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Rationality of the zeta function of the subgroups of abelian p-groups

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Abstract. Given a finite abelian p-group F, we prove an efficient recursive formula for $\sigma_a(F) = \sum_{H \le F} |H|^a$ where H ranges over the subgroups of F. We infer from this formula that the p-component of the corresponding zeta-function on groups of p-rank bounded by some constant r is rational with a simple denominator. We also provide two explicit examples in rank $r = 3$ and $r = 4$, as well as, a closed formula for $\sigma_a(F)$.

1. Introduction

The subgroups of finite abelian p-groups have been intensively studied. An early paper of G. Birkhoff establishes in [8, Theorem 8.1] material to count the number of subgroups of a given type; the version given in [9, (1)] is surely easier to grasp. To fix the notation, our p-groups will be of rank below some fixed r, and are thus isomorphic to a product

$$F = \mathbb{Z}/p^{f_1}\mathbb{Z} \times \mathbb{Z}/p^{f_1+f_2}\mathbb{Z} \times \cdots \times \mathbb{Z}/p^{f_1+f_2+\cdots+f_r}\mathbb{Z}, \tag{1}$$

where f_i are non-negative integers. We write $F = [p; f_1, f_2, \cdots, f_r]$. The *type* of F is the partition $(f_1 + \cdots + f_r, f_1 + \cdots + f_{r-1}, \cdots, f_1)$. The *type* of a subgroup H is its type as an abstract group, while its *cotype* is the type of F/H. In the fifties, P. Hall considered the numbers $g^{\lambda}_{\mu,\nu}(p)$ of subgroups of type μ and cotype ν in a group of type λ, and used them as multiplication constants to form what is now called *the Hall algebra*. The combinatorial aspects have been further

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developed by T. Klein in [16], in the milestone book of I. G. MacDonald [20], and by L. Butler in [10] and [11] concerning the poset formed by these subgroups and the inclusion. This short bibliography is by no means complete! Two closely related fields of enumerative algebra concern the number of subgroups of not-especially abelian groups, e.g. Y. Takegahara in [27], and the number of subgroups of a given index in a fixed group, see [12] by F. J. Grunewald, D. Segal and G. C. Smith, and the book [19] by A. Lubotzky and D. Segal.

Given a finite abelian group F and a complex number a, we concentrate in this paper on the counting function

$$\sigma_a(F) = \sum_{\substack{H\le F,\\ H \text{ subgroup}}} |H|^a. \tag{2}$$

We obviously have $\sigma_a(F_1F_2) = \sigma_a(F_1)\sigma_a(F_2)$ whenever F_1 and F_2 have coprime orders, therefore reducing the study of σ_a to the case of p-groups.

Despite the wealth of work on the question and our restriction to finite abelian groups, it is difficult to get formulae for σ_a that are not (very) intricate. Still we know that, once the type of F is fixed, say equal to λ, the value $\sigma_a(F)$ is a polynomial in p and $q = p^a$, since, by using the Hall polynomials $g^\lambda_{\mu,\nu}$, we have

$$\sigma_a(F) = \sum_{\mu,\nu} g^\lambda_{\mu,\nu}(p) p^{a(\mu_1+\mu_2+\mu_3+\cdots)},$$

the sum being over all possible choices of μ and ν. By combining the expression given in [9, (1)] and the development of the p-binomial coefficient given in [15, Theorem 6.1], we even conclude that $\sigma_a(F)$ is a polynomial in p and q with integer non-negative coefficients. The main novelty of our study is the "simple" recursion formula given in Theorem 3.1. As an interesting consequence, the relevant generating series is shown to be rational; we even provide a fully explicit formula.

Theorem 1.1. *We have*

$$\sum_{f_1,f_2,\cdots,f_r\ge 0} \sigma_a([p; f_1, f_2, \cdots, f_r]) X_1^{f_1}\cdots X_r^{f_r}$$
$$= \sum_{(\epsilon_k)\in\{-1,1\}^r} \prod_{t=1}^{r} \frac{-\epsilon_t p^{\epsilon_t^*(a+\sum_{h=t+1}^r \epsilon_h)}}{p^{a+\sum_{h=t+1}^r \epsilon_h}-1} \frac{1}{1-p^{(a+r-t+1)\sum_{h=t}^r \epsilon_h^* - (\sum_{h=t}^r \epsilon_h^*)^2} X_t},$$

where $\epsilon^* = (1-\epsilon)/2$. *This series belongs in particular to* $\mathbb{Q}(p, p^a, X_1, \cdots, X_r)$ *and a denominator is given in* (16).

