Publ. Math. Debrecen
90/1-2 (2017), 181–194
DOI: 10.5486/PMD.2017.7562

On the number of non-zero digits of integers in multi-base representations

By CSANÁD BERTÓK (Debrecen), LAJOS HAJDU (Debrecen), FLORIAN LUCA (Johannesburg) and DIVYUM SHARMA (Mumbai)

Dedicated to A. Sárközy on the occasion of his 75th birthday

Abstract. We prove various finiteness theorems for integers having only few non-zero digits in different multi-base representations simultaneously.

1. Introduction

It is an old problem to study integers having only 'few' non-zero digits in some classical base b representation, see, e.g. papers by ERDŐS, MAUDUIT, POMERANCE, SÁRKÖZY [6], [7], [15], [16], [17], and the references therein. On the other hand, if a number n has to hold certain other arithmetical property, it may happen that it must have 'many' digits. This is the case when n belongs to some recurrence sequence; see, e.g. BUGEAUD, CIPU and MIGNOTTE [4], LUCA [13] and STEWART [21] for effective results in this direction.

The number of non-zero digits of integers, and that of integers with fixed number of non-zero digits is also investigated with respect to other types of bases, e.g. with respect to linear recurrence number systems, cf. [19], [22]. Another generalization of the classical number systems is given by the so-called multi-base representations, when instead of linear combinations of powers of a fixed number b, one can combine products of powers of fixed primes. For related problems and

Mathematics Subject Classification: 11A63, 11A67, 11D61.
Key words and phrases: multi-base representation, number of non-zero digits, S-unit equations.
Research supported in part by the OTKA grants K100339 and K115479.

results, see, e.g. the papers [1], [2], [5], [10], [11], [12], [18], and the references therein.

It is an interesting question to study integers having only 'few' non-zero digits in different bases simultaneously. Here we mention two results. SENGE and STRAUS [20] proved that the number of those integers whose number of non-zero digits in two different bases b_1 and b_2 with $\log b_1/\log b_2 \notin \mathbb{Q}$ remains under some fixed bound is finite. Later, STEWART [21] gave a more precise, effective version of this result.

In this paper, as a generalization of the problem mentioned in the previous paragraph, we study integers having only 'few' non-zero digits in different multi-base representations simultaneously. To set the problem precisely, we need to introduce some notation.

Let S be a finite set of primes, and write $\mathbb{Z}_S$ (resp. $\mathbb{Z}_S^+$) for the set of integers (resp. positive integers) having no prime divisors outside S. A multi-base representation of an integer n is an expression of the form

$$n = u_1 + \cdots + u_t \tag{1}$$

with $u_1, \ldots, u_t \in \mathbb{Z}_S$. If $S = \{p\}$ and we require that $u_1, \ldots, u_t \in \mathbb{Z}_S^+$, we get several expansions of n as sums of powers of p, with the shortest one (namely, the one with the fewest terms) being the usual expansion of n in base p. For an integer n, we write $w_S(n)$ for the minimal t for which (1) holds with some $u_1, \ldots, u_t \in \mathbb{Z}_S$. If $n > 0$ and we also require that $u_1, \ldots, u_t \in \mathbb{Z}_S^+$, we then write $w_S^+(n)$ instead.

In what follows, we prove various finiteness theorems for integers n with 'small' values of $w_S^+(n)$ with respect to different sets S simultaneously. To prove our results, we use Baker's method for linear forms in logarithms, a deep theorem of EVERTSE [8] bounding the number of non-degenerate solutions of S-unit equations, and a local method of BERTÓK and HAJDU [3] developed for the resolution of exponential equations over $\mathbb{Z}$.

2. New results

Our first theorem concerns the general case.

Theorem 2.1. *Let k be a positive integer, $S_1, \ldots, S_k$ be finite sets of primes such that $S_1 \cap \cdots \cap S_k = \emptyset$. Then, for any T, the inequality*

$$w_{S_1}^+(n) + \cdots + w_{S_k}^+(n) \leq T$$

is valid only for finitely many integers n. Further, the number of such integers n is at most $c_1 = c_1(T, k, s)$, where c_1 is an effectively computable constant depending only on T, k and $s := |S_1 \cup \cdots \cup S_k|$.

Remark 1. Note that the condition $S_1 \cap \cdots \cap S_k = \emptyset$ in the above theorem is necessary. Indeed, if $p \in S_1 \cap \cdots \cap S_k$ held for some prime p, then for $T := k \geq 1$ we would have

$$w^+_{S_1}(n) + \cdots + w^+_{S_k}(n) \leq T$$

for all $n = p^\alpha$ $(\alpha \geq 0)$.

