

Lipschitzian solutions to linear iterative equations

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Dedicated to Professor Zsolt Páles on his 60th birthday

Abstract. We study the problems of the existence, uniqueness and continuous dependence of Lipschitzian solutions φ of equations of the form

$$\varphi(x) = \int_{\Omega} g(\omega)\varphi(f(x, \omega))\mu(d\omega) + F(x),$$

where μ is a measure on a σ -algebra of subsets of Ω .

1. Introduction

Fix a measure space $(\Omega, \mathcal{A}, \mu)$ and a separable metric space (X, ρ) .
Motivated by the appearance of the equation

$$\varphi(x) = \int_{A_1} \varphi(f(x, \omega))\mu(d\omega) + c - \int_{A_2} \varphi(f(x, \omega))\mu(d\omega)$$

with disjoint $A_1, A_2 \in \mathcal{A}$ in the theory of perpetuities and of refinement equations, see section 3.4 of the survey paper [3], we consider problems of the existence, uniqueness and continuous dependence of Lipschitzian solutions φ to the equation

$$\varphi(x) = \int_{\Omega} g(\omega)\varphi(f(x, \omega))\mu(d\omega) + F(x). \quad (1)$$

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Concerning the given functions f, g and F , we assume the following hypotheses in which $\mathcal{B}$ stands for the σ -algebra of all Borel subsets of X and $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

(H₁) Function f maps $X \times \Omega$ into X and for every $x \in X$ the function $f(x, \cdot)$ is $\mathcal{A}$ -measurable, i.e.,

$$\{\omega \in \Omega : f(x, \omega) \in B\} \in \mathcal{A} \quad \text{for all } x \in X \text{ and } B \in \mathcal{B}.$$

(H₂) Function $g: \Omega \rightarrow \mathbb{K}$ is integrable,

$$\int_{\Omega} |g(\omega)| \rho(f(x, \omega), x) \mu(d\omega) < \infty \quad \text{for every } x \in X,$$

and

$$\int_{\Omega} |g(\omega)| \rho(f(x, \omega), f(z, \omega)) \mu(d\omega) \leq \lambda \rho(x, z) \quad \text{for all } x, z \in X \quad (2)$$

with a $\lambda \in [0, 1)$.

(H₃) Function F maps X into a separable Banach space Y over $\mathbb{K}$ and

$$\|F(x) - F(z)\| \leq L \rho(x, z) \quad \text{for all } x, z \in X \quad (3)$$

with an $L \in [0, +\infty)$.

As emphasized in [4, section 0.3] iteration is the fundamental technique for solving functional equations in a single variable, and iterates usually appear in the formulae for solutions. However, as it seems, Lipschitzian solutions are examined rather by the fixed-point method (cf. [4, section 7.2D]). We iterate the operator which transforms a Lipschitzian $F: X \rightarrow Y$ into $\int_{\Omega} g(\omega) F(f(x, \omega)) \mu(d\omega)$; cf. formulas (6) and (8) below. The special case where $g(\omega) = -1$ for every $\omega \in \Omega$ and $\mu(\Omega) = 1$ was examined in [2] on a base of iteration of random-valued functions.

Integrating vector functions we use the Bochner integral.

2. Existence and uniqueness

Putting

$$\gamma = \int_{\Omega} g(\omega) \mu(d\omega), \quad (4)$$

we start with two simple lemmas.

Lemma 2.1. *Assume (H₁) and let $g: \Omega \rightarrow \mathbb{K}$ be integrable with $\gamma \neq 1$. If (2) holds with a $\lambda \in [0, 1)$, then, for any F mapping X into a normed space Y over $\mathbb{K}$, equation (1) has at most one Lipschitzian solution $\varphi: X \rightarrow Y$.*