This formula appears already in the unpublished thesis of G. Bhowmik [1, Section IX], with whom I collaborated at that time. By "a denominator", we mean a polynomial by which we can multiply our series to fall in $\mathbb{Q}[p, p^a, X_1, \cdots, X_r]$. No minimality is assumed. The dependence in $p^a = q$ is maybe better explained by modifying (2) in case of a p-group into

$$\sigma_a(F) = \sum_{\substack{H \leq F, \\ H \text{ subgroup}}} q^{\log |H| / \log p}. \tag{3}$$

We infer from Theorem 1.1 the rather compact closed formula (20) for $\sigma_a(F)$.

A detour in integer matrices arithmetic. Since it is easier to understand the proof below in the framework of integer matrices, let us present this hundred-years-old field, called *Noncommutative Number Theory* by L. N. Vaserstein in [30]. The book [21] of C. C. MacDuffee contains already, in this context, a notion of gcd and lcm that is till under scrutiny, see [28] by R. C. Thompson. A founding result is that, when decomposing a non-singular integer matrix M as a product of two integer matrices $M = AB$, the number of right-classes of A under the action of $SL_r(\mathbb{Z})$ is finite; $A \cdot SL_r(\mathbb{Z})$ is then called a left-divisor of M. From this fact, V. C. Nanda in [22] and [23] introduced a convolution product between functions of integer matrices invariant under the action of $SL_r(\mathbb{Z})$. This algebra is (almost immediately) isomorphic to the Hecke algebra, see the book [17] by A. Krieg. V. C. Nanda detailed examples among which we find an Euler totient function, the divisor function (our σ_0), and a Möbius function. The initial interest for this arithmetic comes from modular forms.

Back to finite abelian groups. Any finite abelian group of rank r can be represented as a quotient $\mathbb{Z}^r/M(\mathbb{Z}^r)$ for some non-singular integer matrix M. This correspondence is shown in [3] to carry through to the subgroups that in return appear as left-divisors of M. The left-divisibility of divisors translates as the inclusion of subgroups, and the right-complementary divisor of any left-divisor H of F is associated to the quotient F/H. In this manner, the arithmetic of subgroups of finite abelian groups and the one of integer matrices locally (i.e. once a home group F is chosen) coincide; for instance, the Möbius function defined on the lattice of subgroups is identical to the one defined on matrices as the convolution inverse of the $\mathbb{1}$-function. More fundamentally, the Hall algebra, the Hecke algebra, and the algebra of *arithmetical* functions on integer matrices coincide. Other connections exist: for instance, the paper [29] of R. C. Thompson converts T. Klein's combinatorial result [16] in terms of divisibility of invariant factors.

Average results. Here, the vocabularies of groups and of matrices get mixed. As shown by G. Bhowmik in [7], the function $\sigma_a(F)$ taken on average under *the determinant condition* $|F| \le x$ exhibits some regularity: when translated in terms of abelian groups, the question is to decide of the asymptotic behavior, when x goes to infinity, of

$$\sum_{|F|\le x} \sigma_a(F) / \sum_{|F|\le x} 1,$$

where F ranges over the finite abelian groups of rank below some fixed r. The sum $\sum_{|F|\le x} 1$ has been the subject of numerous publications, e.g. A. Ivič [14], [26] by O. Robert and P. Sargos, or [18] by H.-Q. Liu. The average order of σ_a is closely related to the behavior of the rather mysterious Dirichlet series

$$D_{r,a}(s) = \prod_{p\ge 2} \sum_{f_1,\cdots,f_r\ge 0} \frac{\sigma_a([p; f_1, \cdots, f_r])}{p^{(rf_1+(r-1)f_2+\cdots+f_r)s}},$$

the product being taken over the primes p. Its abscissa of convergence has been determined in [2], while G. Bhowmik and J. Wu in [6] exhibit a representation of $D_{r,a}(s)$ that yields a larger domain of meromorphic continuation. Since the p-factor of this series is the case $X_t = 1/p^{s(r-t+1)}$ of Theorem 1.1, we now have a completely explicit expression. This series is an analog in the finite group case of the zeta-function, introduced and studied by F. J. Grunewald, D. Segal and G. C. Smith in [12], though these authors work with a fixed group and investigate the generating function associated to the number of subgroups of a given index, as this index varies. In our case, the subgroups are less precisely determined (we do not fix the index) but the sum runs over a family of groups. We further note that it (as well as, the more general version considered in Theorem 1.1) has also been investigated by V. M. Petrogradsky in [25].

As a side-note, we mention another kind of mean-regularity that has been obtained in [5]: we have $\sigma_0(F) = (\log|F|)^{(1+o(1))\log 2}$ for all but $o(x)$ abelian groups of order not more than x. On restricting the set to groups of rank r exactly (there are about $x^{1/r}$ such groups), we show that $\sigma_0(F) = |F|^{[r^2/4]/r}(\log|F|)^{\xi_r+o(1)}$ for all but $o(x^{1/r})$ exceptions, where $\xi_r = (1+(-1)^r)/2$.