Our second result gives an effective bound in a special case.

Theorem 2.2. *Let ℓ be a positive integer, $S_1 = \{p_1, \ldots, p_\ell\}$ and $S_2 = \{q\}$, where $p_1, \ldots, p_\ell$, q are distinct primes. If n is a positive integer with $n > e^{e^e}$ such that $w^+_{S_1}(n) = 1$, then we have*

$$w^+_{S_2}(n) > \frac{c_2 \log\log n}{\log\log\log n},$$

where $c_2 = c_2(\ell, p_1, \ldots, p_\ell, q)$ is an effectively computable positive constant depending only on ℓ, $p_1, \ldots, p_\ell$, q.

Remark 2. The condition $q \notin S_1$ is necessary. This can be easily checked by a similar example as in Remark 1. Further, we note that if S_i are not sets of primes, but sets of multiplicatively independent integers instead, then, after the necessary modifications, our theorems still hold.

Finally, we give a complete list of integers n having only 'few' non-zero digits for some fixed choices of sets S_1, S_2. Note that in the cases considered, though the number of solutions can be bounded, e.g. by results of Evertse [8], there are no tools available which would effectively bound the solutions themselves. To find the solutions explicitly, we apply a method of Bertók and Hajdu [3].

Theorem 2.3. *Let S_1, S_2 be disjoint non-empty sets of primes with $S_1 \cup S_2 = \{2, 3, 5\}$. Then*

$$w^+_{S_1}(n) + w^+_{S_2}(n) \leq 4$$

implies that if

(1) *$S_1 = \{2\}$ and $S_2 = \{3, 5\}$, then $n \in \{1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 16, 18, 20, 24, 25, 32, 34, 36, 40, 48, 72, 80, 81, 96, 128, 130, 136, 144, 160, 258, 260, 288, 384, 640, 1152, 2050, 2052, 4104, 32832\}$;*

(2) $S_1 = \{3\}$ *and* $S_2 = \{2, 5\}$, *then* $n \in \{1, 2, 3, 4, 5, 6, 9, 10, 12, 18, 27, 28, 30, 36, 54, 81, 82, 84, 90, 108, 162, 252, 270, 324, 729, 756, 810, 6561, 6570\}$;

(3) $S_1 = \{5\}$ *and* $S_2 = \{2, 3\}$, *then* $n \in \{1, 2, 3, 5, 6, 10, 25, 26, 27, 30, 50, 125, 126, 130, 150, 625, 630, 650, 3125, 3126, 15625, 78750\}$.

3. Proof of Theorem 2.1

To prove Theorem 2.1, we need to introduce some notions and notations. Let $a_1, \dots, a_\ell \in \mathbb{Q}^*$. Consider the equation

$$a_1x_1 + \cdots + a_\ell x_\ell = 0 \tag{2}$$

in $x_1, \dots, x_\ell \in \mathbb{Z}_S$. A solution $(x_1, \dots, x_\ell)$ of the above equation is said to be *non-degenerate* if

$$\sum_{i \in I} a_i x_i \neq 0 \quad \text{for each non-empty } I \subset \{1, \dots, \ell\},$$

and *degenerate* otherwise. Further, two solutions $(x_1, \dots, x_\ell)$ and $(y_1, \dots, y_\ell)$ of (2) are called *proportional* if, for some $z \in \mathbb{Q}^*$, we have

$$x_i = zy_i \quad \text{for } i = 1, \dots, \ell.$$

Lemma 3.1. *Let* $s = |S|$. *Then equation* (2) *has at most*

$$(2^{35}(\ell - 1)^2)^{(\ell-1)^3 s}$$

non-degenerate solutions $(x_1, \dots, x_\ell) \in \mathbb{Z}_S \times \cdots \times \mathbb{Z}_S$, *no two of which are proportional.*

Proof. The statement is a simple consequence of [8, Theorem 3]. □

Theorem 2.1 can be immediately deduced from the following result.