PROOF. Fix a function F mapping X into a normed space Y over $\mathbb{K}$, let $\varphi_1, \varphi_2: X \rightarrow Y$ be Lipschitzian solutions of (1), and put $\varphi = \varphi_1 - \varphi_2$. Then φ is a Lipschitzian solution of (1) with $F = 0$, and denoting by L_φ the smallest Lipschitz constant for φ , by (2) for all $x, z \in X$, we have

$$\|\varphi(x) - \varphi(z)\| \leq \int_{\Omega} |g(\omega)| \|\varphi(f(x, \omega)) - \varphi(f(z, \omega))\| \mu(d\omega) \leq L_\varphi \lambda \rho(x, z),$$

whence $L_\varphi = 0$ and φ is a constant function. Since γ defined by (4) is different from 1, the only constant solution of (1) with $F = 0$ is the zero function. $\square$

Lemma 2.2. *Under the assumptions (H₁)–(H₃), for every $x \in X$ the function*

$$\omega \mapsto g(\omega)F(f(x, \omega)), \quad \omega \in \Omega,$$

is Bochner integrable, and

$$\left\| \int_{\Omega} g(\omega)F(f(x, \omega)) \mu(d\omega) - \int_{\Omega} g(\omega)F(f(z, \omega)) \mu(d\omega) \right\| \leq L \lambda \rho(x, z) \quad (5)$$

for all $x, z \in X$.

PROOF. The function considered is $\mathcal{A}$ -measurable, for every $\omega \in \Omega$ we have

$$\|g(\omega)F(f(x, \omega))\| \leq L|g(\omega)|\rho(f(x, \omega), x) + L|g(\omega)|\|F(x)\|,$$

and (5) holds for all $x, z \in X$. $\square$

Assuming (H₁)–(H₃) and applying Lemma 2.2, we define

$$F_0(x) = F(x), \quad F_n(x) = \int_{\Omega} g(\omega)F_{n-1}(f(x, \omega)) \mu(d\omega) \quad (6)$$

for all $x \in X$ and $n \in \mathbb{N}$, and we see that

$$\|F_n(x) - F_n(z)\| \leq L \lambda^n \rho(x, z) \quad \text{for all } x, z \in X \text{ and } n \in \mathbb{N}. \quad (7)$$

Our main result reads.

Theorem 2.3. *Assume (H₁)–(H₃). If $\gamma \neq 1$, then equation (1) has exactly one Lipschitzian solution $\varphi: X \rightarrow Y$; it is given by the formula*

$$\varphi(x) = \frac{1}{1-\gamma} \left(\sum_{n=1}^{\infty} (F_n(x) - \gamma F_{n-1}(x)) + F(x) \right) \quad \text{for every } x \in X, \quad (8)$$

$$\|\varphi(x) - \varphi(z)\| \leq \frac{L(1+|\gamma|)}{|1-\gamma|(1-\lambda)} \rho(x, z) \quad \text{for all } x, z \in X, \quad (9)$$

and

$$\|\varphi(x)\| \leq \frac{1}{|1-\gamma|} \left(\frac{L}{1-\lambda} \int_{\Omega} |g(\omega)| \rho(f(x, \omega), x) \mu(d\omega) + \|F(x)\| \right) \quad (10)$$

for every $x \in X$.

PROOF. For the proof of the existence, observe first that by (4), (6) and (7) for all $x \in X$ and $n \in \mathbb{N}$ we have

$$\begin{aligned} \|F_n(x) - \gamma F_{n-1}(x)\| &= \left\| \int_{\Omega} g(\omega) F_{n-1}(f(x, \omega)) \mu(d\omega) - \int_{\Omega} g(\omega) F_{n-1}(x) \mu(d\omega) \right\| \\ &\leq L \lambda^{n-1} \int_{\Omega} |g(\omega)| \rho(f(x, \omega), x) \mu(d\omega). \end{aligned} \quad (11)$$

Consequently, (8) defines a function $\varphi: X \rightarrow Y$. Routine calculations, (8), (7), (2) and (11) show that this function satisfies (9) and (10).

It remains to prove that φ solves (1). To this end, define $M: X \rightarrow [0, \infty)$ by

$$M(x) = L \int_{\Omega} |g(\omega)| \rho(f(x, \omega), x) \mu(d\omega) \quad (12)$$

and fix $x_0 \in X$. An obvious application of (12), (H_2) , (10) and (3) gives

$$M(x) \leq c_1 \rho(x, x_0) + c_2, \quad \|\varphi(x)\| \leq c_1 \rho(x, x_0) + c_2 \quad \text{for every } x \in X \quad (13)$$

with some constants $c_1, c_2 \in [0, \infty)$.

