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Estimates of fractional integral operator with variable kernel

By JIECHENG CHEN (Jinhua), DASHAN FAN (Milwaukee) and TAOTAO ZHENG (Hangzhou)

Abstract. In this paper, we use interpolation and iterative methods to study the fractional integral operator $\mathcal{F}_{\Omega,\alpha}$ with variable kernel. We obtain the sharp size condition on Ω to ensure the (L^q, L^p) boundedness of $\mathcal{F}_{\Omega,\alpha}$ for $0<\alpha<n$, $1<p<\infty$. We also obtain some corresponding estimates of the rough bilinear fractional integral.

1. Introduction and main results

Let S^{n-1} be the unit sphere in Euclidean $\mathbb{R}^n$ $(n \geq 2)$, and $d\sigma$ be the area element on S^{n-1} induced by the Lebesgue measure on $\mathbb{R}^n$. A function $\Omega(x,z)$ defined on $\mathbb{R}^n \times \mathbb{R}^n$ is said to belong to $L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})$, $r \geq 1$, if it satisfies the following conditions: for any x, $z \in \mathbb{R}^n$ and $\lambda \geq 0$,

$$\Omega(x, \lambda z) = \Omega(x, z), \tag{1.1}$$

$$\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})} = \sup_{x\in\mathbb{R}^n} \left(\int_{S^{n-1}} |\Omega(x,z')|^r d\sigma(z') \right)^{\frac{1}{r}} < \infty, \tag{1.2}$$

where $z' = \frac{z}{|z|}$, for any $z \in \mathbb{R}^n \setminus \{0\}$.

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The third author is the corresponding author.

If $\Omega(x,z') \in L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})$ satisfies the mean zero property

$$\int_{S^{n-1}} \Omega(x,z')d\sigma(z') = 0 \quad \text{for all } x \in \mathbb{R}^n, \tag{1.3}$$

then the famous Calderón–Zygmund singular integral operator with variable kernel is defined on the space $\mathcal{S}(\mathbb{R}^n)$ of all Schwartz functions f by

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(x,y)}{|y|^n} f(x-y)dy, \quad f \in \mathcal{S}(\mathbb{R}^n).$$

For the operator T, the following result is well known.

Theorem A ([2], [3], [9]). *Let $n \geq 2$. If the function Ω satisfies conditions* (1.1), (1.3) *and* $\Omega(x,z') \in L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})$, *then the following inequality holds:*

$$\|Tf\|_{L^p(\mathbb{R}^n)} \leq C_{r,p} \|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})} \|f\|_{L^p(\mathbb{R}^n)},$$

provided that

(1) $\frac{1}{r} < \frac{1}{p'} \frac{n}{n-1}$ if $1 < p \leq 2$ $\left(p' = \frac{p}{p-1}\right)$;

(2) $\frac{1}{r} < \frac{1}{p} \frac{1}{n-1} + \frac{1}{p'}$ if $2 \leq p < \infty$.

In this paper, we will study the fractional integral operator

$$\mathcal{F}_{\Omega,\alpha} f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x,y)}{|y|^{n-\alpha}} f(x-y)dy,$$

where $0 < \alpha < n$, $f \in \mathcal{S}(\mathbb{R}^n)$ and Ω satisfies (1.1) and (1.2). In this case, the kernel of $\mathcal{F}_{\Omega,\alpha}$ has less singularity in a neighborhood of the origin than the kernel of singular integral operator T, and one does not need to assume the cancellation condition (1.3) on Ω in the definition of $\mathcal{F}_{\Omega,\alpha}$. On the other hand, when $\Omega = 1$, $\mathcal{F}_{\Omega,\alpha}$ is the Riesz potential

$$\Re_\alpha(f)(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy,$$

which plays significant roles in analysis, partial differential equations, probability theory and in many other fields of mathematics, via the Hardy–Littlewood–Sobolev embedding theory.

In 1971, Muckenhoupt and Wheeden [13] studied the power-weighted (L^q, L^p) boundedness of $\mathcal{F}_{\Omega,\alpha} f$ for all $0 < \alpha < n$. In the unweighted case, their theorem can be stated as follows.

Theorem B ([13]). *Let $n \geq 2$. Suppose that $0 < \alpha < n$, $1 < q < n/\alpha$ and $1/p = 1/q - \alpha/n$. If $\Omega(x, z') \in L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})$ for $r > q'$, then there exists a constant C independent of f and Ω, such that*

$$\|\mathcal{F}_{\Omega,\alpha} f\|_{L^p(\mathbb{R}^n)} \leq C\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})}\|f\|_{L^q(\mathbb{R}^n)},$$

and there is no such C if $r < q'$.

As we see, Muckenhoupt and Wheeden pointed out that the inequality in Theorem B cannot be improved if $r < q'$ for all $0 < \alpha < n$ and all indices p, q satisfying $1 < q < n/\alpha$ and $1/p = 1/q - \alpha/n$. However, when $0 < \alpha < 1/2$, the condition in Theorem B in fact can be improved. To address this interesting phenomenon, Chen, Ding and Fan [4], [5], [10] published a series of papers to study the fractional integral operator $\mathcal{F}_{\Omega,\alpha}$, among other things (see also [1], [6], [8] for some related results). We list one result related to this paper in the following theorem.

Theorem C ([4]). *Let $n \geq 2$, $0 < \alpha < 1/2$ and $\Omega(x, z') \in L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})$. If $r > 2\rho(n-1)/(n-2\alpha)$, where $\rho = (1/2 - \alpha/n)(1/p' - \alpha/n)$, then*

$$\|\mathcal{F}_{\Omega,\alpha} f\|_{L^p(\mathbb{R}^n)} \leq C\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})}\|f\|_{L^q(\mathbb{R}^n)}$$

with $1/p = 1/q - \alpha/n$.