Proposition 3.1. *Let* $k \geq 2$, *and let* $t_1, \dots, t_k \in \mathbb{N}$. *For* $i = 1, \dots, k$, *let*

$$A_i = \{a_{i,1}, \dots, a_{i,t_i}\}$$

be a set of t_i *positive integers. Then the number of positive integers* n *with the property that for each* i, *there exist* $u_{i,1}, \dots, u_{i,t_i} \in \mathbb{Z}_{S_i}^+$ *such that*

$$n = a_{i,1}u_{i,1} + \cdots + a_{i,t_i}u_{i,t_i},$$

is at most

$$(2^{35}(t - 1)^2)^{(k-1)(t-1)^4 s},$$

where $t = t_1 + \cdots + t_k$ *and* $s = |S_1 \cup S_2 \cup \cdots \cup S_k|$.

PROOF. We prove the proposition by induction on k.

Let $S = S_1 \cup S_2 \cup \cdots \cup S_k$. Suppose that $k = 2$. We will prove that the result holds in this case using induction on $t = t_1 + t_2$. Suppose that $t = 2$. Then $t_1 = t_2 = 1$. We now show that the equation

$$a_{1,1}u_{1,1} = a_{2,1}u_{2,1} \tag{3}$$

in $(u_{1,1}, u_{2,1}) \in \mathbb{Z}_{S_1}^+ \times \mathbb{Z}_{S_2}^+$ has at most one solution. Indeed, the equation (3) implies that

$$\frac{u_{1,1}}{u_{2,1}} = \frac{a_{2,1}}{a_{1,1}}.$$

The claim follows by the coprimality of $u_{1,1}$ and $u_{2,1}$. Therefore, the result holds when $k = t = 2$. Let $t \geq 3$, and assume that the result holds whenever $t_1 + t_2 \leq t - 1$. We now consider the case $t_1 + t_2 = t$. We have to count the number of solutions of the S-unit equation

$$a_{1,1}u_{1,1} + \cdots + a_{1,t_1}u_{1,t_1} = a_{2,1}u_{2,1} + \cdots + a_{2,t_2}u_{2,t_2}, \tag{4}$$

where $u_{1,j} \in \mathbb{Z}_{S_1}^+$ and $u_{2,j} \in \mathbb{Z}_{S_2}^+$. By Lemma 3.1, this equation has at most

$$(2^{35}(t-1)^2)^{(t-1)^3 s}$$

non-degenerate solutions. Next, we count the number of degenerate solutions. (Observe that if $t = 3$, then $(t_1, t_2) = (1, 2)$ or $(2, 1)$, and hence all the solutions are non-degenerate. Therefore, while counting degenerate solutions, it is understood that $t \geq 4$.) For a degenerate solution, there exists a non-empty subset I of $\{1, \ldots, t_1\}$ and a non-empty subset J of $\{1, \ldots, t_2\}$ such that

$$\sum_{i \in I} a_{1,i}u_{1,i} - \sum_{j \in J} a_{2,j}u_{2,j} = 0, \tag{5}$$

but no proper subsum in this equation vanishes. Fix I, J. We count the number of solutions of (4) satisfying (5). Since $|I| + |J| \leq t - 2$, it follows from Lemma 3.1 that the S-unit equation (5) has at most

$$(2^{35}(t-3)^2)^{(t-3)^3 s}$$

non-degenerate solutions. Further, by the induction hypothesis, the equation

$$\sum_{i \notin I} a_{1,i}u_{1,i} - \sum_{j \notin J} a_{2,j}u_{2,j} = 0$$

has at most

$$(2^{35}(t-3)^2)^{(t-3)^4 s}$$

solutions. Hence, given I, J, we obtain that there are at most

$$(2^{35}(t-3)^2)^{s(t-3)^3(t-2)}$$

solutions. Varying I and J, we obtain that the total number of degenerate solutions is at most

$$2^t(2^{35}(t-3)^2)^{s(t-3)^3(t-2)} \leq \frac{1}{2}(2^{35}(t-1)^2)^{(t-1)^4 s}.$$

Thus, (4) has at most

$$(2^{35}(t-1)^2)^{(t-1)^3 s} + \frac{1}{2}(2^{35}(t-1)^2)^{(t-1)^4 s} \leq (2^{35}(t-1)^2)^{(t-1)^4 s}$$

solutions. This completes the induction on t. Hence, the result holds for $k = 2$.