In Section 5, we use our method to derive two new explicit formulae: one when $r = 3$, under the determinant condition and a general a, and one when $r = 4$, still under the determinant condition, though this time restricted to the case $a = 0$ to keep the expression within a reasonable size. Finally, in Section 6, we use Theorem 1.1 to derive a closed formula for $\sigma_a(F)$.

2. Duality

The function $\sigma_a(F)$ is defined in (2), and we propose now another expression that is surely not novel but is lacking an easy reference. We present a proof for the sake of completeness.

Lemma 2.1. *When F is a finite abelian group, we have*

$$\sigma_a(F) = \sum_{\substack{H \leq F, \\ H \text{ subgroup}}} |F/H|^a.$$

In terms of divisors of matrices, as explained in the Introduction, the expression (2) can be seen as summing over left-divisors, while the above can be seen as summing over right-divisors. We present an independent proof.

Proof. Since the character group $\hat{F}$ of F is isomorphic to F, we have $\sigma_a(F) = \sigma_a(\hat{F})$. The following function is known to be one-to-one, see [13, Theorem 13.2.3]:

$$\begin{array}{rccl} V_{F\to\hat{F}}: & \text{subgroups of } F & \to & \text{subgroups of } \hat{F} \\ & H & \mapsto & H^\perp = \{\chi / \chi_{|H} = 1\}. \end{array}$$

It is further classical that $H^\perp \cong F/H$. As a consequence, we find that

$$\sigma_a(F) = \sum_{\substack{H \leq G, \\ H \text{ subgroup}}} |H^\perp|^a = \sum_{\substack{H \leq F, \\ H \text{ subgroup of } F}} |F/H|^a$$

as wanted. □

3. Recursion formulae

This section is the heart of the whole paper. The next theorem together with Lemma 2.1 are the only places where we input information on our function. Once this formula is established, the remainder of the proof of Theorem 1.1 is maybe not immediate but is essentially a matter of bookkeeping.

Theorem 3.1. *Let F_r be a finite abelian p-group of rank $r \geq 1$ and exponent p^ℓ. Let e_ℓ be an element of order p^ℓ, and let F_{r-1} be a subgroup such that $F_r = F_{r-1} \oplus \mathbb{Z}e_\ell$. We have*

$$(p^a - 1)\sigma_a(F_r) = p^{a\ell+a}\,|F_{r-1}|\,\sigma_{a-1}(F_{r-1}) - \sigma_{a+1}(F_{r-1}).$$

PROOF. We consider $G_r = F_{r-1} \oplus \mathbb{Z}pe_\ell$. We first prove the following two recursion formulae:

$$\sigma_a(F_r) = p^a \sigma_a(G_r) + \sigma_{a+1}(F_{r-1}) \tag{4}$$

and

$$\sigma_a(F_r) = \sigma_a(G_r) + p^{a\ell} \, |F_{r-1}| \, \sigma_{a-1}(F_{r-1}). \tag{5}$$

A linear combination of both gives the recursion announced in the lemma. The first formula will come from the expression of Lemma 2.1

$$\sigma_a(F_r) = \sum_{H \le F_r} |F_r/H|^a,$$

while the second one will come from the initial expression

$$\sigma_a(F_r) = \sum_{H \le F_r} |H|^a.$$

To do so, we split both summations according to whether H is a subgroup of G_r or not. The first case is readily handled via the two formulae

$$\sum_{\substack{H \le F_r, \\ H \le G_r}} |F_r/H|^a = p^a \sum_{H \le G_r} |G_r/H|^a = p^a \sigma_a(G_r) \tag{6}$$

and

$$\sum_{\substack{H \le F_r, \\ H \le G_r}} |H|^a = \sum_{H \le G_r} |H|^a = \sigma_a(G_r). \tag{7}$$

The second case requires some more analysis. Let K be a subgroup of F_{r-1}. We consider

$$\begin{array}{rccl} \Psi: & \big\{H | H \not\le G_r, H \cap F_{r-1} = K\big\} & \to & F_{r-1}/K \\ & H & \mapsto & y \mod K, \text{where } y \in (H - e_\ell) \cap F_{r-1}. \end{array}$$

