

A scale variational principle of Herglotz

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Abstract. The Herglotz problem is a generalization of the fundamental problem of the calculus of variations. In this paper, we consider a class of non-differentiable functions, where the dynamics is described by a scale derivative. Necessary conditions are derived to determine the optimal solution for the problem. Some other problems are considered, like transversality conditions, the multi-dimensional case, higher-order derivatives and several independent variables.

1. Introduction

The calculus of variations deals with optimization of a given functional, whose analytic expression is the integral of a given function, that depends on time, space and the velocity of the trajectory:

$$x \mapsto \int_a^b L(t, x(t), \dot{x}(t)) dt.$$

The variational principle of Herglotz can be seen as an extension of such classical theories, but instead of an integral, we have the functional as a solution of a differential equation (see [9], [10]):

$$\begin{cases} \dot{z}(t) &= L(t, x(t), \dot{x}(t), z(t)), & \text{with } t \in [a, b], \\ z(a) &= z_a. \end{cases}$$

Without the dependence of z , we can convert this problem into a calculus of variations problem. In fact, integrating the differential equation

$$\dot{z}(t) = L(t, x(t), \dot{x}(t))$$

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from a to b , we obtain

$$z(b) = \int_a^b \left[L(t, x(t), \dot{x}(t)) + \frac{z_a}{b-a} \right] dt.$$

Recently, more advances were made namely proving Noether's type theorems for the variational principle of Herglotz (see e.g. [5], [6], [7], [8], [9], [12]). The aim of this paper is to consider the Herglotz problem, but the trajectories $x(\cdot)$ may be non-differentiable functions. We believe that this situation may model more efficiently certain physical problems, like fractals.

The organization of the paper is the following. In Section 2 we define what a scale derivative is, following the concept as presented in [2], and we present some of its main properties, like the algebraic rules, integration by parts formula, etc. In Section 3 we prove our new results. After presenting the Herglotz scale problem, we prove a necessary condition that every extremizer must fulfill. Some generalizations of the main result are also presented to complete the study.

2. Scale calculus

We review some definitions and the main results from [2] that we will need. For more on the subject, see references [1], [2], [3].

From now on, let α, β, h be reals in $]0, 1[$ with $\alpha + \beta > 1$ and $h \ll 1$, and consider $I := [a - h, b + h]$.

Definition 1. Let $f : I \rightarrow \mathbb{R}$ be a function. The delta derivative of f at t is defined by

$$\Delta_h[f](t) := \frac{f(t+h) - f(t)}{h}, \quad \text{for } t \in [a-h, b],$$

and the nabla derivative of f at t is defined by

$$\nabla_h[f](t) := \frac{f(t) - f(t-h)}{h}, \quad \text{for } t \in [a, b+h].$$

If f is differentiable, then

$$\lim_{h \rightarrow 0} \Delta_h[f](t) = \lim_{h \rightarrow 0} \nabla_h[f](t) = f'(t).$$

These two operators can be combined into a single one, where the real part is the mean value of such operators, and the complex part measures the difference between them.

Definition 2. The h -scale derivative of f at t is given by

$$\frac{\square_h f}{\square t}(t) = \frac{1}{2} [(\Delta_h[f](t) + \nabla_h[f](t)) + i(\Delta_h[f](t) - \nabla_h[f](t))],$$

for $t \in [a, b]$. (1)

For complex valued functions f , such definition is extended by

$$\frac{\square_h f}{\square t}(t) = \frac{\square_h \operatorname{Re} f}{\square t}(t) + i \frac{\square_h \operatorname{Im} f}{\square t}(t).$$

We now explain how to drop the dependence on the parameter h in the definition of the scale derivative. First, consider the set $C_{conv}^0([a, b] \times]0, 1[, \mathbb{C})$ of the functions $g \in C^0([a, b] \times]0, 1[, \mathbb{C})$ for which the limit

$$\lim_{h \rightarrow 0} g(t, h)$$

exists for all $t \in [a, b]$, and let E be a complementary space of $C_{conv}^0([a, b] \times]0, 1[, \mathbb{C})$ in $C^0([a, b] \times]0, 1[, \mathbb{C})$.

Define π the projection of $C_{conv}^0([a, b] \times]0, 1[, \mathbb{C}) \oplus E$ onto $C_{conv}^0([a, b] \times]0, 1[, \mathbb{C})$,

$$\begin{aligned} \pi : C_{conv}^0([a, b] \times]0, 1[, \mathbb{C}) \oplus E &\rightarrow C_{conv}^0([a, b] \times]0, 1[, \mathbb{C}) \\ g := g_{conv} + g_E &\mapsto \pi(g) = g_{conv}. \end{aligned}$$

Using these definitions, we arrive at the main concept of [2].

