

Hölder equivalence of homogeneous Moran sets

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Abstract. For two homogeneous Moran sets $E = \mathcal{C}([0, 1], \{n_k\}, \{c_k\})$ and $E' = \mathcal{C}([0, 1], \{n'_k\}, \{c'_k\})$ with Hausdorff dimensions s and s' with $s' < s$ such that $\{n_k\}$ and $\{n'_k\}$ are bounded and the spacings are uniform in some sense, we prove that there exists a homeomorphism $f : E \rightarrow E'$ such that f is $\left(\frac{s'}{s} - \epsilon\right)$ -Hölder continuous but not $\left(\frac{s'}{s} + \epsilon\right)$ -Hölder continuous for any $\epsilon > 0$.

1. Introduction

The class of homogeneous Moran sets are defined and studied by DEJUN FENG, ZHIYING WEN and JUN WU [4]. Let $n_k \geq 2$ be integers and c_k be positive numbers satisfying that $0 < c_k n_k < 1$ ($k = 1, 2, \dots$). Let $D_k = \prod_{i=1}^k \{1, 2, \dots, n_i\}$ and $D = \cup_{k=0}^{\infty} D_k$, where an element in D_k is denoted by a finite sequence $\sigma_1 \sigma_2 \dots \sigma_k$ of $\sigma_i \in \{1, 2, \dots, n_i\}$ ($i = 1, 2, \dots, k$) and D_0 consists of the empty sequence $\emptyset$. Let $\mathbb{J}_\emptyset = [0, 1]$ and define closed intervals $\mathbb{J}_\sigma \subset [0, 1]$ for $\sigma \in D$ inductively. Let $\sigma = \sigma' i \in D_k$ with $\sigma' \in D_{k-1}$ and $i \in \{1, 2, \dots, n_k\}$. Let $\mathbb{J}_{\sigma'} = [a, b]$ with $b - a = c_1 \dots c_{k-1}$. Then, $\mathbb{J}_{\sigma'1}, \mathbb{J}_{\sigma'2}, \dots, \mathbb{J}_{\sigma'n_k}$ are disjoint closed intervals of length $c_1 \dots c_{k-1} c_k$ contained in $\mathbb{J}_{\sigma'}$ arranged from left to right in this order. To determine these sets, we introduce another quantity, a sequence of positive numbers $(d_k^1, \dots, d_k^{n_k-1})$ called *spacing* satisfying that

$$d_k^1 + \dots + d_k^{n_k-1} + n_k c_k \leq 1.$$

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Then define

$$\mathbb{J}_{\sigma'i} = [a + ((i-1)c_k + d_k^1 + \cdots + d_k^{i-1})\delta, a + (ic_k + d_k^1 + \cdots + d_k^{i-1})\delta] \\ (i = 1, 2, \dots, n_k),$$

where $\delta = c_1 \dots c_{k-1}$. Finally, we define a fractal set

$$E = \bigcap_{k=0}^{\infty} \bigcup_{\sigma \in D_k} \mathbb{J}_{\sigma} \quad (1.1)$$

which we call a *homogeneous Moran set* denoted as $\mathcal{C}([0, 1], \{n_k\}, \{c_k\}, \{d_k^i\})$.

In this paper, we always assume that

$$(*) \quad \sup_{k=1,2,\dots} n_k < \infty, \quad \Delta := \inf_{k=1,2,\dots; i=1,\dots,n^k-1} d_k^i > 0,$$

and

$$s = \lim_{k \rightarrow \infty} \frac{\log N_k}{-\log \delta_k} \text{ exists and } 0 < s < 1,$$

where $\delta_k = c_1 \dots c_k$ and $N_k = n_1 \dots n_k$ ($k = 1, 2, \dots$).