This function is well-defined. Indeed, the set $(H - e_\ell) \cap F_{r-1}$ is non-empty since $H \not\le G_r$, and thus there exists $x = f + ne_\ell \in H$, where $f \in F_{r-1}$ and n is prime to p^ℓ. On multiplying by the inverse of n modulo p, we recover an element of the form $y + e_\ell$ as wanted. Furthermore, the class of y modulo K does not depend on the choice of y. For, if y' also belongs to $(H - e_\ell) \cap F_{r-1}$, then $y - y' = (y + e_\ell) - (y' + e_\ell)$ belongs to $H \cap F_{r-1} = K$. We note that $H = \langle K, y + e_\ell \rangle$,

and that this defines the reverse function to Ψ, proving that Ψ is one-to-one and onto. Note that $F_r/H \cong F_{r-1}/K$. As a corollary, we get

$$\sum_{\substack{H\leq F_r,\\ H\not\leq G_r}} |F_r/H|^a = \sum_{K\leq F_{r-1}} |F_{r-1}/K||F_{r-1}/K|^a = \sigma_{a+1}(F_{r-1}) \tag{8}$$

and

$$\sum_{\substack{H\leq F_r,\\ H\not\leq G_r}} |H|^a = \sum_{K\leq F_{r-1}} p^{a\ell}|F_{r-1}/K||K|^a = p^{a\ell}|F_{r-1}|\sigma_{a-1}(F_{r-1}). \tag{9}$$

Combining (6) together with (8) gives (4), while combining (7) together with (9) gives (5). □

Remark 1. The recursion formula we prove in (4) is already contained in [7], where a proof in terms of matrices is given. The proof below uses the group-theoretical context, offering the advantage that we can re-use the same scheme of proof on the dual group, giving rise to (5). The comparison of both yields the theorem. The reader should notice that this formula offers a very fast manner to compute $\sigma_a(F_r)$: the recursion (4) yields an algorithm of complexity $2^{f_1+f_2+\cdots+f_r}$, while the above reduces this complexity to 2^r.

Remark 2. The part of the proof that involves Ψ is in fact similar to [19, Lemma 1.3.1 (i)], where complements of a given subgroup are being counted.

In the case $r = 1$, Theorem 3.1 recovers, when $a \neq 0$, the classical formula for the sum $\sigma_a(m) = \sum_{d|m} d^a$ of the a-th power of the divisors of the integer m:

$$\sigma_a(p^{f_1}) = \frac{p^{a(f_1+1)}-1}{p^a-1} = \frac{q^{f_1+1}-1}{q-1},$$

and, by continuity, $\sigma_0(p^{f_1}) = f_1 + 1$. We can also use an algebraic argument: the expression for σ_a is a polynomial in $q = p^a$ which we evaluate at $q = 1$.

By the classification of finite abelian groups, $F_{r-1} = [p; f_1, f_2, \cdots, f_{r-1}]$. The situation is, however, not so simple concerning the subgroup G_r introduced in the above proof. Indeed, when $f_r \geq 1$, we have $G_r = [p; f_1, f_2, \cdots, f_r - 1]$, but this is not the case anymore when $f_r = 0$. This fact explains the difficulties met in [2] and in [6]. The novelty of Theorem 3.1 is that it produces a recurrence formula that preserves this representation.

In the case $r = 2$, Theorem 3.1 gives, when $a \neq 0$,

$$(p^a-1)\sigma_a([p; f_1, f_2]) = p^{a(f_1+f_2+1)+f_1}\frac{p^{(a-1)(f_1+1)}-1}{p^{a-1}-1} - \frac{p^{(a+1)(f_1+1)}-1}{p^{a+1}-1}. \tag{10}$$

This formula is generalized in (20).

4. Proof of Theorem 1.1

Let us use our recursion to derive an explicit formula for

$$Q_{r,a}(X_1,\cdots,X_r)=\sum_{f_1,f_2,\cdots,f_r\ge 0}\sigma_a([p;f_1,\cdots,f_r])X_1^{f_1}\cdots X_r^{f_r}, \tag{11}$$

where $r\ge 0$ and the parameter p is fixed. The exponent p^ℓ of the group $\langle f_1,\cdots,f_r\rangle$ is $p^{f_1+\cdots+f_r}$, and we recall that $F_{r-1}=\langle f_1,f_2,\cdots,f_{r-1}\rangle$. An immediate consequence of Theorem 3.1 reads

$$\begin{aligned}(p^a-1)&Q_{r,a}(X_1,\cdots,X_r)\\ &=p^aQ_{r-1,a-1}(p^{a+r-1}X_1,p^{a+r-2}X_2,\cdots,p^{a+1}X_{r-1})\sum_{f_r\ge 0}(p^aX_r)^{f_r}\\ &\quad-Q_{r-1,a+1}(X_1,\cdots,X_{r-1})\sum_{f_r\ge 0}X_r^{f_r}.\end{aligned} \tag{12}$$