Note that $\rho = (1/2 - \alpha/n)(1/p' - \alpha/n)$, $1/p = 1/q - \alpha/n$, the size condition $r > 2\rho(n-1)/(n-2\alpha)$ in Theorem C is equivalent to

$$r > \frac{n-1}{n} q',$$

which is obviously better than the size condition $r > q'$ in Theorem B since S^{n-1} is compact.

We also can show that the size condition Ω in Theorem C is the sharp one.

Theorem 1. *If in the inequality (1.2) we take $r = \frac{n-1}{n} q'$, the transform of $\mathcal{F}_{\Omega,\alpha} f$ of an $f \in L^q$ $(\frac{n}{n-\alpha} < q < \frac{n}{\alpha})$ needs not be in L^p $(1 < p < 2)$.*

Proof. We will modify the proof for a similar problem on the singular integral (see page 223 in [2] by Calderón and Zygmund). Denote $K_{\Omega,\alpha}(x, y) = \frac{\Omega(x,y')}{|y|^{n-\alpha}}$. We have

$$\mathcal{F}_{\Omega,\alpha} f(x) = \int_{\mathbb{R}^n} K_{\Omega,\alpha}(x, x-y) f(y) dy.$$

Take for $f(y)$ the function equal to 1 for $|y| \leq 1$ and equal to zero elsewhere. Then $f \in L^q(\mathbb{R}^n)$ for any $q \geq 1$. Let r be any positive number. We define a function $\Omega(x, y')$ on $\mathbb{R}^n \times S^{n-1}$ by assuming $\Omega(x, y') = 0$ for $|x| \leq 20$. When $|x| > 20$, denote the subset S_x of S^{n-1} by

$$S_x = \left\{ y' \in S^{n-1} : \left| y' - \frac{x}{|x|} \right| < \frac{10}{|x|} \right\},$$

and define $\Omega(x, y')$ as

(a) equal to $|x|^{(n-1)/r}$ if $y' \in S_x$;

(b) equal to zero if $y' \notin S_x$.

Let $A(S_x)$ denote the surface area of S_x. Since $A(S_x) \approx |x|^{-(n-1)}$ uniformly for all $|x| \geq 20$, by the definition, we have that

$$\sup_{x \in \mathbb{R}^n} \int_{S^{n-1}} |\Omega(x, y')|^r d\sigma(y) = \sup_{x \in \mathbb{R}^n} \int_{S_x} |x|^{(n-1)} d\sigma(y) < C.$$

This shows $\Omega(x, y') \in L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})$.

On the other hand, we notice that for sufficiently large $|x|$,

$$\left| (x-y)' - \frac{x}{|x|} \right| = \left| \frac{(x-y)}{|x-y|} - \frac{x}{|x|} \right| < \frac{6}{|x|}$$

uniformly for all $|y| \leq 1$. Hence, by the choice of f and Ω, we have that

$$\begin{aligned} |\mathcal{F}_{\Omega,\alpha} f(x)| &= \left| \int_{\mathbb{R}^n} K_{\Omega,\alpha}(x, x-y) f(y) dy \right| = \left| \int_{|y| \leq 1} \frac{\Omega(x, (x-y)')}{|x-y|^{n-\alpha}} dy \right| \\ &\approx |x|^{-n+\alpha} \left| \int_{|y| \leq 1} \Omega(x, (x-y)') dy \right| \approx \frac{C}{|x|^\eta} \quad \text{as } |x| \to \infty, \end{aligned}$$

where $\eta = n - \alpha - (n-1)/r$. Now, in order to ensure $\mathcal{F}_{\Omega,\alpha} f(x)$ to be in L^p, we must assume that $\eta > n/p$, which is equivalent to $r > \frac{n-1}{n} q'$. This completes the proof. □

Inspired by Theorem A, Theorem B and Theorem C, it naturally raises the following two questions.

Question 1: How to extend Theorem C to the case of $2 < p < \infty$?

Question 2: Does Theorem B hold at the endpoint $r = q'$?

Question 2 was solved in the case $1 < p \leq 2$ (see Theorem D). Thus, similar to Question 1, we need to address this question in the case $2 < p < \infty$.

Now, we state our first main result about the fractional integral operator. The following theorem solves Question 1.

Theorem 2. *For $0 < \alpha < 1/2$, $n \geq 2$, let $1 < q < n/\alpha$, $1/p = 1/q - \alpha/n$ and $\Omega(x, z') \in L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})$. If $2 \leq p < \infty$ and $\frac{1}{r} < \frac{1}{q'} + \frac{n-2\alpha}{pn(n-1)}$, then*

$$\|\mathcal{F}_{\Omega,\alpha} f\|_{L^p(\mathbb{R}^n)} \leq C\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})}\|f\|_{L^q(\mathbb{R}^n)}.$$

It is easy to check that the theorem improves the result in Theorem B in the case $0 < \alpha < 1/2$. We will prove Theorem 2 by using an interpolation between the $L^{\frac{2n}{n+2\alpha}} \to L^2$ estimate and the inequality in Theorem B.

The reader might notice that both Theorem 1 and Theorem C have a restriction $0 < \alpha < \frac{1}{2}$. One naturally expects to remove this restriction. We note that Chen, Ding, Fan in [4] improved Theorem C in the case $1 < p \leq 2$.

Theorem D ([4])**.** *Let $n \geq 2$. Suppose that $0 < \alpha < n$, $1 < q < n/\alpha$, $1/p = 1/q - \alpha/n$, $1 < p \leq 2$ and $\Omega(x, z') \in L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})$. If $r \geq q'$, then*

$$\|\mathcal{F}_{\Omega,\alpha} f\|_{L^p(\mathbb{R}^n)} \leq C\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})}\|f\|_{L^q(\mathbb{R}^n)}.$$

As an application of Theorem 2, in the following theorem we extend Theorem D to the full range $1 < p < \infty$, which solves Question 2.