It is known [4] that for any $E \in \mathcal{C}([0, 1], \{n_k\}, \{c_k\}, \{d_k^i\})$ satisfying $(*)$, the Hausdorff dimension $\dim_H E$ is equal to s as above. For the general notions of fractal geometry, refer to [1], [5]. For the multifractal properties of Moran sets, refer to [3], [9]. For the notions of Hölder equivalence or Lipschitz equivalence, refer to [2], [7], [8]. We prove that

Theorem 1. *For homogeneous Moran sets $E = \mathcal{C}([0, 1], \{n_k\}, \{c_k\}, \{d_k^i\})$ and $E' = \mathcal{C}([0, 1], \{n'_k\}, \{c'_k\}, \{d'^i_k\})$ satisfying the condition $(*)$ with $s = \dim_H E$ and $s' = \dim_H E'$ such that $s \leq s'$. Then, there exists a homeomorphism $f : E \rightarrow E'$ such that*

$$C_1(y-x)^{\frac{s}{s'}+\epsilon} < f(y)-f(x) < C_2(y-x)^{\frac{s}{s'}-\epsilon} \quad (1.2)$$

holds for any $x < y$ in E , where $\epsilon > 0$ is arbitrary and C_1, C_2 are positive constants.

Corollary 1 (QIN WANG and LI-FENG XI [6]). *In the above theorem, if $s = s'$, then E and E' are quasi-Lipschitz equivalent.*

2. Basic lemmas

Lemma 1. *Let k and l be positive integers. Let $U_1, U_2, \dots, U_k$ be a disjoint family of sets having the same number l of elements. Let $k < n \leq kl$. Then, there exists a disjoint family $V_1, V_2, \dots, V_n$ of nonempty sets such that*

- (1) $V_1 \cup V_2 \cup \dots \cup V_n = U_1 \cup U_2 \cup \dots \cup U_k$, and
- (2) for any $j = 1, 2, \dots, n$, there exists $i = 1, 2, \dots, k$ such that $V_j \subset U_i$.
- (3) $\#V_i \leq 3\#V_j$ holds for any $i, j = 1, 2, \dots, n$, where $\#V$ denotes the number of elements in a set V .

PROOF. Let $d = \lfloor n/k \rfloor$ and $n = kd + r$ with $0 \leq r < k$. Then, $n = (k - r)d + r(d + 1)$. We partition each of $U_1, U_2, \dots, U_{k-r}$ into d number of subsets and each of $U_{k-r+1}, U_{k-r+2}, \dots, U_k$ into $d + 1$ number of subsets. These subsets will become $V_1, V_2, \dots, V_n$. Let $h = \lfloor l/d \rfloor$. Since $l = dh + s$ with $0 \leq s < d$, we have $l = (d - s)h + s(h + 1)$. Partition each of $U_1, U_2, \dots, U_{k-r}$ into $d - s$ number of subsets with h elements and s number of subsets with $h + 1$ elements. Let $m = \lfloor l/(d + 1) \rfloor$. Since $l = (d + 1)m + t$ with $0 \leq t < d + 1$, we have $l = (d + 1 - t)m + t(m + 1)$. Partition each of $U_{k-r+1}, U_{k-r+2}, \dots, U_k$ into $d + 1 - t$ number of subsets with m elements and t number of subsets with $m + 1$ elements. The collection of these sets becomes $V_1, V_2, \dots, V_n$. Then, we have (1)(2).

Let us prove (3). We have

$$\max_i \#V_i = \begin{cases} h + 1 & \text{if } l \text{ is not a multiple of } d, \\ h & \text{if } l \text{ is a multiple of } d, \end{cases}$$

$$\min_i \#V_i = \begin{cases} m & \text{if } n \text{ is not a multiple of } k, \\ h & \text{if } n \text{ is a multiple of } k. \end{cases}$$

Case 1. If l is not a multiple of d , then $h = \lfloor l/d \rfloor \leq (l - 1)/d$. Since $l = (d + 1)m + t$, $0 \leq t \leq d$ and $d \geq 1$, we have

$$\begin{aligned} \max_i \#V_i &= h + 1 = \lfloor l/d \rfloor + 1 \leq (l - 1)/d + 1 = ((d + 1)m + t - 1)/d + 1 \\ &\leq ((d + 1)m + d - 1)/d + 1 = m + 1 + (m - 1)/d + 1 \\ &\leq m + 1 + (m - 1) + 1 = 2m + 1 \leq 3m \leq 3 \min_i \#V_i. \end{aligned}$$

Case 2. If l is a multiple of d , then

$$\begin{aligned} \max_i \#V_i &= h = \lfloor l/d \rfloor = l/d = ((d + 1)m + t)/d \leq ((d + 1)m + d)/d \\ &= m + 1 + m/d \leq m + 1 + m = 2m + 1 \leq 3m \leq 3 \min_i \#V_i. \end{aligned} \quad \square$$