Now, let $k \geq 3$. Suppose that the result holds for every k' with $2 \leq k' < k$. That is, given k' in the above range, we assume that the result is valid for all t and for all choices of the sets A_i and S_i. Note that $t = t_1 + \cdots + t_k \geq k \geq 3$. We have to bound the number of solutions of the following system of S-unit equations:

$$\begin{aligned} a_{1,1}u_{1,1} + \cdots + a_{1,t_1}u_{1,t_1} &= a_{2,1}u_{2,1} + \cdots + a_{2,t_2}u_{2,t_2} \\ &\vdots \\ &= a_{k,1}u_{k,1} + \cdots + a_{k,t_k}u_{k,t_k}, \end{aligned} \tag{6}$$

where $u_{i,j} \in \mathbb{Z}_{S_i}^+$. We mention that similar systems of S-unit equations have been studied by Evertse and Győry [9]. However, their theorems cannot be used directly here, so we apply some other results. Namely, by Lemma 3.1, the first equation in (6) has at most

$$(2^{35}(t-2)^2)^{(t-2)^3 s} \tag{7}$$

non-degenerate solutions. For a degenerate solution, there exists a positive integer $l \leq t-2$ and distinct non-empty subsets $I_1, \ldots, I_l \subseteq \{1, \ldots, t_1\}$, $J_1, \ldots, J_l \subseteq \{1, \ldots, t_2\}$ such that for $m = 1, \ldots, l$,

$$\sum_{i \in I_m} a_{1,i}u_{1,i} = \sum_{i \in J_m} a_{2,i}u_{2,i}, \tag{8}$$

but no proper subsum vanishes. Fix $I_1, \dots, I_l$, $J_1, \dots, J_l$. We count the number of solutions of system (6) satisfying the additional equations (8). By Lemma 3.1, for each $m = 1, \dots, l$, (8) has, up to a factor of proportionality, at most

$$(2^{35}(t-3)^2)^{(t-3)^3 s}$$

non-degenerate solutions. Let $((u_{1,i})_{i\in I_m}, (u_{2,i})_{i\in J_m})$ be a solution of (8) with $\gcd((u_{1,i})_{i\in I_m}, (u_{2,i})_{i\in J_m}) = 1$. Set

$$a'_m = \sum_{i\in I_m} a_{1,i}u_{1,i} \left(= \sum_{i\in J_m} a_{2,i}u_{2,i}\right).$$

Then

$$\{((U_m u_{1,i})_{i\in I_m}, (U_m u_{2,i})_{i\in J_m}) : U_m \in \mathbb{Z}^+_{S_1\cap S_2}\}$$

is precisely the set of solutions of (8) which are proportional to $((u_{1,i})_{i\in I_m}, (u_{2,i})_{i\in J_m})$. The problem is thus reduced to considering the following system of equations in the variables $U_1, \dots, U_l$, $u_{3,1} \dots, u_{k,t_k}$:

$$\begin{aligned} a'_1U_1 + \cdots + a'_lU_l &= a_{3,1}u_{3,1} + \cdots + a_{3,t_3}u_{3,t_3} \\ &\vdots \\ &= a_{k,1}u_{k,1} + \cdots + a_{k,t_k}u_{k,t_k}. \end{aligned}$$

Since $(S_1\cap S_2)\cap S_3\cap\cdots\cap S_k = \emptyset$, we apply the induction hypothesis for $k' = k-1$ to conclude that the above system of equations has at most

$$(2^{35}(t-2)^2)^{(k-2)(t-2)^4 s}$$

solutions. Hence, given $I_1, \dots, I_l, J_1, \dots, J_l$, we get at most

$$(2^{35}(t-3)^2)^{(t-3)^3 s(t-2)} \cdot (2^{35}(t-2)^2)^{(k-2)(t-2)^4 s}$$

solutions. Therefore, the number of degenerate solutions is bounded by

$$t^t(2^{35}(t-2)^2)^{(k-1)(t-2)^4 s} \le \frac{1}{2}(2^{35}(t-1)^2)^{(k-1)(t-1)^4 s}. \tag{9}$$

Combining the above bound (9) with (7), we obtain that the total number of solutions is at most

$$(2^{35}(t-1)^2)^{(k-1)(t-1)^4 s}.$$

This completes the induction and the proof of the proposition. □

Proof of Theorem 2.1. Taking $t_i = w^+_{S_i}(n)$ and $A_i = \{1\}$ for all $i = 1, \dots, k$ in Proposition 3.1, the statement immediately follows. □

4. Proof of Theorem 2.2

To prove Theorem 2.2, we use a Baker type estimate due to Matveev [14]. For its formulation, we need to introduce some notation.