Theorem 3. *Let $n \geq 2$, $1 < q < n/\alpha$, $1/p = 1/q - \alpha/n$, $0 < \alpha < n$ and $\Omega(x, z') \in L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})$. If $r \geq q'$, then*

$$\|\mathcal{F}_{\Omega,\alpha} f\|_{L^p(\mathbb{R}^n)} \leq C\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})}\|f\|_{L^q(\mathbb{R}^n)}.$$

We summarize the above results in Theorems 1–3 and Theorems B–D in the following

Theorem 4. *Let $n \geq 2$. Suppose that $1 < q < n/\alpha$ and $1/p = 1/q - \alpha/n$. We have the following conclusions.*

(1) *For $0 < \alpha < 1/2$ and $1 < p \leq 2$, there exists a constant $C > 0$ such that for all $f \in L^q(\mathbb{R}^n)$ and all $\Omega(x, z') \in L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})$*

$$\|\mathcal{F}_{\Omega,\alpha} f\|_{L^p(\mathbb{R}^n)} \leq C\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})}\|f\|_{L^q(\mathbb{R}^n)}$$

if and only if

$$r > \frac{n-1}{n}q'.$$

(2) *For $0 < \alpha < 1/2$ and $2 \leq p < \infty$, there exists a constant $C > 0$ such that for all $f \in L^q(\mathbb{R}^n)$ and all $\Omega(x, z') \in L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})$,*

$$\|\mathcal{F}_{\Omega,\alpha} f\|_{L^p(\mathbb{R}^n)} \leq C\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})}\|f\|_{L^q(\mathbb{R}^n)}$$

if

$$\frac{1}{r} < \frac{1}{q'} + \frac{n-2\alpha}{pn(n-1)}.$$

(3) *For* $1/2 < \alpha \le n$ *and* $1 < p < \infty$, *there exists a constant* $C > 0$ *such that for all* $f \in L^q(\mathbb{R}^n)$ *and all* $\Omega(x, z') \in L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})$,

$$\|\mathcal{F}_{\Omega,\alpha} f\|_{L^p(\mathbb{R}^n)} \le C\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})}\|f\|_{L^q(\mathbb{R}^n)}$$

if and only if $r \ge q'$.

Remark 1. If we consider another fractional integral

$$\mathcal{L}_{\Omega,\alpha} f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x,y)}{|y|^{n-\alpha}} f(x+y)dy,$$

all conclusions in Theorem 4 still hold if we replace $\mathcal{F}_{\Omega,\alpha}$ by $\mathcal{L}_{\Omega,\alpha}$.

We notice that some authors considered fractional Marcinkiewicz integrals with variable kernels. For instance, in [12] (see also [11]), the authors study the fractional Marcinkiewicz integrals with variable kernels in the form of

$$\mu_{\Omega,\alpha}(f)(x) = \left(\int_0^\infty \left|\int_{|x-y|<t} \frac{\Omega(x,x-y)}{|x-y|^{n-1}} f(y)dy\right|^2 \frac{dt}{t^{3-2\alpha}}\right)^{\frac{1}{2}}, \quad 0 < \alpha \le 1.$$

Also, some people study the fractional integral of Marcinkiewicz type

$$M_{\Omega,\alpha}(f)(x) = \left(\int_0^\infty \left|\int_{|x-y|<t} \frac{\Omega(x,x-y)}{|x-y|^{n-1-\alpha}} f(y)dy\right|^2 \frac{dt}{t^3}\right)^{\frac{1}{2}}, \quad 0 < \alpha \le 1.$$

For $1 < p \le 2$, Lin *et al.* [12] gave some estimates on $\mu_{\Omega,\alpha}$.

Theorem E ([12])**.** *Let* $n \ge 2$ *and* $0 < \alpha < \frac{1}{2}$. *If* Ω *satisfies* (1.1), (1.2) *and* (1.3) *for* $r = 2$, *then there exists a constant* C *independent of* f *such that*

$$\|\mu_{\Omega,\alpha}(f)\|_{L^2(\mathbb{R}^n)} \le C\|f\|_{L^{\frac{2n}{n+2\alpha}}(\mathbb{R}^n)}.$$

In addition, for $\frac{n}{n-\alpha} < p < 2$ *and* $1/p = 1/q - \alpha/n$, *if* $\Omega(x,z') \in L^\infty(\mathbb{R}^n) \times L^2(S^{n-1})$ *and satisfies* $L^{2,\alpha}$ *Dini conditions, then there exists a constant* C *independent of* f *such that*

$$\|\mu_{\Omega,\alpha}(f)\|_{L^p(\mathbb{R}^n)} \le C\|f\|_{L^q(\mathbb{R}^n)}.$$

It is easy to see that both integrals $\mu_{\Omega,\alpha}(f)(x)$ and $M_{\Omega,\alpha}(f)(x)$ are pointwise dominated by the fractional integral $\mathcal{F}_{\Omega,\alpha}f$. In the fractional case, we may assume that both Ω and f are non-negative. Thus, by the Minkowski integral inequality, we obtain

$$\begin{aligned}|\mu_{\Omega,\alpha}(f)(x)| &\le \int_{\mathbb{R}^n} \frac{\Omega(x,x-y)}{|x-y|^{n-1}} f(y) \left(\int_0^\infty \chi_{|x-y|<t}(t)\frac{dt}{t^{3-2\alpha}}\right)^{\frac{1}{2}} dy \\ &= \int_{\mathbb{R}^n} \frac{\Omega(x,x-y)}{|x-y|^{n-1}} f(y) \left(\int_{|x-y|}^\infty \frac{dt}{t^{3-2\alpha}}\right)^{\frac{1}{2}} dy \\ &\simeq \int_{\mathbb{R}^n} \frac{\Omega(x,x-y)}{|x-y|^{n-\alpha}} f(y)dy = \mathcal{F}_{\Omega,\alpha}f(x).\end{aligned}$$