Let $D_k = \prod_{i=1}^k \{1, 2, \dots, n_i\}$ and $D'_k = \prod_{i=1}^k \{1, 2, \dots, n'_i\}$ ($k = 0, 1, 2, \dots$). For $\sigma \in D_k$ or $\sigma' \in D'_k$, let $\mathbb{J}_\sigma$ or $\mathbb{J}_{\sigma'}$ be the intervals defined in (1.1) with respect to $E = \mathcal{C}([0, 1], \{n_k\}, \{c_k\}, \{d_k^i\})$ or $E' = \mathcal{C}([0, 1], \{n'_k\}, \{c'_k\}, \{d_k^{i'}\})$, respectively. We call it a *basic interval* of E or E' of level k . We denote by δ'_k , N'_k or Δ' the quantities δ_k , N_k or Δ for E' .

Notation. Denote $C_0 := \sup_i n_i$ and $C'_0 = \sup_i n'_i$.

Definition 1. Let $k = 1, 2, \dots$ and $e = 0, 1, \dots, k$. For basic intervals I and J of E of level k contained in a same basic interval of E of level $k - e$, the minimum interval containing I and J is called an (k, e) -admissible interval of E . Specially, a basic interval of E of level k is a $(k, 0)$ -admissible interval of E . The number of basic intervals of level k contained in an interval H is called the k -size of H and denoted by $\#_k H$. Let $\mathcal{I} = \{I_1, I_2, \dots, I_t\}$ be a set of (k, e) -admissible intervals of E . We call it an (k, e) -admissible partition of F (in E) if

- (1) $I_i \cap I_j = \emptyset$ for any $i \neq j$, and
- (2) $(\cup_{i=1}^t I_i) \cap E = F$.

Specially, the $(k, 0)$ -admissible partition of E , that is, the set of all basic intervals of level k is denoted by $\mathcal{E}_k$ and called the k -basic partition of E . Just by an *admissible partition*, we mean an (k, e) -admissible partition for some e and k . We define the same things for E' and the k -basic partition of E' is denoted by $\mathcal{E}'_k$.

Definition 2. An (l, g) -admissible partition $\mathcal{I}' = \{I_1, I_2, \dots, I_t\}$ of E' is said to be of $\mathcal{E}_k$ -type if $t = \#\mathcal{I}' = N_k$. In this case, there exists a unique order-preserving bijection φ from $\mathcal{E}_k$ to $\mathcal{I}'$, that is, if $x < y$ holds for any $x \in I$ and $y \in J$ with $I, J \in \mathcal{E}_k$, then $x' < y'$ holds for any $x' \in \varphi(I)$ and $y' \in \varphi(J)$. We call φ the *isomorphism* from $\mathcal{E}_k$ to $\mathcal{I}'$. Let $\mathcal{I}'$ and $\mathcal{J}'$ be admissible partitions of E' of $\mathcal{E}_k$ -type and $\mathcal{E}_m$ -type with $k < m$, respectively. They are said to be *consistent* if for any $I \in \mathcal{E}_k$ and $J \in \mathcal{E}_m$ with $I \subset J$, $\varphi(I) \subset \psi(J)$ holds, where φ and ψ are the isomorphisms from $\mathcal{E}_k$ to $\mathcal{I}'$, and from $\mathcal{E}_m$ to $\mathcal{J}'$, respectively.

Lemma 2. (1) For any $k = 1, 2, \dots$, let $N'_{l-1} < N_k \leq N'_l$. Then, there exists an $(l, 1)$ -admissible partition $\mathcal{I}'_l$ of E' of $\mathcal{E}_k$ -type such that $\#_l I \leq 3\#_l J$ holds for any $I, J \in \mathcal{I}'_l$.

(2) Assume that $\mathcal{I}'_l$ is an (l, e) -admissible partition of E' of $\mathcal{E}_k$ -type such that $\#_l I \leq C\#_l J$ for any $I, J \in \mathcal{I}'_l$ with $C > 1$. Let

$$g = \lfloor (\log C + \log C_0) / \log 2 + 1 \rfloor.$$

Then for any integer $h > k$, there exists an (s, g) -admissible partition $\mathcal{I}'_s$ of E' of $\mathcal{E}_m$ -type with $h \leq m < h + \frac{\log C'_0}{\log 2} g$ and some s such that $\mathcal{I}'_l$ and $\mathcal{I}'_s$ are consistent. Moreover, $\#_s I \leq 9C \#_s J$ holds for any $I, J \in \mathcal{I}'_s$.