For an algebraic number α of degree D over $\mathbb{Q}$, the absolute logarithmic height of α is defined by

$$h(\alpha) = \frac{1}{D}\left(\log a_0 + \sum_{i=1}^{D} \log \max(1, |\alpha^{(i)}|)\right),$$

where $a_0 > 0$ is the leading coefficient of the minimal polynomial of α over $\mathbb{Z}$, and the $\alpha^{(i)}$'s are the conjugates of α. Note that in the special case when $\alpha = p/q$ is a non-zero rational number with $\gcd(p, q) = 1$, it follows that $h(\alpha) = h(1/\alpha) = \log \max\{|p|, |q|\}$.

The following result is due to Matveev [14].

Lemma 4.1. *Assume that $\alpha_1, \ldots, \alpha_r$ are positive real algebraic numbers in a real algebraic number field of degree D, $d_1, \ldots, d_r$ are rational integers, and*

$$\Lambda := \alpha_1^{d_1} \ldots \alpha_r^{d_r} - 1$$

is not zero. Set

$$B \geq \max\{|d_1|, \ldots, |d_r|\},$$

and

$$A_i \geq \max\{Dh(\alpha_i), |\log \alpha_i|, 0.16\}, \text{ for all } i = 1, \ldots, r.$$

Then we have

$$|\Lambda| > \exp(-1.4 \cdot 30^{r+3} r^{4.5} D^2 (1 + \log D)(1 + \log B) A_1 \cdots A_r). \tag{10}$$

Now, we are ready to give the proof of Theorem 2.2.

Proof of Theorem 2.2. We combine arguments of Luca [13] and Stewart [21] with some other considerations.

Let n be a positive integer with $w^+_{S_1}(n) = 1$ and $w^+_{S_2}(n) = t$, and write

$$u_1 = n = v_1 + \cdots + v_t, \tag{11}$$

with $u_1 \in \mathbb{Z}^+_{S_1}$ and $v_1, \ldots, v_t \in \mathbb{Z}^+_{S_2}$. Without loss of generality, we may assume that $v_1 \geq \cdots \geq v_t$.

We write

$$u_1 = p_1^{\alpha_1} \dots p_\ell^{\alpha_\ell}, \quad v_i = q^{\beta_i} \quad (i = 1, \dots, t). \tag{12}$$

Let B be the maximum of the exponents appearing in (12).

Equation (11) can be rewritten as

$$u_1 - v_1 = v_2 + \cdots + v_t. \tag{13}$$

Since $u_1 \neq v_1$ and $v_1 \neq 1$ (otherwise $n = 1$ or $t = n$ and the statement is trivial), Lemma 4.1 yields

$$v_1 \exp(-c_4(1 + \log B)) < v_1(u_1 v_1^{-1} - 1), \tag{14}$$

with $c_4 := c_3 h(p_1) \cdots h(p_\ell) h(q)^2$, where

$$c_3 = c_3(\ell + 2) := 1.4 \cdot 30^{\ell+5}(\ell + 2)^{4.5}$$

is the constant appearing in the conclusion of Matveev's theorem (10) when Λ involves $r = \ell + 2$ rational numbers ($r = \ell + 2$ and $D = 1$).
Now, we show that

$$\frac{v_1}{v_j} < \exp\left(2j \log t (c_4(1 + \log B))^{j-1}\right) \quad (j = 2, \dots, t). \tag{15}$$

We prove this claim by induction. Combining the above inequality (14) with

$$v_1(u_1 v_1^{-1} - 1) = u_1 - v_1 < t v_2$$

implied by (13), we get

$$\frac{v_1}{v_2} < \exp\left(\log t + c_4(1 + \log B)\right) \leq \exp\left(2(\log t) c_4(1 + \log B)\right).$$

Let now i be arbitrary with $2 \leq i < t$, and assume by induction that

$$\frac{v_1}{v_j} < \exp\left(2j \log t (c_4(1 + \log B)^{j-1})\right) \quad \text{for all } j = 2, \dots, i. \tag{16}$$

Rewrite (11) as

$$u_1 - v_1 - \cdots - v_i = v_{i+1} + \cdots + v_t. \tag{17}$$

Observe that by (12) and (16) (used with $j = i$), we have

$$h\left(1 + \frac{v_2}{v_1} + \cdots + \frac{v_i}{v_1}\right) = h\left(\frac{q^{\beta_1 - \beta_i} + \cdots + q^{\beta_{i-1} - \beta_i} + 1}{q^{\beta_1 - \beta_i}}\right) =$$

$$= \log(q^{\beta_1-\beta_i} + \cdots + q^{\beta_{i-1}-\beta_i} + 1) \le \log(tq^{\beta_1-\beta_i})$$
$$= \log t + \log\left(\frac{v_1}{v_i}\right) < (2i+1)(\log t)(c_4(1+\log B))^{i-1}.$$