We note for future reference that

$$Q_{1,a}(X_1)=\frac{1}{(1-X_1)(1-p^aX_1)},\quad Q_{0,a}=1. \tag{13}$$

The value at $r=0$ follows from the definition (11). We also check directly that the relation (12) holds true also when $r=1$, though we will not use it. We rewrite the above, when $a\neq 0$ and $r\ge 1$, in the form

$$\begin{aligned}Q_{r,a}&(X_1,\cdots,X_r)\\ &=\frac{p^a}{(p^a-1)(1-p^aX_r)}Q_{r-1,a-1}(p^{a+r-1}X_1,p^{a+r-2}X_2,\cdots,p^{a+1}X_{r-1})\\ &\quad-\frac{1}{(p^a-1)(1-X_r)}Q_{r-1,a+1}(X_1,\cdots,X_{r-1}).\end{aligned}$$

We can reiterate this process to obtain a rational fraction, provided that the parameter a that appears does not vanish, which we assume. We will argue by continuity later. Each time we use the above formula, we change the parameter r to $r-1$, the parameter a to $a+\epsilon$ where $\epsilon=\pm 1$, and the parameters X_i to $p^{\epsilon^*(a+r-i)}X_i$ where $\epsilon^*=(1-\epsilon)/2$. We, furthermore, multiply $Q_{r-1,a+\epsilon}$ by $w_\epsilon(p^a,X_r)$, where

$$w_\epsilon(q,Y)=-\epsilon\frac{q^{\epsilon^*}}{q-1}\frac{1}{1-q^{\epsilon^*}Y}. \tag{14}$$

With these notations, the above relation reads

$$Q_{r,a}((X_i))=\sum_{\epsilon\in\{-1,1\}}w_\epsilon(p^a,X_r)Q_{r-1,a+\epsilon}\big((p^{\epsilon^*(a+r-i)}X_i)_i\big). \tag{15}$$

In this form, it is easily iterated and yields the next lemma.

Lemma 4.1. *When $r \geq 1$, we have*

$$Q_{r,a}((X_i)) = \sum_{(\epsilon_k)} \prod_{s=0}^{r-1} w_{\epsilon_{r-s}}\Big(p^{a+\sum_{k=0}^{s-1}\epsilon_{r-k}}, p^{\sum_{k=0}^{s-1}\epsilon^*_{r-k}(a+s-k+\sum_{\ell=0}^{k-1}\epsilon_{r-\ell})} X_{r-s}\Big),$$

where the sum runs over $(\epsilon_k)_{1\leq k\leq r} \in \{-1,1\}^r$ and $\epsilon^* = (1-\epsilon)/2$.

Proof. To prove the above formula, we use recursion, starting from $r = 1$, where it is readily checked. We employ (15) to get

$$\begin{aligned} &Q_{r,a}((X_i)) \\ &\quad = \sum_{\epsilon_r\in\{\pm1\}} w_{\epsilon_r}(p^a, X_r) \sum_{(\epsilon_k)_{1\leq k\leq r-1}\in\{\pm1\}^{r-1}} \prod_{s=0}^{r-2} w_{\epsilon_{r-1-s}}\Big(p^{a+\epsilon_r+\sum_{k=0}^{s-1}\epsilon_{r-1-k}}, \\ &\qquad p^{\sum_{k=0}^{s-1}\epsilon^*_{r-1-k}(a+\epsilon_r+s-k+\sum_{\ell=0}^{k-1}\epsilon_{r-1-\ell})} p^{\epsilon^*_r(a+\epsilon_r+r-(r-1-s))} X_{r-1-s}\Big). \end{aligned}$$

With $s' = s+1$, $k' = k+1$ and $\ell' = \ell+1$, the right-hand side reads:

$$\begin{aligned} &\sum_{\epsilon_r\in\{\pm1\}} w_{\epsilon_r}(p^a, X_r) \sum_{(\epsilon_k)_{1\leq k\leq r-1}\in\{\pm1\}^{r-1}} \prod_{s'=1}^{r-1} w_{\epsilon_{r-s'}}\Big(p^{a+\epsilon_r+\sum_{k'=1}^{s'-1}\epsilon_{r-k'}}, \\ &\qquad p^{\sum_{k'=1}^{s'-1}\epsilon^*_{r-k'}(a+\epsilon_r+s'-k'+\sum_{\ell'=1}^{k'-1}\epsilon_{r-\ell})} p^{\epsilon^*_r(a+\epsilon_r+s')} X_{r-s'}\Big). \end{aligned}$$