PROOF. (1) We consider a basic interval I of E' of level $l-1$ to be the set of basic intervals J of E' of level l such that $J \subset I$. There are N'_{l-1} number of sets as this which are denoted by $U_1, U_2, \dots, U_{N'_{l-1}}$. All of them have n'_l number of elements. Since $N'_{l-1} < N_k \leq n'_l N'_{l-1} = N'_l$, applying Lemma 1, we have a disjoint family $V_1, V_2, \dots, V_{N_k}$ of nonempty sets such that (1)(2)(3) of Lemma 1 hold with $k = N'_{l-1}$, $n = N_k$ and $l = n'_l$. Moreover, we may assume that each of $V_1, V_2, \dots, V_{N_k}$ consists of neighboring basic intervals of level l , so that the admissible intervals generated by them are disjoint. Hence, they define a $(l, 1)$ -admissible partition $\mathcal{I}'_l$ of E' of $\mathcal{E}_k$ -type satisfying that $\#_l I \leq 3\#_l J$ for any $I, J \in \mathcal{I}'_l$.

(2) Denote $N_{k,h} = n_{k+1} \cdots n_h$ for $h = k+1, k+2, \dots$ and $N'_{l,s} = n'_{l+1} \cdots n'_s$ for $s = l+1, l+2, \dots$. If $h \leq k$ or $s \leq l$, we define $N_{k,h} = N'_{l,s} = 1$. Let $p = \min_{I \in \mathcal{I}'_l} \#_l I$ and $q = \max_{I \in \mathcal{I}'_l} \#_l I$.

Take any $h > k$ and take an integer s such that $pN'_{l,s-g} < N_{k,h} \leq pN'_{l,s}$. Since

$$\frac{pN'_{l,s}}{qN'_{l,s-g}} \geq \frac{n'_{s-g+1} \cdots n'_s}{C} \geq \frac{2^g}{C} \geq \frac{2^{(\log C + \log C_0)/\log 2}}{C} = C_0,$$

there exists m such that

$$qN'_{l,s-g} < N_{k,m} \leq pN'_{l,s}. \quad (2.1)$$

If $qN'_{l,s-g} < N_{k,h}$, then we can take $m = h$. Otherwise, since $N_{k,h} \leq qN'_{l,s-g} < N_{k,m}$, we must have $h < m$. Moreover, since $pN'_{l,s-g} < N_{k,h}$ and $N_{k,m} \leq pN'_{l,s}$, we have

$$\frac{N_{k,m}}{N_{k,h}} < \frac{pN'_{l,s}}{pN'_{l,s-g}} = n'_{s-g+1} \cdots n'_s \leq C_0'^g.$$

Therefore,

$$2^{m-h} \leq n_{h+1} \cdots n_m = \frac{N_{k,m}}{N_{k,h}} < C_0'^g,$$

and hence, $m < h + \frac{\log C'_0}{\log 2} g$. Thus, there exists m satisfying (2.1) together with

$$h \leq m < h + \frac{\log C'_0}{\log 2} g.$$

Construct an (s, g) -admissible partition $\mathcal{I}'_s$ of E' of $\mathcal{E}_m$ -type such that $\mathcal{I}'_l$ and $\mathcal{I}'_s$ are consistent and $\#_s I \leq 9C\#_s J$ holds for any $I, J \in \mathcal{I}'_s$.