Definition 3. The scale derivative of $f \in C^0(I, \mathbb{C})$, denoted by $\frac{\square f}{\square t}$, is defined by

$$\frac{\square f}{\square t}(t) := \left\langle \frac{\square_h f}{\square t} \right\rangle(t), \quad t \in [a, b], \quad (2)$$

where

$$\left\langle \frac{\square_h f}{\square t} \right\rangle(t) := \lim_{h \rightarrow 0} \pi \left(\frac{\square_h f}{\square t}(t) \right).$$

Definition 4. Given $f : I^n = [a - nh, b + nh] \rightarrow \mathbb{C}$, define the higher-order scale derivative of f by

$$\frac{\square^n f}{\square t^n}(t) = \frac{\square}{\square t} \left(\frac{\square^{n-1} f}{\square t^{n-1}} \right)(t), \quad t \in [a, b],$$

where $\frac{\square^1 f}{\square t^1} := \frac{\square f}{\square t}$ and $\frac{\square^0 f}{\square t^0} := f$.

We will adopt the notation $\square^n f(t)$ instead of $\frac{\square^n f}{\square t^n}(t)$ when there is no danger of confusion.

Scale partial derivatives are also considered here. They are defined as in the standard case.

Definition 5. Let $f : \prod_{i=1}^n [a_i - h, b_i + h] \rightarrow \mathbb{R}$ be a function. Define, for each $i \in \{1, \dots, n\}$,

$$\begin{aligned} \Delta_h^i[f](t_1, \dots, t_n) \\ := \frac{f(t_1, \dots, t_{i-1}, t_i + h, t_{i+1}, \dots, t_n) - f(t_1, \dots, t_{i-1}, t_i, t_{i+1}, \dots, t_n)}{h}, \end{aligned}$$

for $t_i \in [a_i - h, b_i]$ and for $t_j \in [a_j - h, b_j + h]$ if $j \neq i$, and

$$\begin{aligned} \nabla_h^i[f](t_1, \dots, t_n) \\ := \frac{f(t_1, \dots, t_{i-1}, t_i, t_{i+1}, \dots, t_n) - f(t_1, \dots, t_{i-1}, t_i - h, t_{i+1}, \dots, t_n)}{h}, \end{aligned}$$

for $t_i \in [a_i, b_i + h]$ and for $t_j \in [a_j - h, b_j + h]$, if $j \neq i$. The h -scale partial derivative of f with respect to the i -th coordinate is given by

$$\frac{\square_h f}{\square t_i}(t_1, \dots, t_n) = \frac{1}{2} [(\Delta_h^i[f] + \nabla_h^i[f]) + i(\Delta_h^i[f] - \nabla_h^i[f])],$$

for $t_i \in [a_i, b_i]$.

The definition of partial scale derivatives $\square f / \square t_i$ is clear.

In what follows, we will denote

$$\begin{aligned} C_{\square}^n([a, b], \mathbb{K}) &:= \{f \in C^0(I^n, \mathbb{K}) \mid \frac{\square^k f}{\square t^k} \in C^0(I^{n-k}, \mathbb{C}), k = 1, 2, \dots, n\}, \\ \mathbb{K} &= \mathbb{R} \text{ or } \mathbb{K} = \mathbb{C}. \end{aligned}$$

Definition 6. Let $f \in C^0(I, \mathbb{C})$ and $\alpha \in]0, 1[$. We say that f is Hölderian of Hölder exponent α if there exists a constant $C > 0$ such that, for all $s, t \in I$,

$$|f(t) - f(s)| \leq C|t - s|^\alpha,$$

and we write $f \in H^\alpha(I, \mathbb{C})$, or simply $f \in H^\alpha$ when there is no danger of mislead.

We say that $f(t_1, \dots, t_n) \in H^\alpha$ if $f(t_1, \dots, t_{i-1}, \cdot, t_{i+1}, \dots, t_n) \in H^\alpha$, for all $i \in \{1, \dots, n\}$ and for all $t_j \in [a_j, b_j]$, $j \neq i$.

