Publ. Math. Debrecen
46 / 1-2 (1995), 63–70

Spectral geometry on certain almost Hermitian Einstein manifolds

By LEW FRIEDLAND (New York)

Abstract. On compact Riemannian and Kähler manifolds the spectra of the real and complex Laplacians determine the geometry of the manifolds to a considerable extent, though not completely, as isospectral manifolds need not be isometric. The literature on the geometric consequences of isospectrality is extensive (e.g., [1], [2], [4]–[8], [10], [11]). In this paper we consider the spectrum of the real Laplacian on three classes of almost Hermitian Einstein manifolds which include almost Kähler and nearly Kähler Einstein manifolds. We prove that a compact almost Hermitian Einstein manifold of positive scalar curvature ϱ with $\varrho \le \varrho^*$, or negative scalar curvature ϱ with $\varrho \ge \varrho^*$, or nonzero scalar curvature ϱ with $\varrho = \varrho^*$ (where ϱ^* is the $*$–scalar curvature) and isospectral to a Kähler manifold of constant holomorphic sectional curvature is Kähler and the manifolds are holomorphically isometric.

1. Preliminaries

Let (M, g) be a Riemannian manifold of real dimension $m = 2n \ge 2$ with metric $g = (g_{ij})$. If $R = (R_{hijk})$ is the Riemann curvature tensor, $Rc = (R_{ij}) = g^{hk} R_{hijk}$ the Ricci curvature tensor and $\varrho = g^{ij} R_{ij}$ the scalar curvature, then the Einstein tensor $E = (E_{ij})$ is given by

$$E_{ij} \equiv R_{ij} - \frac{\varrho}{m} g_{ij} \,, \tag{1.1}$$

where M is Einstein if $E = 0$.

If (M, g) is a compact connected C^∞ manifold and $\Delta = -(d\delta + \delta d)$ is the Laplace operator on p–forms, $0 \le p \le 2n$, (0–forms corresponding to differentiable functions on M) with respect to the metric g, then the spectrum of the Laplacian are the eigenvalues of Δ,

$$\mathrm{Spec}^p(M, g) = \{\lambda_{ip} \mid 0 \ge \lambda_{1,p} \ge \lambda_{2,p} \ge \cdots \ge \lambda_{k,p} \ge \ldots \downarrow -\infty\} \,, \tag{1.2}$$

AMS Subject Classification (1991): 53C25, 53C55 and 58G25.

where each eigenvalue is repeated as often as its multiplicity. Further, $\mathrm{Spec}^{2n-p}(M,g) = \mathrm{Spec}^p(M,g)$ when M is orientable.

Relevant to the study of the spectrum is the Minakshisundaram-Pleijel-Gaffney asymptotic formula

$$\sum_{k=0}^{\infty} \exp(\lambda_{k,p} t) \underset{t\to 0}{\sim} \frac{1}{(4\pi t)^n} \sum_{i=0}^{\infty} a_{i,p} t^i \,, \tag{1.3}$$

where the first three coefficients are given by [8]

$$a_{0,p} = \binom{2n}{p} \int_M dM = \binom{2n}{p} \mathrm{vol}(M)\,, \tag{1.4}$$

$$a_{1,p} = \left[\frac{1}{6}\binom{2n}{p} - \binom{2n-2}{p-1}\right] \int_M \varrho dM\,, \tag{1.5}$$

$$a_{2,p} = \int_M \left[c_1(2n,p)\varrho^2 + c_2(2n,p)|Rc|^2 + c_3(2n,p)|R|^2\right] dM; \tag{1.6}$$

where

$$c_1(2n,p) = \frac{1}{72}\binom{2n}{p} - \frac{1}{6}\binom{2n-2}{p-1} + \frac{1}{2}\binom{2n-4}{p-2}\,, \tag{1.7}$$

$$c_2(2n,p) = -\frac{1}{180}\binom{2n}{p} + \frac{1}{2}\binom{2n-2}{p-1} - 2\binom{2n-4}{p-2}\,, \tag{1.8}$$

$$c_3(2n,p) = \frac{1}{180}\binom{2n}{p} - \frac{1}{12}\binom{2n-2}{p-1} + \frac{1}{2}\binom{2n-4}{p-2}\,. \tag{1.9}$$

2. Almost Hermitian manifolds and the Bochner curvature tensor

Let (M,g,J) be an almost Hermitian manifold of real dimension $2n \geq 2$ with almost complex structure $J = (F_i{}^j)$ and almost Hermitian metric $g = (g_{ij})$; that is, $g(JX,JY) = g(X,Y)$ for all X, Y in the tangent space of M at p, $T_p(M)$. In dimension $2n = 2$, (M,g) is Kähler Einstein and has holomorphic sectional curvature $\kappa = \frac{\varrho}{2}$.