Take any $K \in \mathcal{I}'_l$. Then, we have

$$\#_l K N'_{l,s-g} \leq q N'_{l,s-g} < N_{k,m} \leq p N'_{l,s} \leq \#_l K N'_{l,s-g}.$$

Hence by the same argument as in the proof of (1) applying Lemma 1, there exists a (s, g) -admissible partition $\mathcal{K}_K$ of K in E' with $\#\mathcal{K}_K = N_{k,m}$. Moreover, $\#_s I \leq 3\#_s J$ holds for any $I, J \in \mathcal{K}_K$. Let $\mathcal{I}'_s = \cup_{K \in \mathcal{I}'_l} \mathcal{K}_K$. Then, it is clear that $\mathcal{I}'_s$ is a (s, g) -admissible partition of E' of $\mathcal{E}_m$ -type which is consistent with $\mathcal{I}'_l$. Take any $I, J \in \mathcal{I}'_s$. Let $I \in \mathcal{K}_K$ and $J \in \mathcal{K}_L$. Then, since

$$\#_s I \leq \frac{1}{\#\mathcal{K}_K} \sum_{I' \in \mathcal{K}_K} 3\#_s I' = \frac{3}{N_{k,m}} \sum_{I' \in \mathcal{K}_K} \#_s I' = \frac{3}{N_{k,m}} N'_{l,s} \#_l K$$

and

$$\#_s J \geq \frac{1}{\#\mathcal{K}_L} \sum_{I' \in \mathcal{K}_L} (1/3)\#_s I' = \frac{1/3}{N_{k,m}} \sum_{I' \in \mathcal{K}_L} \#_s I' = \frac{1/3}{N_{k,m}} N'_{l,s} \#_l L,$$

we have

$$\#_s J \leq 9 \frac{\#_l K}{\#_l L} \#_s J \leq 9C\#_s J,$$

which completes the proof. $\square$

Corollary 2. *There exist sequences of positive integers $\{k_i\}$, $\{g_i\}$ and $\{s_i\}$ increasing to ∞ such that*

- (i) $\lim_{i \rightarrow \infty} k_i/i = \infty$ and $\lim_{i \rightarrow \infty} k_{i+1}/k_i = 1$,
- (ii) $(1/2)^{k_{i+1}-k_i} < \Delta$ ($i = 1, 2, \dots$),
- (iii) $\sup g_i/i < \infty$, and
- (iv) *there exists a consistent family of (s_i, g_i) -admissible partitions $\mathcal{I}'_i$ ($i = 1, 2, \dots$) of E' of $\mathcal{E}_{k_i}$ -type.*

PROOF. Take j such that $2^{-j} < \Delta$. We construct $k_1 < k_2 < \dots$ inductively starting by an arbitrary k_1 . For $k = k_1$, there exists $(l, 1)$ -admissible partitions $\mathcal{I}'_l$ of E' of $\mathcal{E}_k$ -type by (1) of Lemma 2. Let $(s_1, g_1) = (l, 1)$ and $\mathcal{I}'_1 = \mathcal{I}'_l$. Assume that k_i , (s_i, g_i) and $\mathcal{I}'_i$ are determined. For $k = k_i$, $h = k + j$ and $\mathcal{I}'_i$, apply Lemma 2 and get m and (s, g) -admissible partitions $\mathcal{I}'_s$ of E' of $\mathcal{E}_k$ -type. Define $k_{i+1} = m$, $(s_{i+1}, g_{i+1}) = (s, g)$ and $\mathcal{I}'_{i+1} = \mathcal{I}'_s$. Then, we have (i)(ii). We also have (iii) since

$$g_i \leq \frac{i \log 9 + \log C_0}{\log 2} + 1 \quad (i = 1, 2, \dots).$$

$\square$

3. Proof of the main theorem

Take a sequence $k_1 < k_2 < \dots$ and a consistent family of (s_i, g_i) -admissible partitions $\mathcal{I}'_i$ ($i = 1, 2, \dots$) as in Corollary 2. Note that since $\lim_{i \rightarrow \infty} g_i/k_i = 0$ and $N_{s_i - g_i} \leq N_{k_i} \leq N_{s_i}$, we have

$$0 < \liminf_{i \rightarrow \infty} N'_{s_i - g_i} / N_{k_i} \leq \limsup_{i \rightarrow \infty} N'_{s_i} / N_{k_i} < \infty.$$

In particular, we have

$$(iv) \quad \lim_{i \rightarrow \infty} s_i/i = \infty \text{ and } \lim_{i \rightarrow \infty} s_{i+1}/s_i = 1.$$

We may also assume that

$$(v) \quad (1/2)^{s_{i+1} - s_i} < \Delta' \quad (i = 1, 2, \dots).$$

For $x \in E$, let $I^i(x)$ be $I \in \mathcal{E}_{k_i}$ such that $x \in I$. Let φ_i be the isomorphism from $\mathcal{E}_{k_i}$ to $\mathcal{I}'_i$. Since $\lim_{i \rightarrow \infty} N'_{s_i - g_i} = \infty$, $y \in E'$ such that $y \in \varphi_i(I^i(x))$ for any $i = 1, 2, \dots$ is determined, which is denoted by $f(x)$.