We transform the above expression with:

$$\begin{aligned} a+\epsilon_r+\sum_{k'=1}^{s'-1}\epsilon_{r-k'} &= a+\sum_{k'=0}^{s'-1}\epsilon_{r-k'}, \\ \sum_{k'=1}^{s'-1}\epsilon^*_{r-k'}\Big(a+\epsilon_r+s'-k'+\sum_{\ell'=1}^{k'-1}\epsilon_{r-\ell}\Big) &= \sum_{k'=1}^{s'-1}\epsilon^*_{r-k'}\Big(a+s'-k'+\sum_{\ell'=0}^{k'-1}\epsilon_{r-\ell}\Big), \\ \sum_{k'=1}^{s'-1}\epsilon^*_{r-k'}\Big(a+s'-k'+\sum_{\ell'=0}^{k'-1}\epsilon_{r-\ell}\Big) \;+\; \epsilon^*_r(a+\epsilon_r+s') & \\ = \sum_{k'=0}^{s'-1}\epsilon^*_{r-k'}\Big(a+s'-k'+\sum_{\ell'=0}^{k'-1}\epsilon_{r-\ell}\Big). & \end{aligned}$$

The factor $w_{\epsilon_r}(p^a, X_r)$ gets readily incorporated in the product over s from 1 to $r-1$ as the value for $s = 0$. This completes the proof. □

On using the definition given by (14) on the expression given by Lemma 4.1, we get a fully explicit formula:

$$Q_{r,a}((X_i))$$
$$= \sum_{(\epsilon_k)\in\{\pm1\}^r} \prod_{s=0}^{r-1} \frac{-\epsilon_{r-s} p^{\epsilon^*_{r-s}(a+\sum_{k=0}^{s-1}\epsilon_{r-k})}}{p^{a+\sum_{k=0}^{s-1}\epsilon_{r-k}}-1} \frac{1}{1-p^{\sum_{k=0}^{s}\epsilon^*_{r-k}(a+s-k+\sum_{\ell=0}^{k-1}\epsilon_{r-\ell})}X_{r-s}}.$$

Some beautification is called for. We first notice that

$$-k+\sum_{\ell=0}^{k-1}\epsilon_{r-\ell} = \sum_{\ell=0}^{k-1}(\epsilon_{r-\ell}-1) = -2\sum_{\ell=0}^{k-1}\epsilon^*_{r-\ell},$$

yielding that $Q_{r,a}((X_i))$ is equal to

$$\sum_{(\epsilon_k)\in\{\pm1\}^r} \prod_{s=0}^{r-1} \frac{-\epsilon_{r-s} p^{\epsilon^*_{r-s}(a+\sum_{k=0}^{s-1}\epsilon_{r-k})}}{p^{a+\sum_{k=0}^{s-1}\epsilon_{r-k}}-1} \frac{1}{1-p^{\sum_{k=0}^{s}\epsilon^*_{r-k}(a+s-2\sum_{\ell=0}^{k-1}\epsilon^*_{r-\ell})}X_{r-s}}.$$

The indices of shape $r-s$, $r-k$ and $r-\ell$ were useful for the recursion, but introduce now a useless level of complexity. We set $t=r-s$, $h=r-k$ and $g=r-\ell$, and get, for $Q_{r,a}((X_i))$, the expression

$$\sum_{(\epsilon_k)\in\{\pm1\}^r} \prod_{t=1}^{r} \frac{-\epsilon_t p^{\epsilon^*_t(a+\sum_{h=t+1}^{r}\epsilon_h)}}{p^{a+\sum_{h=t+1}^{r}\epsilon_h}-1} \frac{1}{1-p^{\sum_{h=t}^{r}\epsilon^*_h(a+r-t-2\sum_{g=h+1}^{r}\epsilon^*_g)}X_t}.$$

The proof of Theorem 1.1 is almost complete. We only need to use the identity

$$\sum_{h=t}^{r}\epsilon^*_h\Big(a+r-t-2\sum_{g=h+1}^{r}\epsilon^*_g\Big) = (a+r-t)\sum_{h=t}^{r}\epsilon^*_h - \Big(\sum_{h=t}^{r}\epsilon^*_h\Big)^2 + \sum_{h=t}^{r}\epsilon^*_h,$$

which is valid because ${\epsilon^*_h}^2 = \epsilon^*_h$.