Fix $x \in X$. According to Lemma 2.2, the function

$$\omega \mapsto g(\omega) \varphi(f(x, \omega)), \quad \omega \in \Omega,$$

is Bochner integrable. Moreover, by (11)–(13),

$$\begin{aligned} &\left\| g(\omega) \left(F_n(f(x, \omega)) - \gamma F_{n-1}(f(x, \omega)) \right) \right\| \\ &\leq \lambda^{n-1} |g(\omega)| M(f(x, \omega)) \leq \lambda^{n-1} |g(\omega)| (c_1 \rho(f(x, \omega), x_0) + c_2) \\ &\leq \lambda^{n-1} |g(\omega)| (c_1 \rho(f(x, \omega), x) + c_1 \rho(x, x_0) + c_2) \end{aligned}$$

for all $n \in \mathbb{N}$ and $\omega \in \Omega$. Hence, making use of (H_2) , the dominated convergence theorem and (6), we see that

$$\begin{aligned} &\int_{\Omega} \sum_{n=1}^{\infty} g(\omega) \left(F_n(f(x, \omega)) - \gamma F_{n-1}(f(x, \omega)) \right) \mu(d\omega) \\ &= \sum_{n=1}^{\infty} \int_{\Omega} g(\omega) \left(F_n(f(x, \omega)) - \gamma F_{n-1}(f(x, \omega)) \right) \mu(d\omega) \\ &= \sum_{n=1}^{\infty} (F_{n+1}(x) - \gamma F_n(x)). \end{aligned} \quad (14)$$

Applying now (8), (14) and (6), we get

$$\begin{aligned}
& \int_{\Omega} g(\omega) \varphi(f(x, \omega)) \mu(d\omega) \\
&= \frac{1}{1-\gamma} \int_{\Omega} \left[\sum_{n=1}^{\infty} g(\omega) (F_n(f(x, \omega)) - \gamma F_{n-1}(f(x, \omega))) + g(\omega) F(f(x, \omega)) \right] \mu(d\omega) \\
&= \frac{1}{1-\gamma} \left[\sum_{n=1}^{\infty} (F_{n+1}(x) - \gamma F_n(x)) + F_1(x) \right] \\
&= \frac{1}{1-\gamma} \left[\sum_{n=1}^{\infty} (F_n(x) - \gamma F_{n-1}(x)) + \gamma F(x) \right] = \varphi(x) - F(x).
\end{aligned}$$

The proof is complete. $\square$

3. Examples

Example 3.1. Given $\lambda \in (0, 1)$ and an integrable $\xi: \Omega \rightarrow \mathbb{R}$, consider the equation

$$\varphi(x) = \lambda^2 \int_{\Omega} \varphi\left(\frac{1}{\lambda}x + \xi(\omega)\right) \mu(d\omega)$$

with $\mu(\Omega) = 1$. According to Lemma 2.1, the zero function is its only Lipschitzian solution $\varphi: \mathbb{R} \rightarrow \mathbb{R}$. Note, however, that if

$$\int_{\Omega} \xi(\omega) \mu(d\omega) = 0 \quad \text{and} \quad \int_{\Omega} \xi(\omega)^2 \mu(d\omega) < \infty,$$

then this equation solves also the function $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ given by

$$\varphi(x) = x^2 + \frac{\lambda^2}{1-\lambda^2} \int_{\Omega} \xi(\omega)^2 \mu(d\omega).$$

Example 3.2. Given $\lambda \in (0, 1)$, consider the equation

$$\varphi(x) = 2\varphi(\lambda\sqrt{x} + 1 - \lambda) + \log \frac{x}{(\lambda\sqrt{x} + 1 - \lambda)^2}.$$

According to Lemma 2.1 (in this case $f(x, \omega) = \lambda\sqrt{x} + 1 - \lambda$, $g(\omega) = 2$ and $F(x) = \log \frac{x}{(\lambda\sqrt{x} + 1 - \lambda)^2}$ for all $x \in [1, \infty)$ and $\omega \in \Omega$, $\mu(\Omega) = 1$), the logarithmic function restricted to $[1, \infty)$ is the only Lipschitzian solution $\varphi: [1, \infty) \rightarrow \mathbb{R}$ to this equation, and it is unbounded in spite of the fact that F is bounded.