We prove that f satisfies the required conditions. By the construction, it is clear that f is strictly increasing. Take any $x \in E$. If $x + 0 \in E$, then x is not the right end point of $I^i(x)$ for any $i = 1, 2, \dots$. Hence, $f(x)$ and $f(x + 0)$ stay in a same (s_i, g_i) -admissible interval of E' as $s_i - g_i \rightarrow \infty$. Hence, $f(x + 0) = f(x) + 0$. Thus, f is right continuous. The same argument holds for $x - 0$. Thus, f is continuous.

Now, we prove the inequality (1.2). Let $x, y \in E$ satisfy that $x < y$ and $y - x$ is sufficiently small. Take $i = 1, 2, \dots$ such that $\delta_{k_{i+2}} < y - x \leq \delta_{k_{i+1}}$. Then, since

$$y - x \leq \delta_{k_{i+1}} \leq (1/2)^{k_{i+1} - k_i + 1} \delta_{k_i - 1} < \Delta \delta_{k_i - 1} \leq d_{k_i}^j \delta_{k_i - 1}$$

for any $j \in \{1, 2, \dots, n_{k_i - 1} - 1\}$. Hence, there exists a basic interval I of E of level k_i such that $\{x, y\} \subset I$. Therefore, $f(x)$ and $f(y)$ belong to a same (s_i, g_i) -admissible interval of E' , and hence, in a same basic interval of E' of level $s_i - g_i$. Denote $l = s_i - g_i$ and $h = s_i$. Since $f(x)$ and $f(y)$ belong to a same basic interval of E' of level l , we have $f(y) - f(x) \leq \delta'_l$. For any $\epsilon > 0$, there exists a sufficiently small $\lambda > 0$ such that

$$(1 - \lambda)^2 (s' + \lambda)^{-1} (s - \lambda) < \frac{s}{s'} - \epsilon.$$

Take i_0 such that

$$|s - \frac{\log N_{k_{i+2}}}{-\log \delta_{k_{i+2}}}| < \lambda, \quad |s' - \frac{\log N'_l}{-\log \delta'_l}| < \lambda,$$

$$\frac{(k_{i+2} - k_i) \log C_0}{k_{i+2} \log 2} < \lambda \quad \text{and} \quad \frac{(h - l) \log C'_0}{h \log 2} < \lambda$$

for any $i \geq i_0$. Assume that $i \geq i_0$. Since $N'_l \leq N_{k_i} \leq N'_h$, we have

$$\begin{aligned} -\log(f(y) - f(x)) &\geq -\log \delta'_l > (s' + \lambda)^{-1} \log N'_l \\ &= (s' + \lambda)^{-1} \log N'_h \left(1 - \frac{\log(n'_{l+1} \cdots n'_h)}{\log N'_h} \right) \\ &\geq (s' + \lambda)^{-1} \log N'_h \left(1 - \frac{(h - l) \log C'_0}{h \log 2} \right) \\ &> (s' + \lambda)^{-1} \log N'_h (1 - \lambda) \geq (1 - \lambda)(s' + \lambda)^{-1} \log N_{k_i} \\ &= (1 - \lambda)(s' + \lambda)^{-1} \log N_{k_{i+2}} \left(1 - \frac{\log(n_{k_i+1} \cdots n_{k_{i+2}})}{\log N_{k_{i+2}}} \right) \\ &\geq (1 - \lambda)(s' + \lambda)^{-1} \log N_{k_{i+2}} \left(1 - \frac{(k_{i+2} - k_i) \log C_0}{k_{i+2} \log 2} \right) \\ &> (1 - \lambda)^2 (s' + \lambda)^{-1} \log N_{k_{i+2}} \\ &> (1 - \lambda)^2 (s' + \lambda)^{-1} (s - \lambda) (-\log \delta_{k_{i+2}}) \\ &> (1 - \lambda)^2 (s' + \lambda)^{-1} (s - \lambda) (-\log(y - x)) \\ &> \left(\frac{s}{s'} - \epsilon \right) (-\log(y - x)) \end{aligned}$$

Hence, for any $\epsilon > 0$, $f(y) - f(x) < (y - x)^{\frac{s}{s'} - \epsilon}$ holds for any $x < y$ in E such that $y - x$ is sufficiently small. Thus, for any $\epsilon > 0$, there exists C_1 such that

$$f(y) - f(x) < C_1 (y - x)^{\frac{s}{s'} - \epsilon}$$

for any $x < y$ in E .