Similarly,

$$\begin{aligned}|M_{\Omega,\alpha}(f)(x)| &\le \int_{\mathbb{R}^n} \frac{\Omega(x,x-y)}{|x-y|^{n-1-\alpha}} f(y) \left(\int_0^\infty \chi_{\{|x-y|<t\}}(t)\frac{dt}{t^3}\right)^{\frac{1}{2}} dy \\ &= \int_{\mathbb{R}^n} \frac{\Omega(x,x-y)}{|x-y|^{n-1-\alpha}} f(y) \left(\int_{|x-y|}^\infty \frac{dt}{t^3}\right)^{\frac{1}{2}} dy \\ &\simeq \int_{\mathbb{R}^n} \frac{\Omega(x,x-y)}{|x-y|^{n-\alpha}} f(y)dy = \mathcal{F}_{\Omega,\alpha}f(x).\end{aligned}$$

As an application of our results, we will study the rough bilinear fractional integral

$$\widetilde{B}_{\Omega,\alpha}(f,g)(x) = \int_{\mathbb{R}^n} f(x+y)g(x-y)\frac{\Omega(x,y')}{|y|^{n-\alpha}}dy,$$

where $0 < \alpha < n$. The following result is an easy consequence of our results together with an application of Hölder's inequality, which is an improvement of Proposition 1 in [7].

Theorem 5. *Let $n \ge 2$. Suppose that $p > \frac{n-\alpha}{n}, 1 < p_1, p_2 < \infty$, $1/p = 1/p_1 + 1/p_2 - \alpha/n$ and $1/\sigma = 1/p_1 + 1/p_2$. We have the following conclusions.*

(1) *For $0 < \alpha < 1/2$ and $1 < p \le 2$, there exists a constant $C > 0$ such that for all $f \in L^{p_1}(\mathbb{R}^n)$, $g \in L^{p_2}(\mathbb{R}^n)$ and all $\Omega(x,z') \in L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})$,*

$$\|\widetilde{B}_{\Omega,\alpha}(f,g)\|_{L^p(\mathbb{R}^n)} \le C\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})}\|f\|_{L^{p_1}(\mathbb{R}^n)}\|g\|_{L^{p_2}(\mathbb{R}^n)}$$

if

$$r > \frac{n-1}{n}\sigma'.$$

(2) *For* $0 < \alpha < 1/2$ *and* $2 \leq p < \infty$, *there exists a constant* $C > 0$ *such that for all* $f \in L^{p_1}(\mathbb{R}^n)$, $g \in L^{p_2}(\mathbb{R}^n)$ *and all* $\Omega(x, z') \in L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})$,

$$\|\widetilde{B}_{\Omega,\alpha}(f,g)\|_{L^p(\mathbb{R}^n)} \leq C\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})}\|f\|_{L^{p_1}(\mathbb{R}^n)}\|g\|_{L^{p_2}(\mathbb{R}^n)}$$

if

$$\frac{1}{r} < \frac{1}{\sigma'} + \frac{n-2\alpha}{pn(n-1)}.$$

(3) *For* $1/2 < \alpha \leq n$ *and* $1 < p < \infty$, *there exists a constant* $C > 0$ *such that for all* $f \in L^{p_1}(\mathbb{R}^n)$, $g \in L^{p_2}(\mathbb{R}^n)$ *and all* $\Omega(x, z') \in L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})$,

$$\|\widetilde{B}_{\Omega,\alpha}(f,g)\|_{L^p(\mathbb{R}^n)} \leq C\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})}\|f\|_{L^{p_1}(\mathbb{R}^n)}\|g\|_{L^{p_2}(\mathbb{R}^n)}$$

if $r \geq \sigma'$.

This paper is organized as follows. In Section 2, we prove Theorem 2. Theorem 3 and Theorem 5 will be proved in Section 3.

Throughout the paper, the letter C always denotes a positive constant that may vary at each occurrence, but is independent of all essential variables.

2. Proof of Theorem 2

We invoke the interpolation methods used in [3]. Suppose now $2 \leq p < \infty$, consider the kernel

$$\Omega(x, z', \xi) = |\Omega(x, z')|^{\ell_1(\xi)} \operatorname{sgn} \Omega(x, z'),$$

where ξ is a complex parameter. For $f \in C_0^\infty$, consider also the function $f(x,\xi)$ and $g(x,\xi)$,

$$f(x,\xi) = |f(x)|^{\ell_2(\xi)} \operatorname{sgn} f(x), \quad g(x,\xi) = |g(x)|^{\ell_3(\xi)} \operatorname{sgn} g(x),$$

where $g(x)$ is a simple function, and $\ell_i(\xi) = a_i\xi + b_i$ $(i = 1, 2, 3)$ are linear functions whose coefficients will be determined later.

We set

$$G(\xi) = \int_{x\in\mathbb{R}^n}\int_{y\in\mathbb{R}^n} \frac{\Omega(x, x-y, \xi)}{|x-y|^{n-\alpha}} f(y,\xi)dy\ g(x,\xi)dx.$$

Consider the point $\left(\frac{1}{p}, \frac{1}{r}\right)$ in the square

$$(Q) \quad 0 \leq s \leq 1, \quad 0 \leq t \leq 1, \quad \text{where } \frac{1}{r} < \frac{1}{q'} + \frac{n-2\alpha}{pn(n-1)}.$$

The point $\left(\frac{1}{p}, \frac{1}{r}\right)$ lies below the segment joining $\left(\frac{1}{2}, \frac{n-2\alpha}{2(n-1)}\right)$ and $\left(0, \frac{n-\alpha}{n}\right)$. We may assume $\frac{1}{r} > \frac{1}{q'} = 1 - \frac{\alpha}{n} - \frac{1}{p}$, therefore, $\left(\frac{1}{p}, \frac{1}{r}\right)$ lies on the segment from point $\left(\frac{1}{p_1}, \frac{1}{r_1}\right)$ to point $\left(\frac{1}{2}, \frac{1}{r_0}\right)$, where $\frac{1}{r_1} < 1 - \frac{\alpha}{n} - \frac{1}{p_1}$ and $\frac{1}{r_0} < \frac{n-2\alpha}{2(n-1)}$.