Theorem 1. For all $f \in H^\alpha$ and $g \in H^\beta$, we have

$$\frac{\square(f \cdot g)}{\square t}(t) = \frac{\square f}{\square t}(t) \cdot g(t) + f(t) \cdot \frac{\square g}{\square t}(t), \quad t \in [a, b].$$

Theorem 2. Let $f \in C^1_{\square}([a, b], \mathbb{R})$ be such that

$$\lim_{h \rightarrow 0} \int_a^b \left(\frac{\square_h f}{\square t} \right)_E(t) dt = 0, \quad (3)$$

where

$$\frac{\square_h f}{\square t} = \left(\frac{\square_h f}{\square t} \right)_{conv} + \left(\frac{\square_h f}{\square t} \right)_E.$$

Then,

$$\int_a^b \frac{\square f}{\square t}(t) dt = f(b) - f(a).$$

As a consequence, we have the following integration by parts formula. If

$$\lim_{h \rightarrow 0} \int_a^b \left(\frac{\square_h(f \cdot g)}{\square t} \right)_E(t) dt = 0,$$

where $f \in H^\alpha$ and $g \in H^\beta$, then

$$\int_a^b \frac{\square f}{\square t}(t) \cdot g(t) dt = [f(t)g(t)]_a^b - \int_a^b f(t) \cdot \frac{\square g}{\square t}(t) dt.$$

3. The scale variational principle of Herglotz

The (classical) variational principle of Herglotz is described in the following way. Consider the differential equation

$$\begin{cases} \dot{z}(t) &= L(t, x(t), \dot{x}(t), z(t)), & \text{with } t \in [a, b] \\ z(a) &= z_a \\ x(a) &= x_a, x(b) = x_b, \end{cases}$$

where x, z and L are smooth functions. We wish to find x (and the correspondent solution z of the system) such that $z(b)$ attains an extremum. The necessary condition is a second-order differential equation:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x} + \frac{\partial L}{\partial z} \frac{\partial L}{\partial \dot{x}},$$

for all $t \in [a, b]$. This can be seen as an extension of the basic problem of calculus of variations. If L does not depend on z , then integrating the differential equation along the interval $[a, b]$, we get

$$\begin{cases} \int_a^b \left[L(t, x(t), \dot{x}(t)) + \frac{z_a}{b-a} \right] dt \rightarrow \text{extremize} \\ x(a) = x_a, \quad x(b) = x_b. \end{cases}$$

As is well known, many physical phenomena are characterized by non-differentiable functions (e.g. generic trajectories of quantum mechanics [4], scale-relativity without the hypothesis of space-time differentiability [11]). The usual procedure is to replace the classical derivative by a scale derivative, and consider the space of continuous (and non-differentiable) functions. The scale calculus of variations approach was studied in [1], [2], [3] for a certain concept of scale derivative $\square x(t)$:

$$\begin{cases} \int_a^b L(t, x(t), \square x(t)) \rightarrow \text{extremize} \\ x(a) = x_a, \quad x(b) = x_b. \end{cases}$$

Motivated by this problem, we define the fundamental scale variational principle of Herglotz. First we need to define what extremum is.

Definition 7. We say that $z \in C^1([a, b], \mathbb{C})$ attains an extremum at $t = b$ if $z'(b) = 0$.

The problem is then stated in the following way. Consider the system

$$\begin{cases} \dot{z}(t) = L(t, x(t), \square x(t), z(t)), & \text{with } t \in [a, b] \\ z(a) = z_a \\ x(a) = x_a, \quad x(b) = x_b. \end{cases} \quad (4)$$

For simplicity, define

$$[x, z](t) := (t, x(t), \square x(t), z(t)).$$

We assume that

- (1) the trajectories x are in $H^\alpha \cap C_\square^1([a, b], \mathbb{R})$, $\square x \in H^\alpha$ and the functional z in $C^2([a, b], \mathbb{C})$,
- (2) for each x , there exists a unique solution z of the system (4),
- (3) z_a, x_a, x_b are fixed numbers,

(4) the Lagrangian $L : [a, b] \times \mathbb{R} \times \mathbb{C}^2 \rightarrow \mathbb{C}$ is of class C^2 .

Observe that the solution $z(t)$ actually is a function on three variables, to know $z = z(t, x(t), \square x(t))$. When there is no danger of mislead, we will simply write $z(t)$. We are interested in finding a trajectory x for which the corresponding solution z is such that $z(b)$ attains an extremum; and, in particular, what necessary conditions such solutions must fulfill. These equations are called Euler–Lagrange equation types. Again, problem (4) can be reduced to the scale variational problem in case L is independent of z :

$$\int_a^b L \left[(t, x(t), \square x(t)) + \frac{z_a}{b-a} \right] dt \rightarrow \text{extremize.}$$

Theorem 3. *If the pair (x, z) is a solution of problem (4), and $\frac{\partial L}{\partial \square x}[x, z] \in H^\alpha(I, \mathbb{C})$ ($\alpha \in]0, 1[$), then (x, z) is a solution of the equation*

$$\frac{\square}{\square t} \left(\frac{\partial L}{\partial \square x}[x, z](t) \right) = \frac{\partial L}{\partial x}[x, z](t) + \frac{\partial L}{\partial z}[x, z](t) \frac{\partial L}{\partial \square x}[x, z](t), \quad (5)$$

for all $t \in [a, b]$.