Define the Bochner curvature tensor $B = (B_{hijk})$ by

$$\begin{aligned} B_{hijk} \equiv R_{hijk} - \frac{1}{2n+4} &(R_{ij}g_{hk} - R_{ik}g_{hj} + R_{hk}g_{ij} - R_{hj}g_{ik} \\ &+F_{ij}F_h{}^r R_{rk} - F_{ik}F_h{}^r R_{rj} + F_{hk}F_i{}^r R_{rj} - F_{hj}F_i{}^r R_{rk} \\ &-2F_{jk}F_h{}^r R_{ri} - 2F_{hi}F_j{}^r R_{rk}) \end{aligned} \tag{2.1}$$

$$+\frac{\varrho}{(2n+2)(2n+4)}\left(g_{ij}g_{hk}-g_{ik}g_{hj}+F_{ij}F_{hk}-F_{ik}F_{hj}-2F_{hi}F_{jk}\right).$$

Lemma 2.1. (see, e.g., [3]). *If (M,g) is a Kähler manifold of nonzero constant holomorphic sectional curvature κ, then the Riemann curvature tensor, Ricci curvature tensor and scalar curvature are given, respectively, by*

$$R_{hijk}=\frac{\kappa}{4}\left(g_{hk}g_{ij}-g_{hj}g_{ik}+F_{hk}F_{ij}-F_{hj}F_{ik}-2F_{hi}F_{jk}\right), \tag{2.2}$$

$$R_{ij}=\frac{n+1}{2}\kappa g_{ij}, \tag{2.3}$$

$$\varrho=n(n+1)\kappa. \tag{2.4}$$

Hence, (M,g) is Einstein and $B=0$.

By Schur's theorem, a Kähler manifold of dimension $2n\geq 4$ with constant holomorphic sectional curvature κ is of constant holomorphic curvature; that is, κ is a global constant on the manifold.

Lemma 2.2. [3]. *An almost Hermitian manifold (M,g) of dimension $2n\geq 4$ with curvature tensor given by (2.2) is Kähler and of constant holomorphic sectional curvature $\kappa=\frac{\varrho}{n(n+1)}$.*

Consequently, we have

Corollary 2.3. *If (M,g) is an almost Hermitian Einstein manifold of dimension $2n\geq 4$ with $B=0$ and $\varrho\neq 0$, then the conclusion of Lemma 2.2 follows.*

Lemma 2.4. *If (M,g) is an almost Hermitian Einstein manifold, then the square length of the Bochner tensor is given by*

$$|B|^2=|R|^2+\frac{\varrho^2-3\varrho\varrho^*}{n(n+1)}, \tag{2.5}$$

where $\varrho^*\equiv F^{kq}F^{jr}R_{krqj}$.

Proof. The equation follows from a calculation of $|B_{hijk}|^2$ using (1.1) and (2.1).

3. Spectral geometry on almost Hermitian Einstein manifolds

An almost Kähler manifold (M,g,J) is an almost Hermitian manifold such that the differential form $\omega\equiv F_{ij}dx^i\wedge dx^j$ is closed; that is, $d\omega=0$

and hence, $\delta\omega = 0$ and ω is harmonic. This is equivalent to $\nabla_i F_{jk} + \nabla_j F_{ki} + \nabla_k F_{ij} = 0$, where ∇ is the covariant derivative with respect to the Riemannian connection on M. A nearly Kähler manifold is an almost Hermitian manifold satisfying $\nabla_i {F_j}^k + \nabla_j {F_i}^k = 0$.

Lemma 3.1. (see, e.g., [3]). *On an almost Kähler manifold*

$$\varrho \leq \varrho^* \,; \tag{3.1}$$

while on a nearly Kähler manifold

$$\varrho \geq \varrho^* \,. \tag{3.2}$$

Further, equality holds in each case if and only if the manifold is Kähler.