There is an s, $0 < s < 1$, such that

$$\frac{1}{r} = \frac{1-s}{r_0} + \frac{s}{r_1}, \quad \frac{1}{p} = \frac{1-s}{2} + \frac{s}{p_1}, \quad \frac{1}{q} = \frac{1-s}{q_0} + \frac{s}{q_1}, \tag{2.1}$$

where $q_0 = \frac{2n}{n+2\alpha}$, $\frac{1}{q_1} = \frac{1}{p_1} + \frac{\alpha}{n}$.

Let

$$\lambda_1(\xi) = \frac{1-\xi}{r_0} + \frac{\xi}{r_1}, \quad \lambda_2(\xi) = \frac{1-\xi}{2} + \frac{\xi}{p_1}, \quad \lambda_3(\xi) = \frac{1-\xi}{q_0} + \frac{\xi}{q_1},$$

and define functions ℓ_1, ℓ_2, ℓ_3 by

$$\ell_1(\xi) = r\lambda_1(\xi), \quad \ell_2(\xi) = q\lambda_3(\xi), \quad \ell_3(\xi) = \frac{p}{p-1}[1 - \lambda_2(\xi)].$$

Then, for $\xi = s$ we have $\ell_1(s) = \ell_2(s) = \ell_3(s) = 1$.

For $\operatorname{Re}\xi = 0$,

$$\operatorname{Re}\ell_1(\xi) = \frac{r}{r_0}, \quad \operatorname{Re}\ell_2(\xi) = \frac{q}{q_0}, \quad \operatorname{Re}\ell_3(\xi) = \frac{p'}{2}.$$

Using Hölder's inequality and the L^2 boundedness of $\mathcal{F}_{\Omega,\alpha}$ ($p = 2$ in Theorem C), we obtain

$$\begin{aligned}
|G(\xi)| &\le \|g(\cdot,\xi)\|_{L^2(\mathbb{R}^n)} \left(\int_{x\in\mathbb{R}^n} \left| \int_{y\in\mathbb{R}^n} \frac{\Omega(x, x-y, \xi)}{|x-y|^{n-\alpha}} f(y,\xi) dy \right|^2 dx \right)^{\frac{1}{2}} \\
&\le C \left\| |g|^{\frac{p'}{2}} \right\|_{L^2(\mathbb{R}^n)} \left\| |f|^{\frac{q}{q_0}} \right\|_{L^{q_0}(\mathbb{R}^n)} \|\Omega(\cdot,\cdot,\xi)\|_{L^\infty(\mathbb{R}^n)\times L^{r_0}(S^{n-1})} \\
&\le C \|g\|_{L^{p'}(\mathbb{R}^n)}^{\frac{p'}{2}} \|f\|_{L^q(\mathbb{R}^n)}^{\frac{q}{q_0}} \sup_x \left(\int_{S^{n-1}} |\Omega(x,z')|^r d\sigma(z') \right)^{\frac{1}{r_0}}.
\end{aligned}$$

For $\operatorname{Re}\xi = 1$,

$$\operatorname{Re}\ell_1(\xi) = \frac{r}{r_1}, \quad \operatorname{Re}\ell_2(\xi) = \frac{q}{q_1}, \quad \operatorname{Re}\ell_3(\xi) = \frac{p'}{p_1'}.$$

For $0<\alpha<\frac{1}{2}$, by Hölder's inequality and Theorem B, we have

$$\begin{aligned}|G(\xi)| &\le \|g(\cdot,\xi)\|_{L^{p_1'}(\mathbb{R}^n)}\left(\int_{x\in\mathbb{R}^n}\left|\int_{y\in\mathbb{R}^n}\frac{\Omega(x,x-y,\xi)}{|x-y|^{n-\alpha}}f(y,\xi)dy\right|^{p_1}dx\right)^{\frac{1}{p_1}}\\ &\le C\left\||g|^{\frac{p'}{p_1'}}\right\|_{L^{p_1'}(\mathbb{R}^n)}\left\||f|^{\frac{q}{q_1}}\right\|_{L^{q_1}(\mathbb{R}^n)}\|\Omega(\cdot,\cdot,\xi)\|_{L^\infty(\mathbb{R}^n)\times L^{r_1}(S^{n-1})}\\ &\le C\|g\|_{L^{p'}(\mathbb{R}^n)}^{\frac{p'}{p_1'}}\|f\|_{L^q(\mathbb{R}^n)}^{\frac{q}{q_1}}\sup_x\left(\int_{S^{n-1}}|\Omega(x,z')|^r d\sigma(z')\right)^{\frac{1}{r_1}}.\end{aligned}$$

By the three-line theorem, we get

$$|G(s)|\le C\|g\|_{L^{p'}(\mathbb{R}^n)}^{p'\left(\frac{1-s}{2}+\frac{s}{p_1'}\right)}\|f\|_{L^q(\mathbb{R}^n)}^{q\left(\frac{1-s}{q_0}+\frac{s}{q_1}\right)}\sup_x\left\{\left(\int_{S^{n-1}}|\Omega(x,z')|^r d\sigma(z')\right)\right\}^{\frac{1-s}{r_0}+\frac{s}{r_1}}.$$

According to (2.1), the exponent of both $\|g\|_{L^{p'}(\mathbb{R}^n)}$ and $\|f\|_{L^q(\mathbb{R}^n)}$ is 1, and the exponent of $\sup_x\{\cdots\}$ is $\frac{1}{r}$.

