], [20, Theorem 1.4], [21, Theorems 1.3 and 1.4], [24, Theorem]. Moreover, we point out that [16, Theorem 3.9] and [19, Theorem 3.3] follow directly from Proposition 1.8.

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