PROOF. Let ϵ be an arbitrary real, and consider variation functions of x of type $x(t) + \epsilon \eta(t)$, with $\eta \in H^\beta(I, \mathbb{R}) \cap C_\square^1([a, b], \mathbb{R})$ ($\beta \in]0, 1[$), $\eta(a) = \eta(b) = \square \eta(a) = 0$, and

$$\lim_{h \rightarrow 0} \int_a^b \left(\frac{\square_h}{\square t} \left(\lambda(t) \frac{\partial L}{\partial \square x}[x, z](t) \eta(t) \right) \right)_E dt = 0.$$

The corresponding rate of change of z , caused by the change of x in the direction of η , is given by

$$\theta(t) = \frac{d}{d\epsilon} z(t, x(t) + \epsilon \eta(t), \square x(t) + \epsilon \square \eta(t))|_{\epsilon=0}.$$

Then

$$\begin{aligned} \dot{\theta}(t) &= \frac{d}{dt} \frac{d}{d\epsilon} z(t, x(t) + \epsilon \eta(t), \square x(t) + \epsilon \square \eta(t))|_{\epsilon=0} \\ &= \frac{d}{d\epsilon} L(t, x(t) + \epsilon \eta(t), \square x(t) + \epsilon \square \eta(t), z(t, x(t) + \epsilon \eta(t), \square x(t) + \epsilon \square \eta(t)))|_{\epsilon=0} \\ &= \frac{\partial L}{\partial x}[x, z](t) \eta(t) + \frac{\partial L}{\partial \square x}[x, z](t) \square \eta(t) + \frac{\partial L}{\partial z}[x, z](t) \theta(t). \end{aligned}$$

We obtain a first order linear differential equation on θ , whose solution is

$$\lambda(b)\theta(b) - \theta(a) = \int_a^b \lambda(t) \left[\frac{\partial L}{\partial x}[x, z](t)\eta(t) + \frac{\partial L}{\partial \square x}[x, z](t)\square\eta(t) \right] dt,$$

where

$$\lambda(t) := \exp \left(- \int_a^t \frac{\partial L}{\partial z}[x, z](\tau) d\tau \right).$$

Using the fact that $\theta(a) = \theta(b) = 0$, we get

$$\int_a^b \lambda(t) \left[\frac{\partial L}{\partial x}[x, z](t)\eta(t) + \frac{\partial L}{\partial \square x}[x, z](t)\square\eta(t) \right] dt = 0.$$

Integrating by parts the second term, we obtain

$$\begin{aligned} \int_a^b \left[\lambda(t) \frac{\partial L}{\partial x}[x, z](t) - \frac{\square}{\square t} \left(\lambda(t) \frac{\partial L}{\partial \square x}[x, z](t) \right) \right] \eta(t) dt \\ + \left[\eta(t) \lambda(t) \frac{\partial L}{\partial \square x}[x, z](t) \right]_a^b = 0. \end{aligned}$$

Since $\eta(a) = \eta(b) = 0$, and η is an arbitrary function elsewhere,

$$\lambda(t) \frac{\partial L}{\partial x}[x, z](t) - \frac{\square}{\square t} \left(\lambda(t) \frac{\partial L}{\partial \square x}[x, z](t) \right) = 0,$$

for all $t \in [a, b]$. Since the function $t \mapsto \lambda(t)$ is differentiable, and the function $t \mapsto \frac{\partial L}{\partial \square x}[x, z](t)$ is in H^α , it follows that

$$\lambda(t) \left(\frac{\partial L}{\partial x}[x, z](t) + \frac{\partial L}{\partial z}[x, z](t) \frac{\partial L}{\partial \square x}[x, z](t) - \frac{\square}{\square t} \left(\frac{\partial L}{\partial \square x}[x, z](t) \right) \right) = 0.$$

Finally, since $\lambda(t) > 0$, for all t , we get

$$\frac{\square}{\square t} \left(\frac{\partial L}{\partial \square x}[x, z](t) \right) = \frac{\partial L}{\partial x}[x, z](t) + \frac{\partial L}{\partial z}[x, z](t) \frac{\partial L}{\partial \square x}[x, z](t),$$

for all $t \in [a, b]$. □

Remark 1. Assume that the set of state functions x is $C^1([a, b], \mathbb{R})$. Then equation (5) becomes

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}}[x, z](t) \right) = \frac{\partial L}{\partial x}[x, z](t) + \frac{\partial L}{\partial z}[x, z](t) \frac{\partial L}{\partial \dot{x}}[x, z](t),$$

which is the generalized variational principle of HERGLOTZ as in [10].