Now, recalling that ℓ_1, ℓ_2, ℓ_3 are all equal to 1 at $\xi=s$, we have

$$\|\mathcal{F}_{\Omega,\alpha}f\|_{L^p(\mathbb{R}^n)}\le C\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})}\|f\|_{L^q(\mathbb{R}^n)} \text{ with } \frac{1}{r}<\frac{1}{q'}+\frac{n-2\alpha}{pn(n-1)}.$$

Theorem 2 is proved. □

3. Proof of Theorem 3 and Theorem 5

3.1. Proof of Theorem 3. By Theorem D, it suffices to show the case $2<p<\infty$. Without loss of generality, we may assume that both Ω and f are non-negative. Since we have obtained a better result when $0<\alpha<1/2$ in Theorem 2, our strategy is to use a two-step iteration to extend α to the full range $(0,n)$. The first step is to use the result of Theorem 2 to obtain the boundedness of $\mathcal{F}_{\Omega,\alpha}$ under the condition $r=q'$ when $1/2\le\alpha\le n/2$. Then continue this process to obtain the theorem for $\alpha\in(n/2,n)$. We now begin our proof of step 1 by fixing an $\varepsilon_0\in[\sqrt{2}-1,\frac{1}{2})$ and letting $p_0=(2+\varepsilon_0)n$. For this choice of p_0, it is easy to see that, for any $\frac{1}{2}\le\alpha\le\frac{n}{2}$, we have $\alpha-\varepsilon_0\le n\left(\frac{1}{2}-\frac{1}{p}\right)$ if $p\ge p_0$. Let δ be a number to be chosen later. We write

$$\mathcal{F}_{\Omega,\alpha}f(x)\approx\int_{\mathbb{R}^n}\frac{\Omega(x,y)}{|y|^{n-\alpha}}f(x-y)dy$$

$$= \int_{|y|\le\delta} \frac{\Omega(x,y)}{|y|^{n-\alpha}} f(x-y)dy + \int_{|y|>\delta} \frac{\Omega(x,y)}{|y|^{n-\alpha}} f(x-y)dy := H_1 + H_2.$$

To estimate H_1, it is easy to see that

$$\begin{aligned} H_1 &= \int_{|y|\le\delta} \frac{|y|^{\alpha-\varepsilon_0}\,\Omega(x,y)}{|y|^{n-\varepsilon_0}} f(x-y)dy \\ &\le \delta^{\alpha-\varepsilon_0} \int_{\mathbb{R}^n} \frac{\Omega(x,y)}{|y|^{n-\varepsilon_0}} f(x-y)dy \le \delta^{\alpha-\varepsilon_0} \mathcal{F}_{\Omega,\varepsilon_0} f(x). \end{aligned}$$

For H_2, using Hölder's inequality and noticing that the condition $1/p = 1/q - \alpha/n$ implies $(\alpha-n)q' + n < 0$, we obtain

$$\begin{aligned} H_2 &\le \left(\int_{|y|>\delta} \left(\frac{\Omega(x,y)}{|y|^{n-\alpha}} \right)^{q'} dy \right)^{\frac{1}{q'}} \|f\|_{L^q(\mathbb{R}^n)} \\ &= \left(\int_\delta^\infty \int_{S^{n-1}} \Omega(x,y')^{q'} r^{(\alpha-n)q'} r^{n-1} dy' dr \right)^{\frac{1}{q'}} \|f\|_{L^q(\mathbb{R}^n)} \\ &\le \|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^{q'}(S^{n-1})} \left(\int_\delta^\infty r^{(\alpha-n)q'+n-1} dr \right)^{\frac{1}{q'}} \|f\|_{L^q(\mathbb{R}^n)} \\ &\le \|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})} \delta^{\alpha-\frac{n}{q}} \|f\|_{L^q(\mathbb{R}^n)}. \end{aligned}$$

Thus,

$$\begin{aligned} \mathcal{F}_{\Omega,\alpha} f(x) &\le \delta^{\alpha-\varepsilon_0} \mathcal{F}_{\Omega,\varepsilon_0} f(x) + \delta^{\alpha-\frac{n}{q}} \|f\|_{L^q(\mathbb{R}^n)} \|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})} \\ &= \delta^{\alpha-\varepsilon_0} \big(\mathcal{F}_{\Omega,\varepsilon_0} f(x) + \delta^{\varepsilon_0-\frac{n}{q}} \|f\|_{L^q(\mathbb{R}^n)} \|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})} \big). \end{aligned}$$

Now we take

$$\delta = \left(\frac{\mathcal{F}_{\Omega,\varepsilon_0} f(x)}{\|f\|_{L^q(\mathbb{R}^n)} \|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})}} \right)^{\frac{1}{\varepsilon_0-\frac{n}{q}}},$$

and denote $\kappa_0 = \frac{\alpha-\varepsilon_0}{\varepsilon_0-\frac{n}{q}}$. It follows that

$$\mathcal{F}_{\Omega,\alpha} f(x) \le (\mathcal{F}_{\Omega,\varepsilon_0} f(x))^{1+\kappa_0} \left(\|f\|_{L^q(\mathbb{R}^n)} \|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})} \right)^{-\kappa_0}. \tag{3.1}$$

In order to obtain the L^p boundedness of $\mathcal{F}_{\Omega,\alpha}$, we should make some estimates on $\mathcal{F}_{\Omega,\varepsilon_0}$. For $p \ge p_0 = (2+\varepsilon_0)n$, it is not difficult to see that

$$\left\| (\mathcal{F}_{\Omega,\varepsilon_0} f)^{1+\kappa_0} \right\|_{L^p(\mathbb{R}^n)} = \|\mathcal{F}_{\Omega,\varepsilon_0} f\|^{1+\kappa_0}_{L^{p(1+\kappa_0)}(\mathbb{R}^n)}.$$