Example 3.3. To see that assumptions (H_1) – (H_3) do not guarantee the existence of a *continuous* solution $\varphi: X \rightarrow Y$ to equation (1), given $\alpha \in (-1, 1)$, a bounded and $\mathcal{A}$ -measurable $\xi: \Omega \rightarrow \mathbb{R}$, and a Lipschitzian $F: \mathbb{R} \rightarrow [0, \infty)$ such that $F^{-1}(\{0\})$ is a singleton, consider the equation

$$\varphi(x) = \int_{\Omega} \varphi(\alpha x + \xi(\omega)) \mu(d\omega) + F(x) \quad (15)$$

with $\mu(\Omega) = 1$. Assume a continuous $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ solves it. We shall see that then ξ is a.e. constant. To this end, fix an $M \in (0, \infty)$ such that $|\xi(\omega)| \leq M$ for every $\omega \in \Omega$, and a real number $a \geq \frac{M}{1-|\alpha|}$ such that $F^{-1}(\{0\}) \subset [-a, a]$. Then

$$|\alpha x + \xi(\omega)| \leq a \quad \text{for all } x \in [-a, a] \text{ and } \omega \in \Omega,$$

and so $\varphi|_{[-a, a]}$ is a continuous, hence also bounded, solution of (15). According to [1, Corollary 4.1(ii) and Example 4.1], it is possible only if ξ is a.e. constant.

4. Continuous dependence

Given a normed space $(Y, \|\cdot\|)$, consider now the linear space $\text{Lip}(X, Y)$ of all Lipschitzian functions mapping X into Y , and its linear subspace $\text{BL}(X, Y)$ of all Lipschitzian and bounded functions mapping X into Y . Fix $x_0 \in X$ and define $\|\cdot\|_{\text{Lip}}: \text{Lip}(X, Y) \rightarrow [0, \infty)$ by

$$\|u\|_{\text{Lip}} = \|u(x_0)\| + \|u\|_L,$$

where $\|u\|_L$ stands for the smallest Lipschitz constant for u . Clearly, $\|\cdot\|_{\text{Lip}}$ is a norm in $\text{Lip}(X, Y)$. It depends on the given point x_0 , but for different points such norms are equivalent. It is well known that if $(Y, \|\cdot\|)$ is Banach, then so is $(\text{Lip}(X, Y), \|\cdot\|_{\text{Lip}})$. In the linear space $\text{BL}(X, Y)$ we consider the norm $\|\cdot\|_{\text{BL}}$ given by

$$\|u\|_{\text{BL}} = \sup \{ \|u(x)\| : x \in X \} + \|u\|_L.$$

It is also well known that if $(Y, \|\cdot\|)$ is Banach, then so is $(\text{BL}(X, Y), \|\cdot\|_{\text{BL}})$.

Assume (H_1) and (H_2) , $\gamma \neq 1$, and let Y be a separable Banach space over $\mathbb{K}$.

According to Theorem 2.3, for every $F \in \text{Lip}(X, Y)$ the formula

$$\varphi^F(x) = \frac{1}{1-\gamma} \left(\sum_{n=1}^{\infty} (F_n(x) - \gamma F_{n-1}(x)) + F(x) \right) \quad \text{for every } x \in X \quad (16)$$

defines the only Lipschitzian solution φ^F of equation (1),

$$\|\varphi^F\|_L \leq \frac{1 + |\gamma|}{|1 - \gamma|(1 - \lambda)} \|F\|_L \quad (17)$$

and

$$\|\varphi^F(x)\| \leq \frac{1}{|1 - \gamma|} \left(\frac{\|F\|_L}{1 - \lambda} \int_{\Omega} |g(\omega)| \rho(f(x, \omega), x) \mu(d\omega) + \|F(x)\| \right) \quad (18)$$