Theorem 4. *Let the pair (x, z) be a solution of the problem (4), but now $x(b)$ is free. Then (x, z) is a solution of the equation*

$$\frac{\square}{\square t} \left(\frac{\partial L}{\partial \square x} [x, z](t) \right) = \frac{\partial L}{\partial x} [x, z](t) + \frac{\partial L}{\partial z} [x, z](t) \frac{\partial L}{\partial \square x} [x, z](t),$$

for all $t \in [a, b]$, and verifies the transversality condition

$$\frac{\partial L}{\partial \square x} [x, z](b) = 0.$$

PROOF. Following the proof of Theorem 3, the Euler–Lagrange equation is deduced. Then

$$\left[\eta(t) \lambda(t) \frac{\partial L}{\partial \square x} [x, z](t) \right]_a^b = 0.$$

Since $\eta(a) = 0$ and $\eta(b)$ is arbitrary, we obtain the transversality condition. $\square$

Multi-dimensional case

For simplicity, we considered so far one state function x only, but the multi-dimensional case $(x_1, \dots, x_n)$ is easily studied.

Theorem 5. *Let $\alpha \in]0, 1[$ and let the vector $(x_1, \dots, x_n, z)$ be a solution of the problem: find $(x_1, \dots, x_n)$ that extremizes $z(b)$, with*

$$\begin{cases} \dot{z}(t) &= L(t, x_1(t), \dots, x_n(t), \square x_1(t), \dots, \square x_n(t), z(t)), & \text{with } t \in [a, b] \\ z(a) &= z_a \\ x_i(a) &= x_{ia}, x_i(b) = x_{ib} \end{cases} \quad (6)$$

where, for all $i \in \{1, \dots, n\}$,

- (1) the trajectories x_i are in $H^\alpha \cap C_\square^1([a, b], \mathbb{R})$, $\square x_i \in H^\alpha$ and the functional z in $C^2([a, b], \mathbb{C})$,
- (2) z_a, x_{ia}, x_{ib} are fixed numbers,
- (3) $\frac{\partial L}{\partial \square x_i} [x_1, \dots, x_n, z] \in H^\alpha(I, \mathbb{C})$
- (4) the Lagrangian $L : [a, b] \times \mathbb{R}^n \times \mathbb{C}^{n+1} \rightarrow \mathbb{C}$ is of class C^2 .

Then, for all $i \in \{1, \dots, n\}$, $(x_1, \dots, x_n, z)$ is a solution of the equation

$$\begin{aligned} & \frac{\square}{\square t} \left(\frac{\partial L}{\partial \square x_i} [x_1, \dots, x_n, z](t) \right) \\ &= \frac{\partial L}{\partial x_i} [x_1, \dots, x_n, z](t) + \frac{\partial L}{\partial z} [x_1, \dots, x_n, z](t) \frac{\partial L}{\partial \square x_i} [x_1, \dots, x_n, z](t), \end{aligned}$$

for all $t \in [a, b]$.

Theorem 6. *Let the vector $(x_1, \dots, x_n, z)$ be a solution of the problem as stated in Theorem 5, but now $x_i(b)$ is free, for all $i \in \{1, \dots, n\}$. Then, for all $i \in \{1, \dots, n\}$, $(x_1, \dots, x_n, z)$ is a solution of the equation*

$$\begin{aligned} & \frac{\square}{\square t} \left(\frac{\partial L}{\partial \square x_i} [x_1, \dots, x_n, z](t) \right) \\ &= \frac{\partial L}{\partial x_i} [x_1, \dots, x_n, z](t) + \frac{\partial L}{\partial z} [x_1, \dots, x_n, z](t) \frac{\partial L}{\partial \square x_i} [x_1, \dots, x_n, z](t), \end{aligned}$$

for all $t \in [a, b]$, and verifies the transversality condition

$$\frac{\partial L}{\partial \square x_i} [x_1, \dots, x_n, z](b) = 0.$$

Higher-order derivatives case

Theorem 7. *Let $\alpha \in]0, 1[$ and let the pair (x, z) be a solution of the problem: find x that extremizes $z(b)$, with*

$$\begin{cases} \dot{z}(t) &= L(t, x, \square x(t), \dots, \square^n x(t), z(t)), & \text{with } t \in [a, b] \\ z(a) &= z_a \\ \square^i x(a) &= x_{ia}, \square^i x(b) = x_{ib}, & \text{for all } i \in \{0, \dots, n-1\}, \end{cases}$$