Note that $p(1+\kappa_0) \geq 2$, and

$$\frac{1}{\tilde{q}'} < \frac{1}{\tilde{q}'} + \frac{n-2\varepsilon_0}{p\,(1+\kappa_0)n(n-1)}$$

for any $\tilde{q} > 1$. Using Theorem 2, we obtain

$$\begin{aligned}\left\|(\mathcal{F}_{\Omega,\varepsilon_0} f)^{1+\kappa_0}\right\|_{L^p(\mathbb{R}^n)} &= \|\mathcal{F}_{\Omega,\varepsilon_0} f\|^{1+\kappa_0}_{L^{p(1+\kappa_0)}(\mathbb{R}^n)} \\ &\leq C\|\Omega\|^{1+\kappa_0}_{L^\infty(\mathbb{R}^n)\times L^{\tilde{q}'}(S^{n-1})}\|f\|^{1+\kappa_0}_{L^{\tilde{q}}(\mathbb{R}^n)},\end{aligned} \tag{3.2}$$

where

$$\frac{1}{\tilde{q}} = \frac{1}{p(1+\kappa_0)} + \frac{\varepsilon_0}{n}.$$

The condition $\frac{1}{p} = \frac{1}{q} - \frac{\alpha}{n}$ and a trivial calculation yield

$$\frac{1}{p(1+\kappa_0)} + \frac{\varepsilon_0}{n} = \frac{1}{p\left(1+\frac{\alpha-\varepsilon_0}{\varepsilon_0-\frac{n}{q}}\right)} + \frac{\varepsilon_0}{n} = \frac{n-q\varepsilon_0}{qn} + \frac{\varepsilon_0}{n} = \frac{1}{q},$$

which implies $\tilde{q} = q$.

Combining the above conclusion with the estimates of (3.1) and (3.2), for $\frac{1}{2} \leq \alpha \leq \frac{n}{2}$ and $p \geq p_0 = (2+\varepsilon_0)n$, we obtain that

$$\begin{aligned}\|\mathcal{F}_{\Omega,\alpha} f\|_{L^p(\mathbb{R}^n)} &\leq C\left\|(\mathcal{F}_{\Omega,\varepsilon_0} f)^{1+\kappa_0}\right\|_{L^p(\mathbb{R}^n)}\left(\|f\|_{L^q(\mathbb{R}^n)}\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})}\right)^{-\kappa_0} \\ &\leq C\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})}\|f\|_{L^q(\mathbb{R}^n)},\end{aligned} \tag{3.3}$$

where $1/q = 1/p + \alpha/n$ and $r = q'$.

In the following, we need to discuss the boundedness of $\mathcal{F}_{\Omega,\alpha}$ for $2 < p < p_0$. We also note that the following $L^{q_1} \to L^2$ boundedness of $\mathcal{F}_{\Omega,\alpha}$ for $\alpha \leq \frac{n}{2}$ was established in [4]:

$$\|\mathcal{F}_{\Omega,\alpha} f\|_{L^2(\mathbb{R}^n)} \leq C\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^{r_1}(S^{n-1})}\|f\|_{L^{q_1}(\mathbb{R}^n)}, \tag{3.4}$$

where $q_1 = \frac{2n}{n+2\alpha}$ and $r_1 = q_1'$.

Interpolating between (3.4) and the following inequality

$$\|\mathcal{F}_{\Omega,\alpha} f\|_{L^{p_0}(\mathbb{R}^n)} \leq C\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^{r_0}(S^{n-1})}\|f\|_{L^{q_0}(\mathbb{R}^n)},$$

we get that, for all $2 < p < p_0$,

$$\|\mathcal{F}_{\Omega,\alpha} f\|_{L^p(\mathbb{R}^n)} \leq C\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})}\|f\|_{L^q(\mathbb{R}^n)}, \tag{3.5}$$

where
$$\frac{1}{p}=\frac{\theta}{p_0}+\frac{1-\theta}{2},\quad \frac{1}{r}=\frac{\theta}{r_0}+\frac{1-\theta}{r_1},\quad \frac{1}{q}=\frac{\theta}{q_0}+\frac{1-\theta}{q_1}.$$
Thus, a trivial calculation yields $\theta=1-2/p$ and $r=q'$. Therefore, we complete the proof of the theorem for $1/2\le\alpha\le n/2$.

Our second step is to extend α to the range $\frac{n}{2}<\alpha<n$, by invoking the result obtained in the previous step. For any $\alpha\in(\frac{n}{2},n)$, we can find a small positive number ϵ such that $\frac{n}{2}<\alpha\le n-\epsilon$. Let $\varepsilon_1=\frac{n}{2}$ and $p_1=\frac{n}{\epsilon}$. When $p\ge p_1$, we have $\alpha-\varepsilon_1<n\left(\frac{1}{2}-\frac{1}{p}\right)$. Let δ_1 be a number to be chosen later and write
$$\mathcal{F}_{\Omega,\alpha}f(x)=\int_{|y|\le\delta_1}\frac{\Omega(x,y)}{|y|^{n-\alpha}}f(x-y)dy+\int_{|y|>\delta_1}\frac{\Omega(x,y)}{|y|^{n-\alpha}}f(x-y)dy:=H_3+H_4.$$
It yields that, by a similar argument as the estimate of H_1 and H_2,
$$H_3\le\int_{|y|\le\delta_1}\frac{|y|^{\alpha-\varepsilon_1}\Omega(x,y)}{|y|^{n-\varepsilon_1}}f(x-y)dy\le{\delta_1}^{\alpha-\varepsilon_1}\mathcal{F}_{\Omega,\varepsilon_1}f(x),$$
$$H_4\le\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})}{\delta_1}^{\alpha-\frac{n}{q}}\|f\|_{L^q(\mathbb{R}^n)}.$$
Thus,
$$\mathcal{F}_{\Omega,\alpha}f(x)\le{\delta_1}^{\alpha-\varepsilon_1}(\mathcal{F}_{\Omega,\varepsilon_1}f(x)+{\delta_1}^{\varepsilon_1-\frac{n}{q}}\|f\|_{L^q(\mathbb{R}^n)}\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})}).$$
We now take
$$\delta_1=\left(\frac{\mathcal{F}_{\Omega,\varepsilon_1}f(x)}{\|f\|_{L^q(\mathbb{R}^n)}\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})}}\right)^{\frac{1}{\varepsilon_1-\frac{n}{q}}},$$
and denote $\kappa_1=\frac{\alpha-\varepsilon_1}{\varepsilon_1-\frac{n}{q}}$. It is not difficult to see that
$$\mathcal{F}_{\Omega,\alpha}f(x)\le(\mathcal{F}_{\Omega,\varepsilon_1}f(x))^{1+\kappa_1}\left(\|f\|_{L^q(\mathbb{R}^n)}\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})}\right)^{-\kappa_1}\tag{3.6}$$
and
$$\left\|(\mathcal{F}_{\Omega,\varepsilon_1}f)^{1+\kappa_1}\right\|_{L^p(\mathbb{R}^n)}=\|\mathcal{F}_{\Omega,\varepsilon_1}f\|^{1+\kappa_1}_{L^{p(1+\kappa_1)}(\mathbb{R}^n)}.$$
In order to use the result of step 1, it is necessary to ensure that $p(1+\kappa_1)\ge2$. Note that in this case the inequality $\alpha-\varepsilon_1\le n\left(\frac{1}{2}-\frac{1}{p}\right)$ is equivalent to $p(1+\kappa_1)\ge2$ if α lies in the interval $\left(\frac{n}{2},n\right)$. Hence, for $p\ge p_1=\frac{n}{\epsilon}$, by virtue of the result of step 1, we get
$$\|\mathcal{F}_{\Omega,\varepsilon_1}f\|^{1+\kappa_1}_{L^{p(1+\kappa_1)}(\mathbb{R}^n)}\le C\|\Omega\|^{1+\kappa_1}_{L^\infty(\mathbb{R}^n)\times L^{\tilde{\tilde{q}}'}(S^{n-1})}\|f\|^{1+\kappa_1}_{L^{\tilde{\tilde{q}}}(\mathbb{R}^n)},\tag{3.7}$$