Theorem 3.2. *Suppose that (M, g, J) and (M', g', J') are compact almost Hermitian manifolds with* $\mathrm{Spec}^p(M, g) = \mathrm{Spec}^p(M', g')$ *for $p = 0, 1$ or 2.*

a) *In dimension $2n = 2$, (M, g) is of constant holomorphic curvature κ if and only if (M', g') is.*

b) *In dimension $2n \geq 4$, if (M', g') is Einstein and of positive scalar curvature ϱ' with $\varrho' \leq {\varrho'}^*$, or negative scalar curvature ϱ' with $\varrho' \geq {\varrho'}^*$, or nonzero scalar curvature ϱ' with $\varrho' = {\varrho'}^*$, and (M, g) is Kähler and of constant holomorphic sectional curvature κ, then (M', g') is a Kähler manifold of constant holomorphic sectional curvature $\kappa' = \kappa$ in the following cases: $p = 0$ and $2n \geq 4$; $p = 1$ and $2n \geq 16$; $p = 2$ and $2n = 4, 6, 8, 14$ or $2n \geq 18$.*

Proof. Letting $p = 0$ in (1.4)–(1.9) gives

$$a_{0,0} = \int_M dM = \mathrm{vol}(M)\,, \tag{3.3}$$

$$a_{1,0} = \frac{1}{6}\int_M \varrho dM\,, \tag{3.4}$$

$$a_{2,0} = \frac{1}{360}\int_M \left[5\varrho^2 - 2|Rc|^2 + 2|R|^2\right] dM\,. \tag{3.5}$$

a) In dimension $2n = 2$, $|R|^2 = \varrho^2$ and since g is an Einstein metric, then by (1.1), $|Rc|^2 = \frac{\varrho^2}{2}$ and $a_{2,0} = \frac{1}{60}\int_M \varrho^2 dM$. Since $a_{0,0} = a'_{0,0}$, it follows that $\mathrm{vol}(M) = \mathrm{vol}(M')$. If, say, (M, g) has constant holomorphic curvature κ, since $a_{1,0} = a'_{1,0}$ and $a_{2,0} = a'_{2,0}$, then $\int_{M'} \varrho' dM' = 2\kappa\,\mathrm{vol}(M')$ and $\int_{M'} {\varrho'}^2 dM' = 4\kappa^2\,\mathrm{vol}(M')$. We then have equality in the Schwarz inequality $\left(\int_{M'} \varrho' dM'\right)^2 \leq \left(\int_{M'} {\varrho'}^2 dM'\right)\left(\int_{M'} dM'\right)$, so that ϱ' is constant and $\varrho' = \varrho$.

The proofs in the remaining cases are similar upon taking $p = 1$ and 2, respectively, in (1.4)–(1.9). For $p = 2$ the result follows, as well, as a consequence of the case $p = 0$ since $\mathrm{Spec}^0(M, g) = \mathrm{Spec}^0(M', g')$ by duality.

b) Suppose (M', g') is almost Hermitian Einstein of either positive scalar curvature ϱ' with $\varrho' \leq \varrho'^*$ or negative scalar curvature ϱ' with $\varrho' \geq \varrho'^*$. In dimension $2n \geq 4$, substitute (2.5) in (3.5). Then, since $E = 0$ and $a_{2,0} = a'_{2,0}$, we have

$$\begin{aligned}
&\int_M \left[\frac{(5n^2+4n-3)\varrho^2 + 6\varrho\varrho^*}{n(n+1)} + 2|B|^2\right] dM \\
&\quad = \int_M \left[\frac{(5n^2+4n+3)\varrho^2}{n(n+1)} + 2|B|^2\right] dM \\
&\quad = \int_{M'} \left[\frac{(5n^2+4n-3){\varrho'}^2 + 6\varrho'\varrho'^*}{n(n+1)} + 2|B'|^2\right] dM' \\
&\quad \geq \int_{M'} \left[\frac{(5n^2+4n+3){\varrho'}^2}{n(n+1)} + 2|B'|^2\right] dM' \,.
\end{aligned} \tag{3.6}$$

Since (M, g, J) is Kähler with constant holomorphic sectional curvature κ, then by Lemma 2.1, $B = 0$. Further, since ϱ is constant, then $a_{0,0} = a'_{0,0}$, $a_{1,0} = a'_{1,0}$ and the Schwarz inequality imply that $\left(\int_{M'} {\varrho'}^2 dM'\right)\left(\int_{M'} dM'\right) \geq \left(\int_{M'} \varrho' dM'\right)^2 = \left(\int_M \varrho dM\right)^2 = \varrho^2[\mathrm{vol}(M)]^2 = \varrho^2 \,\mathrm{vol}(M')\,\mathrm{vol}(M) = \mathrm{vol}(M') \int_M \varrho^2 dM$. Then by (3.6), $|B'|^2 = 0$. Hence, by Corollary 2.3, (M', g', J') is Kähler and of constant holomorphic sectional curvature $\kappa' = \kappa$.