for every $x \in X$. Putting

$$c_0 = \frac{1}{1 - \lambda} \left(\int_{\Omega} |g(\omega)| \rho(f(x_0, \omega), x_0) \mu(d\omega) + 1 + |\gamma| \right), \quad c = \max\{1, c_0\}, \quad (19)$$

and applying (17) and (18), we see that if $F \in \text{Lip}(X, Y)$, then

$$\begin{aligned} \|\varphi^F\|_{\text{Lip}} &= \|\varphi^F(x_0)\| + \|\varphi^F\|_L \leq \frac{1}{|1 - \gamma|} (c_0 \|F\|_L + \|F(x_0)\|) \\ &\leq \frac{c}{|1 - \gamma|} \|F\|_{\text{Lip}}. \end{aligned}$$

Moreover, if d_0 defined by

$$d_0 = \sup \left\{ \int_{\Omega} |g(\omega)| \rho(f(x, \omega), x) \mu(d\omega) : x \in X \right\} \quad (20)$$

is finite, then putting

$$d = \max \left\{ 1, \frac{d_0 + 1 + |\gamma|}{1 - \lambda} \right\} \quad (21)$$

and applying (18) and (17), again we see also that if $F \in \text{BL}(X, Y)$, then $\varphi^F \in \text{BL}(X, Y)$ as well, and

$$\|\varphi^F\|_{\text{BL}} \leq \frac{1}{|1 - \gamma|} \left(\frac{d_0 + 1 + |\gamma|}{1 - \lambda} \|F\|_L + \sup \{ \|F(x)\| : x \in X \} \right) \leq \frac{d}{|1 - \gamma|} \|F\|_{\text{BL}}.$$

Theorem 4.1. Assume (H_1) , (H_2) , and let γ defined by (4) be different from 1. If Y is a separable Banach space over $\mathbb{K}$, then:

(i) for any $F \in \text{Lip}(X, Y)$, the function $\varphi^F : X \rightarrow Y$ defined by (16) and (6) is the only Lipschitzian solution of (1), the operator

$$F \mapsto \varphi^F, \quad F \in \text{Lip}(X, Y), \quad (22)$$

is a linear homeomorphism of $(\text{Lip}(X, Y), \|\cdot\|_{\text{Lip}})$ onto itself, and

$$\|\varphi^F\|_{\text{Lip}} \leq \frac{c}{|1-\gamma|} \|F\|_{\text{Lip}} \quad \text{for every } F \in \text{Lip}(X, Y),$$

with c given by (19);

(ii) if, additionally, d_0 defined by (20) is finite, then the restriction of the operator (22) to $\text{BL}(X, Y)$ is a linear homeomorphism of $(\text{BL}(X, Y), \|\cdot\|_{\text{BL}})$ onto itself, and

$$\|\varphi^F\|_{\text{BL}} \leq \frac{d}{|1-\gamma|} \|F\|_{\text{BL}} \quad \text{for every } F \in \text{BL}(X, Y),$$

with d given by (21).

PROOF. By the above considerations and the Banach inverse mapping theorem, it remains to show that operator (22) is one-to-one, maps $\text{Lip}(X, Y)$ onto $\text{Lip}(X, Y)$, and $\text{BL}(X, Y)$ onto $\text{BL}(X, Y)$.

The first property follows from the fact that for any $F \in \text{Lip}(X, Y)$ the function φ^F is a solution of (1): if $\varphi^F = 0$, then $F = 0$. To get the next two, observe that if $\psi \in \text{Lip}(X, Y)$, then, by Lemma 2.2, the function $F: X \rightarrow Y$ given by

$$F(x) = \psi(x) - \int_{\Omega} g(\omega) \psi(f(x, \omega)) \mu(d\omega)$$

belongs to $\text{Lip}(X, Y)$, if ψ is also bounded, then so is F , and, since both ψ and φ^F solve (1), $\psi = \varphi^F$ by Lemma 2.1. $\square$

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