where

- (1) the trajectories x are in $H^\alpha \cap C_\square^n([a, b], \mathbb{R})$, $\square x \in H^\alpha$ and the functional z in $C^2([a, b], \mathbb{C})$,
- (2) z_a, x_{ia}, x_{ib} are fixed numbers, for all $i \in \{0, \dots, n-1\}$,
- (3) $\frac{\partial L}{\partial \square^i x} [x, z] \in H^\alpha(I^n, \mathbb{C})$, for all $i \in \{1, \dots, n\}$,
- (4) $[x, z](t) = (t, x, \square x(t), \dots, \square^n x(t), z(t))$ and $[x](t) = (t, x, \square x(t), \dots, \square^n x(t))$,
- (5) the Lagrangian $L : [a, b] \times \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ is of class C^2 .

Then, (x, z) is a solution of the equation

$$\lambda(t) \frac{\partial L}{\partial x} [x, z](t) + \sum_{i=1}^n (-1)^i \frac{\square^i}{\square t^i} \left(\lambda(t) \frac{\partial L}{\partial \square^i x} [x, z](t) \right) = 0,$$

for all $t \in [a, b]$.

PROOF. Let $x(t) + \epsilon \eta(t)$ be a variation function of x , with $\epsilon \in \mathbb{R}$ and $\eta \in H^\beta \cap C_\square^n([a, b], \mathbb{R})$ ($\beta \in]0, 1[$). Also, assume that the variations fulfill the conditions:

- (1) for all $i = 0, \dots, n-1$, $\square^i \eta(a) = \square^i \eta(b) = 0$, and $\square^n \eta(a) = 0$,

(2) for all $i = 1, 2, \dots, n$ and $k = 0, 1, \dots, i-1$,

$$\lim_{h \rightarrow 0} \int_a^b \left(\frac{\square_h}{\square t} \left(\lambda(t) \frac{\square^k}{\square t^k} \left(\frac{\partial L}{\partial \square^i x} [x, z](t) \right) \square^{i-k-1} \eta(t) \right) \right)_E dt = 0.$$

Define

$$\theta(t) = \frac{d}{d\epsilon} z(t, x(t) + \epsilon \eta(t), \square x(t) + \epsilon \square \eta(t), \dots, \square^n x(t) + \epsilon \square^n \eta(t))|_{\epsilon=0}.$$

Then

$$\dot{\theta}(t) = \frac{\partial L}{\partial x} [x, z](t) \eta(t) + \sum_{i=1}^n \frac{\partial L}{\partial \square^i x} [x, z](t) \square^i \eta(t) + \frac{\partial L}{\partial z} [x, z](t) \theta(t).$$

Solving this linear ODE, we arrive at

$$\int_a^b \lambda(t) \left[\frac{\partial L}{\partial x} [x, z](t) \eta(t) + \sum_{i=1}^n \frac{\partial L}{\partial \square^i x} [x, z](t) \square^i \eta(t) \right] dt = 0,$$

where

$$\lambda(t) := \exp \left(- \int_a^t \frac{\partial L}{\partial z} [x, z](\tau) d\tau \right).$$

Integrating by parts n times, we obtain the following:

$$\begin{aligned} & \int_a^b \left[\lambda(t) \frac{\partial L}{\partial x} [x, z](t) + \sum_{i=1}^n (-1)^i \frac{\square^i}{\square t^i} \left(\lambda(t) \frac{\partial L}{\partial \square^i x} [x, z](t) \right) \right] \eta(t) dt \\ & + \left[\sum_{i=1}^n \sum_{k=0}^{i-1} (-1)^k \frac{\square^k}{\square t^k} \left(\lambda(t) \frac{\partial L}{\partial \square^i x} [x, z](t) \right) \square^{i-1-k} \eta(t) \right]_a^b = 0, \end{aligned}$$

and rearranging the terms, we get

$$\begin{aligned} & \int_a^b \left[\lambda(t) \frac{\partial L}{\partial x} [x, z](t) + \sum_{i=1}^n (-1)^i \frac{\square^i}{\square t^i} \left(\lambda(t) \frac{\partial L}{\partial \square^i x} [x, z](t) \right) \right] \eta(t) dt \\ & + \left[\sum_{i=1}^n \left[\sum_{k=i}^n (-1)^{k-i} \frac{\square^{k-i}}{\square t^{k-i}} \left(\lambda(t) \frac{\partial L}{\partial \square^k x} [x, z](t) \right) \right] \square^{i-1} \eta(t) \right]_a^b = 0. \end{aligned}$$