5. Consequences on Dirichlet series

Let us investigate a possible denominator for the series $Q_{r,a}((X_i))$ of Theorem 1.1. The index t being fixed, for each (ϵ_k), only one factor has the variable X_t. All these factors are of the shape $1-p^{(a+r-t+1)j-j^2}X_t$ for some j in $\{0,\cdots,r-t+1\}$. A possible denominator is thus simply

$$B_r(p,q,X_1,\cdots,X_r) = \prod_{t=1}^{r}\prod_{j=0}^{r-t+1}\big(1-q^j p^{(r-t+1)j-j^2}X_t\big), \tag{16}$$

by which we mean that the product $A_r = B_r(p, q, X_1, \cdots, X_r)Q_{r,a}((X_i))$ falls a priori inside $\mathbb{Q}(p, q)[X_1, \cdots, X_r]$. However, the only possible remaining poles are for $q = p^b$ for some integer b, and this is not possible, since, when $s = 2 + |b|$ and $X_t = 1/p^{s(r-t+1)}$, the series $D_{r,a}(s)$ is bounded. It would be helpful to get a better description of A_r, and at minimum show that it is prime to B_r. Furthermore, its coefficients are integers, and thus likely to have a combinatorial expression. We will see below that these coefficients may vary in signs.

When we restrict our attention to the case $q = 1$ (i.e. $a = 0$) and $X_t = 1/p^{s(r-t+1)}$, the denominator (16) becomes (Careful! We have replaced j by i and then used $j = r - t + 1$ to be able to compare with [6, Theorem 1]):

$$\prod_{j=1}^{r}\prod_{i=0}^{j}(1 - p^{-js+ji-i^2}).$$

The zeta product extracted in [6, Theorem 1] corresponds to $i = j/2$ when j is even, and to $i = (j \pm 1)/2$ when j is odd.

We checked the formula given by Theorem 1.1 in the case $r = 1$ with (13), and in the case $r = 2$ with [2, Corollary 3] that we recall:

$$B_2(p, q, X_1, X_2)\, Q_{2,a}(X_1, X_2) = 1 + qX_1 - q(q + 1)X_1X_2. \tag{17}$$

In the case $a = 0$ and $r = 3$, the erratum [4, (4.17)] gives the proper formula that we have also checked against our expression.

We finally investigated formulae for $r = 3$, $r = 4$ and $r = 5$. The formulae are huge in general. We can, however, record two new explicit formulae to help test conjectures. When $r = 3$, we can keep a arbitrary and still have a manageable formula under the determinant condition:

$$\begin{aligned} B_3(p, q, X^3, X^2, X)\, Q_{3,a}(X^3, X^2, X) \\ &= 1 + qX^2 + p(q + 1)qX^3 - (q^2 + (p + 1)q + 1)qX^4 \\ &\quad - ((p + 1)(q^3 + 1) + (p^2 + p + 1)q(q + 1))qX^5 \\ &\quad + (q^4 + (p + 1)q(q^2 + 1) + (p^2 + p + 1)q^2 + 1)qX^6 \\ &\quad - (q^2 + q + 1)pq^3X^8 + (q^3 + (p + 1)q(q + 1) + 1)pq^3X^9 \\ &\quad + (pq^2 + (p + 1)q + p)pq^4X^{10} - (pq^3 + (p + 1)q(q + 1) + p)pq^4X^{11}. \end{aligned} \tag{18}$$

We have used a GP-Pari [24] script to run the computations based on (15) rather than on Theorem 1.1. Since we know a possible denominator, we have used a data

structure of the form [Numerator, Denominator-Vector], where Denominator-Vector was a list of triplets $[u, v, k]$ meaning that the denominator was the product of $1 - p^u q^v X_k$, taken over all the triplets of the list. The addition of any two such structures is readily handled. The computations took essentially no time, while a brute force algorithm using Theorem 1.1 and relying on the arithmetic of rational fractions was taking a very long time when $r = 4$. We checked that the final minimal denominator was indeed $B_r(p, q, X_1, \cdots, X_r)$. When $r = 4$, we use the determinant condition and stick to $a = 0$ to get