Letting $p = 1$ in (1.4)–(1.9) gives

$$a_{0,1} = 2n \int_M dM = 2n \,\mathrm{vol}(M)\,, \tag{3.7}$$

$$a_{1,1} = \frac{n-3}{3} \int_M \varrho dM\,, \tag{3.8}$$

$$\begin{aligned} a_{2,1} = \frac{1}{180} \int_M \Big[&(5n-30)\varrho^2 + (-2n+90)|Rc|^2 \\ &+ (2n-15)|R|^2\Big] dM\,. \end{aligned} \tag{3.9}$$

In dimension $2n \geq 16$, substitute (2.5) in (3.9). Then, since $E = 0$ and $a_{2,1} = a'_{2,1}$, we have

$$\begin{aligned}
&\int_M \left[\frac{(5n^3 - 26n^2 + 12n + 60)\varrho^2 + (6n - 45)\varrho\varrho^*}{n(n+1)} + (2n-15)|B|^2 \right] dM \\
&\quad = \int_M \left[\frac{(5n^3 - 26n^2 + 18n + 15)\varrho^2}{n(n+1)} + (2n-15)|B|^2 \right] dM \\
&\quad = \int_{M'} \left[\frac{(5n^3 - 26n^2 + 12n + 60){\varrho'}^2 + (6n - 45)\varrho'{\varrho'}^*}{n(n+1)} \right. \\
&\qquad\qquad \left. + (2n-15)|B'|^2 \right] dM' \\
&\quad \geq \int_{M'} \left[\frac{(5n^3 - 26n^2 + 18n + 15){\varrho'}^2}{n(n+1)} + (2n-15)|B'|^2 \right] dM' \,.
\end{aligned} \tag{3.10}$$

The result then follows in a similar way as in the case $p = 0$.

Letting $p = 2$ in (1.4)–(1.9) gives

$$a_{0,2} = (2n^2 - n)\int_M dM = (2n^2 - n)\,\mathrm{vol}(M)\,, \tag{3.11}$$

$$a_{1,2} = \frac{2n^2 - 13n + 12}{6}\int_M \varrho dM\,, \tag{3.12}$$

$$\begin{aligned}
a_{2,2} = \frac{1}{360}\int_M &\Big[(10n^2 - 125n + 300)\varrho^2 \\
&+ (-4n^2 + 362n - 1080)|Rc|^2 \\
&+ (4n^2 - 62n + 240)|R|^2\Big] dM\,.
\end{aligned} \tag{3.13}$$

In dimension $2n = 4, 6, 8, 14$ or $2n \geq 18$, substitute (2.5) in (3.13). Then, since $E = 0$ and $a_{2,2} = a'_{2,2}$, we have

$$\begin{aligned}
&\int_M \left[\frac{(10n^4 - 117n^3 + 350n^2 + 3n - 780)\varrho^2 + (12n^2 - 186n + 720)\varrho\varrho^*}{n(n+1)} \right. \\
&\qquad\qquad \left. + (4n^2 - 62n + 240)|B|^2 \right] dM
\end{aligned}$$

$$= \int_M \left[\frac{(10n^4 - 117n^3 + 362n^2 - 183n - 60)\varrho^2}{n(n+1)} \right.$$

$$\left. +(4n^2 - 62n + 240)|B|^2 \right] dM \tag{3.14}$$

$$= \int_{M'} \left[\frac{(10n^4 - 117n^3 + 350n^2 + 3n - 780){\varrho'}^2 + (12n^2 - 186n + 720)\varrho'{\varrho'}^*}{n(n+1)} \right.$$

$$\left. +(4n^2 - 62n + 240)|B'|^2 \right] dM'$$

$$\geq \int_{M'} \left[\frac{(10n^4 - 117n^3 + 362n^2 - 183n - 60){\varrho'}^2}{n(n+1)} \right.$$

$$\left. +(4n^2 - 62n + 240)|B'|^2 \right] dM' .$$

The result then follows in a similar way as in the case $p = 0$.