Since $\square^i \eta(a) = \square^i \eta(b) = 0$, for all $i \in \{0, \dots, n-1\}$ and η is arbitrary elsewhere, we get

$$\lambda(t) \frac{\partial L}{\partial x} [x, z](t) + \sum_{i=1}^n (-1)^i \frac{\square^i}{\square t^i} \left(\lambda(t) \frac{\partial L}{\partial \square^i x} [x, z](t) \right) = 0,$$

for all $t \in [a, b]$. □

Theorem 8. *Let the pair (x, z) be a solution of the problem as stated in Theorem 7, but now $\square^i x(b)$ is free, for all $i \in \{0, \dots, n-1\}$. Then, (x, z) is a solution of the equation*

$$\lambda(t) \frac{\partial L}{\partial x}[x, z](t) + \sum_{i=1}^n (-1)^i \frac{\square^i}{\square t^i} \left(\lambda(t) \frac{\partial L}{\partial \square^i x}[x, z](t) \right) = 0,$$

for all $t \in [a, b]$, and verifies the transversality condition

$$\sum_{k=i}^n (-1)^{k-i} \frac{\square^{k-i}}{\square t^{k-i}} \left(\lambda(t) \frac{\partial L}{\partial \square^k x}[x, z](t) \right) = 0 \quad \text{at } t = b,$$

for all $i \in \{1, \dots, n\}$.

Several independent variables case

We generalize Theorem 3 for several independent variables. First we fix some notations. The variable time is $t \in [a, b]$, $x = (x_1, \dots, x_n) \in \Omega := \prod_{i=1}^n [a_i, b_i]$ are the space coordinates and the state function is $u := u(t, x)$.

Theorem 9. *Let $\alpha \in]0, 1[$ and let the pair (u, z) be a solution of the problem: find u that extremizes $z(b)$, with*

$$\begin{cases} \dot{z}(t) = \int_{\Omega} L \left(t, x, u, \frac{\square u}{\square t}, \frac{\square u}{\square x_1}, \dots, \frac{\square u}{\square x_n}, z(t) \right) d^n x, & \text{with } t \in [a, b] \\ z(a) = z_a \\ u(t, x) \text{ takes fixed values, } & \forall t \in [a, b] \forall x \in \partial \Omega \\ u(t, x) \text{ takes fixed values, } & \forall t \in \{a, b\} \forall x \in \Omega, \end{cases} \quad (7)$$

where, for all $i \in \{1, \dots, n\}$,

- (1) the trajectories u are in $H^\alpha(I \times \Omega, \mathbb{R}) \cap C_{\square}^1([a, b] \times \Omega, \mathbb{R})$, $\frac{\square u}{\square t}, \frac{\square u}{\square x_i} \in H^\alpha([a, b] \times \Omega, \mathbb{C})$ and the functional z in $C^2([a, b], \mathbb{C})$,
- (2) z_a is a fixed number,
- (3) $d^n x = dx_1 \dots dx_n$,
- (4) $\frac{\partial L}{\partial \square t}[u, z], \frac{\partial L}{\partial \square x_i}[u, z] \in H^\alpha(I \times \Omega, \mathbb{C})$, where $\frac{\partial L}{\partial \square t}[u, z]$ denotes the partial derivative of L with respect to the variable $\frac{\square u}{\square t}$, and $\frac{\partial L}{\partial \square x_i}[u, z]$ denotes the partial derivative of L with respect to the variable $\frac{\square u}{\square x_i}$, and $[u, z](t) = (t, x, u, \frac{\square u}{\square t}, \frac{\square u}{\square x_1}, \dots, \frac{\square u}{\square x_n}, z(t))$,
- (5) $L : [a, b] \times \Omega \times \mathbb{R} \times \mathbb{C}^{n+2} \rightarrow \mathbb{C}$ is of class C^2 .

Then, (u, z) is a solution of the equation

$$\begin{aligned} \frac{\partial L}{\partial u}[u, z](t) + \frac{\partial L}{\partial \square t}[u, z](t) \int_{\Omega} \frac{\partial L}{\partial \square z}[u, z](t) d^n x - \frac{\square}{\square t} \left(\frac{\partial L}{\partial \square t}[u, z](t) \right) \\ - \sum_{i=1}^n \frac{\square}{\square x_i} \left(\frac{\partial L}{\partial \square x_i}[u, z](t) \right) = 0, \end{aligned} \quad (8)$$

for all $t \in [a, b]$ and for all $x \in \Omega$.