Let us prove the inequality of the opposite direction. Let $x, y \in E$ satisfy that $x < y$ and $y - x$ is sufficiently small. Take $i = 1, 2, \dots$ such that $\delta_{k_{i-1}} < y - x \leq \delta_{k_i}$. Then, there exists a basic interval I of level k_{i-1} such that $x \in I$ but $y \notin I$. Therefore, $f(x)$ and $f(y)$ belong to distinct (s_{i-1}, g_{i-1}) -admissible intervals in $\mathcal{I}'_{i-1}$, and hence, belong to distinct basic interval of E' of level s_{i-1} . Therefore, by (v),

$$f(y) - f(x) \geq d_{s_{i-1}}^{j'} \delta'_{s_{i-1}-1} \geq \Delta \delta'_{s_{i-1}-1} > \delta'_{s_i}.$$

Denote $l = s_i - g_i$ and $h = s_i$. For any $\epsilon > 0$, there exists a sufficiently small $\lambda > 0$ such that

$$(1 + \lambda)^2 (s' - \lambda)^{-1} (s + \lambda) < \frac{s}{s'} + \epsilon.$$

Take i_0 such that

$$|s - \frac{\log N_{k_{i-2}}}{-\log \delta_{k_{i-2}}}| < \lambda, \quad |s' - \frac{\log N'_h}{-\log \delta'_h}| < \lambda,$$

$$\frac{(k_i - k_{i-2}) \log C_0}{k_{i-2} \log 2} < \lambda \quad \text{and} \quad \frac{(h - l) \log C'_0}{l \log 2} < \lambda$$

for any $i \geq i_0$. Assume that $i \geq i_0$. Since $N'_l \leq N_{k_i} \leq N'_h$, we have

$$\begin{aligned} -\log(f(y) - f(x)) &< -\log \delta'_h < (s' - \lambda)^{-1} \log N'_h \\ &= (s' - \lambda)^{-1} \log N'_l \left(1 + \frac{\log(n'_{l+1} \cdots n'_h)}{\log N'_l} \right) \\ &\leq (s' - \lambda)^{-1} \log N'_l \left(1 + \frac{(h - l) \log C'_0}{l \log 2} \right) \\ &< (s' - \lambda)^{-1} \log N'_l (1 + \lambda) \leq (1 + \lambda)(s' - \lambda)^{-1} \log N_{k_i} \\ &= (1 + \lambda)(s' - \lambda)^{-1} \log N_{k_{i-2}} \left(1 + \frac{\log(n_{k_{i-2}+1} \cdots n_{k_i})}{\log N_{k_{i-2}}} \right) \\ &\leq (1 + \lambda)(s' + \lambda)^{-1} \log N_{k_{i-2}} \left(1 + \frac{(k_i - k_{i-2}) \log C_0}{k_{i-2} \log 2} \right) \\ &< (1 + \lambda)^2 (s' - \lambda)^{-1} \log N_{k_{i-2}} \\ &< (1 + \lambda)^2 (s' - \lambda)^{-1} (s + \lambda) (-\log \delta_{k_{i-2}}) \\ &\leq (1 + \lambda)^2 (s' - \lambda)^{-1} (s + \lambda) (-\log(y - x)) \\ &< \left(\frac{s}{s'} + \epsilon \right) (-\log(y - x)) \end{aligned}$$

Hence, for any $\epsilon > 0$, $f(y) - f(x) > (y - x)^{\frac{s}{s'} + \epsilon}$ holds for any $x < y$ in E such that $y - x$ is sufficiently small. Thus, for any $\epsilon > 0$, there exists C_2 such that

$$f(y) - f(x) > C_2(y - x)^{\frac{s}{s'} + \epsilon}$$

for any $x < y$ in E .

Thus, we complete the proof.

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