If (M', g') is almost Hermitian Einstein of nonzero scalar curvature ϱ' with $\varrho' = {\varrho'}^*$, then we have equality in (3.6), (3.10) and (3.14) and the results follow in the same way.

Corollary 3.3. *If $(\mathbb{C}P^n, g_0, J_0)$ is complex projective space with the Fubini–Study metric, (M', g', J') a compact almost Hermitian Einstein manifold of either positive scalar curvature ϱ' with $\varrho' \leq {\varrho'}^*$ or nonzero scalar curvature ϱ' with $\varrho' = {\varrho'}^*$, and $\mathrm{Spec}^p(M', g', J') = \mathrm{Spec}^p(\mathbb{C}P^n, g_0, J_0)$ for $p = 0, 1$ or 2, then (M', g', J') is Kähler and holomorphically isometric to $(\mathbb{C}P^n, g_0, J_0)$ in the following cases: $p = 0$ and $2n \geq 2$, hence $(\mathbb{C}P^n, g_0, J_0)$ is characterized by the spectrum in every dimension in these classes of manifolds; $p = 1$ and $2n = 2$ or $2n \geq 16$; $p = 2$ and $2n = 2, 4, 6, 8, 14$ or $2n \geq 18$.*

Proof. Since $(\mathbb{C}P^n, g_0, J_0)$ is the only Kähler manifold with a metric of positive constant holomorphic sectional curvature κ, then (M', g', J') is Kähler with constant holomorphic sectional curvature κ and (M', g', J') and $(\mathbb{C}P^n, g_0, J_0)$ are holomorphically isometric.

Corollary 3.4. *If (M, g, J) is a compact Kähler manifold of constant holomorphic sectional curvature κ and (M', g', J') is either compact almost Kähler Einstein with positive scalar curvature or compact nearly Kähler*

Einstein with negative scalar curvature and $\mathrm{Spec}^p(M,g) = \mathrm{Spec}^p(M',g')$ *for* $p = 0, 1$ *or 2, then the conclusion of Theorem 3.2b holds.*

Proof. The result follows immediately from Lemma 3.1 and Theorem 3.2.

We note here that a compact almost Kähler Einstein manifold of non-negative scalar curvature is Kähler [9].

References

[1] M. Berger, P. Gauduchon and E. Mazet, Le spectre d'une variété riemannienne, Lecture Notes in Math., vol. 194, *Springer Verlag, Berlin and New York*, 1971.

[2] B.–Y. Chen and L. Vanhecke, The spectrum of the Laplacian of Kähler manifolds, *Proc. Amer. Math. Soc.* **79** (1980), 82–86.

[3] L. Friedland and C. C. Hsiung, A certain class of almost Hermitian manifolds, *Tensor N. S.* **48** (1989), 252–263.

[4] P. Gilkey, Spectral geometry and the Kähler condition for complex manifolds, *Inventiones* **26** (1974), 231–258.

[5] P. Gilkey, The spectral geometry of real and complex manifolds, *Proc. Symposia Pure Math., Amer. Math. Soc.* **27** (1975), 265–280.

[6] P. Gilkey and J. Sacks, Spectral geometry and manifolds of constant holomorphic sectional curvature, *ibid.*, 281–285.

[7] S. I. Goldberg, A characterization of complex projective space, *C. R. Math. Rep. Acad. Sci. Canada* **6** (1984), 193–198.

[8] V. K. Patodi, Curvature and the fundamental solution of the heat operator, *J. Ind. Math. Soc.* **34** (1970), 269–285.

[9] K. Sekigawa, On some compact Einstein almost Kähler manifolds, *J. Math. Soc. Japan* **39** (1987), 677–684.

[10] S. Tanno, Eigenvalues of the Laplacian of Riemannian manifolds, *Tôhoku Math. J.* **25** (1973), 391–403.

[11] S. Tanno, The spectrum of the Laplacian for 1-forms, *Proc. Amer. Math. Soc.* **45** (1974), 125–129.

LEW FRIEDLAND
DEPARTMENT OF MATHEMATICS
STATE UNIVERSITY OF NEW YORK
GENESEO, NEW YORK 14454
U.S.A

(Received January 6, 1994)