$$\begin{aligned}
&B_4(p, 1, X^4, X^3, X^2, X)\, Q_{4,0}(X^4, X^3, X^2, X)\\
&\quad = (7p^3 + 5p^2 + 8p + 4)p^6X^{26} - (6p^3 + 4p^2 + 6p + 2)p^6X^{25}\\
&\quad - (5p^3 + 10p^2 + 9p + 8)p^6X^{24} + (6p^4 - 10p^3 - 4p^2 - 4p - 2)p^4X^{23}\\
&\quad + (5p^4 + 15p^3 + 7p^2 + 8p + 1)p^4X^{22} + (4p^5 + 6p^3 + 12p^2 - 2p + 4)p^4X^{21}\\
&\quad - 3(3p^2 + p + 1)p^4X^{20} - 2(2p^6 + 3p^5 + 12p^4 + 5p^3 + 5p^2 + 2p + 1)pX^{19}\\
&\quad + (p^7 + 3p^6 + 12p^5 + 23p^4 + 6p^3 + 8p^2 + 3p + 1)pX^{18}\\
&\quad + 2(p^8 + 2p^7 + 5p^6 + 7p^5 + 18p^4 + 18p^3 + 10p^2 + 6p + 2)pX^{17}\\
&\quad - (3p^8 + 6p^7 + 15p^6 + 16p^5 + 16p^4 + 18p^3 - 4p^2 + p - 2)pX^{16}\\
&\quad - (2p^6 + 6p^5 + 16p^4 + 18p^3 + 24p^2 + 24p + 4)p^2X^{15}\\
&\quad + (3p^7 + p^6 + p^5 - 20p^4 - 13p^3 - 21p^2 - 26p - 9)pX^{14}\\
&\quad + 2(3p^6 + 4p^5 + 5p^4 + 4p^3 + 4p^2 + 4p - 3)pX^{13}\\
&\quad + (9p^5 + 11p^4 + 23p^3 + 15p^2 + 12p + 10)pX^{12}\\
&\quad - 6(p^2 + p - 1)p^3X^{11} - (3p^5 + 2p^4 + 8p^3 - p^2 + 5p + 9)X^{10}\\
&\quad + 4(p^5 + p^3 + 2p + 2)X^9 + (6p^4 + 4p^3 + 9p^2 + 2p + 7)X^8\\
&\quad - 2(p^3 + 3p^2 + 2p + 4)p^2X^7 - (3p^2 + 2p + 3)pX^6\\
&\quad - 2(2p^2 + 3p + 4)X^5 + (p^2 + 4p + 2)pX^4 + 2pX^3 + X^2 + 1 \qquad (19)
\end{aligned}$$

This expression shows that the polynomial in p in front of each power of X is not a sum of monomials of constant signs, as could have been thought from the expressions for $r \le 3$. We computed similarly the p-factor $Q_{5,0}(X^5, X^4, X^3, X^2, X)$, and obtained a quotient of a polynomial in $\mathbb{Z}[p, X]$ of degree 50 in X and 25 in p, the largest monomial being $11p^{25}X^{50}$, by $B_5(p, 1, X^5, X^4, X^3, X^2, X)$ as expected.

6. A closed formula

We exploit Theorem 1.1 to express $\sigma_a([p; f_1, \cdots, f_r])$. We use the expansion

$$\frac{1}{1-q^{au}p^vX_t} = \sum_{f_t\geq 0} q^{auf_t}p^{vf_t}X_t^{f_t}$$

to find that

$$\begin{aligned}
&\sum_{f_1,f_2,\cdots,f_r\geq 0} \sigma_a([p; f_1, f_2, \cdots, f_r])X_1^{f_1}\cdots X_r^{f_r}\\
&\quad= \sum_{(\epsilon_k)\in\{-1,1\}^r}\prod_{t=1}^{r}\frac{-\epsilon_t q^{\epsilon_t^*}p^{\epsilon_t^*\sum_{h=t+1}^r\epsilon_h}}{qp^{\sum_{h=t+1}^r\epsilon_h}-1}\times\\
&\quad\prod_{t=1}^{r}\sum_{f_t\geq 0} q^{\sum_{h=t}^r\epsilon_h^*f_t}p^{\left((r-t+1)\sum_{h=t}^r\epsilon_h^*-\left(\sum_{h=t}^r\epsilon_h^*\right)^2\right)f_t}X_t^{f_t}.
\end{aligned}$$

On identifying the coefficients, we find that

$$\begin{aligned}
&\sigma_a([p; f_1, f_2, \cdots, f_r])\\
&\quad= \sum_{(\epsilon_k)\in\{-1,1\}^r}\prod_{t=1}^{r}\frac{-\epsilon_t q^{\epsilon_t^*}p^{\epsilon_t^*\sum_{h=t+1}^r\epsilon_h}}{qp^{\sum_{h=t+1}^r\epsilon_h}-1}\times\\
&\quad q^{\sum_{t=1}^r\sum_{h=t}^r\epsilon_h^*f_t}p^{\sum_{t=1}^r\left((r-t+1)\sum_{h=t}^r\epsilon_h^*-\left(\sum_{h=t}^r\epsilon_h^*\right)^2\right)f_t}.
\end{aligned}\tag{20}$$

As a mean of verification, we note that (3) gives us $\sigma_a([p; f_1, f_2, \cdots, f_r]) = 1$ when $q = 0$. In the expression above, the only contribution when $q = 0$ occurs when $\epsilon_t^* = 0$ for every $t \in \{1, \cdots, r\}$, i.e. when $\epsilon_t = 1$ for every $t \in \{1, \cdots, r\}$. In this case $\sum_{h=t}^r \epsilon_h^* = r-t+1$, and the above formula gives the value 1 as required. The case $r = 2$ is given in (10).

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