PROOF. Let $u(t, x) + \epsilon \eta(t, x)$ be a variation function of u , with $\epsilon \in \mathbb{R}$ and $\eta \in H^\beta(I \times \Omega, \mathbb{R}) \cap C_0^1([a, b] \times \Omega, \mathbb{R})$ ($\beta \in]0, 1[$). Also, assume that the variations fulfill the conditions:

- (1) $\eta(t, x) = 0, \quad \forall t \in [a, b] \forall x \in \partial\Omega,$
- (2) $\eta(t, x) = 0, \quad \forall t \in \{a, b\} \forall x \in \Omega,$
- (3) $\frac{\square \eta}{\square t}(a, x) = \frac{\square \eta}{\square x_i}(a, x) = 0, \quad \forall x \in \Omega,$
- (4) for all $i = 1, 2, \dots, n,$

$$\lim_{h \rightarrow 0} \int_a^b \left(\frac{\square_h}{\square t} \left(\lambda(t) \frac{\partial L}{\partial \square t}[u, z](t) \eta(t) \right) \right)_E dt = 0.$$

and

$$\lim_{h \rightarrow 0} \int_a^b \left(\frac{\square_h}{\square x_i} \left(\lambda(t) \frac{\partial L}{\partial \square x_i}[u, z](t) \eta(t) \right) \right)_E dt = 0,$$

where

$$\lambda(t) := \exp \left(- \int_a^t \int_{\Omega} \frac{\partial L}{\partial z}[u, z](\tau) d^n x d\tau \right).$$

Let

$$\theta(t) = \frac{d}{d\epsilon} z \left(t, x, u + \epsilon \eta, \frac{\square u}{\square t} + \epsilon \frac{\square \eta}{\square t}, \frac{\square u}{\square x_1} + \epsilon \frac{\square \eta}{\square x_1}, \dots, \frac{\square u}{\square x_n} + \epsilon \frac{\square \eta}{\square x_n} \right) \Big|_{\epsilon=0}.$$

Proceeding with some calculations, we arrive at the ODE

$$\begin{aligned} \dot{\theta}(t) - \int_{\Omega} \frac{\partial L}{\partial z}[u, z](t) d^n x \theta(t) \\ = \int_{\Omega} \frac{\partial L}{\partial u}[u, z](t) \eta + \frac{\partial L}{\partial \square t}[u, z](t) \frac{\square \eta}{\square t} + \sum_{i=1}^n \frac{\partial L}{\partial \square x_i}[u, z](t) \frac{\square \eta}{\square x_i} d^n x. \end{aligned}$$

Solving the ODE, and taking into consideration that $\theta(a) = \theta(b) = 0$, we get

$$\int_a^b \int_{\Omega} \lambda(t) \left[\frac{\partial L}{\partial u}[u, z](t) \eta + \frac{\partial L}{\partial \square t}[u, z](t) \frac{\square \eta}{\square t} + \sum_{i=1}^n \frac{\partial L}{\partial \square x_i}[u, z](t) \frac{\square \eta}{\square x_i} \right] d^n x dt = 0.$$

Integrating by parts, and considering the boundary conditions over η , we get

$$\int_a^b \int_{\Omega} \left[\lambda(t) \frac{\partial L}{\partial u}[u, z](t) - \frac{\square}{\square t} \left(\lambda(t) \frac{\partial L}{\partial \square t}[u, z](t) \right) - \sum_{i=1}^n \frac{\square}{\square x_i} \left(\lambda(t) \frac{\partial L}{\partial x_i}[u, z](t) \right) \right] \eta d^n x dt = 0.$$

By the arbitrariness of η , it follows that for all $t \in [a, b]$ and for all $x \in \Omega$,

$$\lambda(t) \frac{\partial L}{\partial u}[u, z](t) - \frac{\square}{\square t} \left(\lambda(t) \frac{\partial L}{\partial \square t}[u, z](t) \right) - \sum_{i=1}^n \frac{\square}{\square x_i} \left(\lambda(t) \frac{\partial L}{\partial x_i}[u, z](t) \right) = 0.$$

Since $\lambda(t) > 0$, this condition implies that

$$\begin{aligned} \frac{\partial L}{\partial u}[u, z](t) + \frac{\partial L}{\partial \square t}[u, z](t) \int_{\Omega} \frac{\partial L}{\partial \square z}[u, z](t) d^n x - \frac{\square}{\square t} \left(\frac{\partial L}{\partial \square t}[u, z](t) \right) \\ - \sum_{i=1}^n \frac{\square}{\square x_i} \left(\frac{\partial L}{\partial \square x_i}[u, z](t) \right) = 0, \end{aligned}$$

for all $t \in [a, b]$ and for all $x \in \Omega$, and the theorem is proved. $\square$

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