where $\frac{1}{\tilde{\tilde{q}}} = \frac{1}{p(1+\kappa_1)} + \frac{\varepsilon_1}{n}$. Also, a trivial calculation yields $\frac{1}{p(1+\kappa_1)} + \frac{\varepsilon_1}{n} = \frac{1}{q}$, which implies $\tilde{\tilde{q}} = q$.

The estimates of (3.6) and (3.7) give us that, for all $p \geq p_1$,

$$\begin{aligned} \|\mathcal{F}_{\Omega,\alpha} f\|_{L^p(\mathbb{R}^n)} &\leq C \left\| (\mathcal{F}_{\Omega,\varepsilon_1} f)^{1+\kappa_1} \right\|_{L^p(\mathbb{R}^n)} \left(\|f\|_{L^q(\mathbb{R}^n)} \|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})} \right)^{-\kappa_1} \\ &\leq C \|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})} \|f\|_{L^q(\mathbb{R}^n)}. \end{aligned} \tag{3.8}$$

Thus, it remains to prove the boundedness of $\mathcal{F}_{\Omega,\alpha}$ for $2 < p < p_1$. Analogous to the proof in step 1, interpolating between (3.4) and (3.8) for $\alpha \in (\frac{n}{2}, n)$, we get the desired result.

3.2. Bilinear fractional integral. We will only prove (1) in Theorem 5, since the proof for other parts is the same. Let $q' = 1 + \frac{p_2}{p_1}$ and $q = 1 + \frac{p_1}{p_2}$, by Hölder's inequality we have

$$\widetilde{B}_{\Omega,\alpha}(f,g)(x) \leq \mathcal{F}_{\Omega,\alpha}(f^q)(x)^{1/q} \; \mathcal{L}_{\Omega,\alpha}(g^{q'})(x)^{1/q'}.$$

By Hölder's inequality again, we have

$$\left\| \widetilde{B}_{\Omega,\alpha}(f,g) \right\|_{L^p(\mathbb{R}^n)}^p \leq \|\mathcal{F}_{\Omega,\alpha}(f^q)\|_{L^p(\mathbb{R}^n)}^{p/q} \left\| \mathcal{L}_{\Omega,\alpha}(g^{q'}) \right\|_{L^p(\mathbb{R}^n)}^{p/q'}.$$

By Theorem 4, we have a constant $C > 0$ such that

$$\begin{aligned} &\|\mathcal{F}_{\Omega,\alpha}(f^q)\|_{L^p(\mathbb{R}^n)}^{p/q} \left\| \mathcal{L}_{\Omega,\alpha}(g^{q'}) \right\|_{L^p(\mathbb{R}^n)}^{p/q'} \\ &\quad \leq C^p \|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})}^p \|f^q\|_{L^\sigma(\mathbb{R}^n)}^{p/q} \left\| g^{q'} \right\|_{L^\sigma(\mathbb{R}^n)}^{p/q'}, \end{aligned}$$

where

$$r > \frac{n-1}{n}\sigma' \text{ and } 1/p = 1/\sigma - \frac{\alpha}{n}.$$

Noting that

$$\|f^q\|_{L^\sigma(\mathbb{R}^n)}^{p/q} \left\| g^{q'} \right\|_{L^\sigma(\mathbb{R}^n)}^{p/q'} = \|f\|_{L^{\sigma q}(\mathbb{R}^n)}^p \|g\|_{L^{\sigma q'}(\mathbb{R}^n)}^p,$$

and

$$1/\sigma = \frac{p_1 + p_2}{p_1 p_2},$$

we easily see that

$$\left\| \widetilde{B}_{\Omega,\alpha}(f,g) \right\|_{L^p(\mathbb{R}^n)}^p \leq C \|f\|_{L^{p_1}(\mathbb{R}^n)}^p \|g\|_{L^{p_2}(\mathbb{R}^n)}^p.$$

The theorem is proved. □

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JIECHENG CHEN
DEPARTMENT OF MATHEMATICS
ZHEJIANG NORMAL UNIVERSITY
321000 JINHUA
P. R. CHINA

E-mail: jcchen@zjnu.edu.cn

DASHAN FAN
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF WISCONSIN–MILWAUKEE
MILWAUKEE, 53201 WI
USA

E-mail: fan@uwm.edu

TAOTAO ZHENG
DEPARTMENT OF MATHEMATICS
ZHEJIANG UNIVERSITY OF
SCIENCE AND TECHNOLOGY
310023 HANGZHOU
P. R. CHINA

E-mail: taotzheng@126.com

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