Hence, Lemma 4.1 yields

$$\begin{aligned} &v_1 \exp(-(2i+1)(c_4(1+\log B))^i) \\ &< v_1\left(u_1 v_1^{-1}\left(1+\frac{v_2}{v_1}+\cdots+\frac{v_i}{v_1}\right)^{-1}-1\right) \\ &< v_1\left(1+\frac{v_2}{v_1}+\cdots+\frac{v_i}{v_1}\right)\left(u_1 v_1^{-1}\left(1+\frac{v_2}{v_1}+\cdots+\frac{v_i}{v_1}\right)^{-1}-1\right). \end{aligned}$$

The above inequality, together with

$$\begin{aligned} &v_1\left(1+\frac{v_2}{v_1}+\cdots+\frac{v_i}{v_1}\right)\left(u_1 v_1^{-1}\left(1+\frac{v_2}{v_1}+\cdots+\frac{v_i}{v_1}\right)^{-1}-1\right) \\ &= u_1 - v_1 - \cdots - v_i < tv_{i+1} \end{aligned}$$

obtained from (17), implies the inequality

$$\begin{aligned} \frac{v_1}{v_{i+1}} &< \exp\left(\log t + (2i+1)(\log t)(c_4(1+\log B))^i\right) \\ &< \exp\left((2i+2)(\log t)(c_4(1+\log B))^i\right), \end{aligned}$$

which completes the induction step. Hence, our claim (15) follows. Now, note that either $B = \beta_1$ or $B \in \{\alpha_1, \ldots, \alpha_\ell\}$. In the latter case we have $2^B \le n \le tq^{\beta_1}$, so $\beta_1 \ge c_5 B - c_6 \log t$, where $c_5 := \log 2/\log q$ and $c_6 := 1/\log q$. Since $q \ge 2$, it follows that the inequality

$$\beta_1 \ge c_5 B - c_6 \log t$$

holds both when $B = \beta_1$ and when $B \in \{\alpha_1, \ldots, \alpha_\ell\}$.
Further, $\log n/\log(p_1 \cdots p_\ell) \le B \le \log n/\log 2$, showing that

$$\log\log n - c_7 \le \log B \le \log\log n + c_8, \tag{18}$$

where $c_7 := \log\log\max\{3, p_1 \ldots p_\ell\}$ and $c_8 := -\log\log 2$. Note now that since $q \notin \{p_1, \ldots, p_\ell\}$, it follows that $v_t = 1$. Setting $j = t$ in (15) and taking logarithms, we get

$$c_5 B - c_6 \log t \le \beta_1 \le c_9 t \log t (c_4(1+\log B))^{t-1}, \tag{19}$$

where we can take $c_9 := \max\{1, 2(\log q)^{-1}\}$. If the left-hand side of (19) is smaller than $c_5 B/2$, we then get that

$$\log t > c_{10} B,$$

where $c_{10} := c_5/(2c_6)$, therefore,

$$t > e^{c_{10} B} > n^{c_{11}},$$

where $c_{11} := c_{10}/\log(p_1 \cdots p_\ell)$, which for large n is better than the inequality we are after. If the left-hand side of (19) is at least $c_5 B/2$, then by taking logarithms we get

$$\log B - c_{12} < (t-1)\log(c_4(1+\log B)) + \log t + \log\log t,$$

where $c_{12} := -\log(c_5/2) + \log c_9$. From here, we get right–away that in fact

$$t > (1+o(1))\frac{\log B}{\log\log B}$$

as $B \to \infty$. Combining this with (18), we get that for every $\varepsilon > 0$, taking $c_{13} := 1-\varepsilon$, the inequality

$$t > c_{13}\frac{\log\log n}{\log\log\log n}$$

holds for all $n > n_0(\varepsilon)$, where $n_0(\varepsilon)$ is effectively computable in terms of ε and $p_1, \ldots, p_\ell, q$. Hence, the statement follows. □

5. Proof of Theorem 2.3

To prove Theorem 2.3 we use the method of Bertók and Hajdu, described in [3].

Proof of Theorem 2.3. If $w^+_{S_1}(n) + w^+_{S_2}(n) = 2$, then it is clear that the only solution is $n = 1$, so we suppose that $w^+_{S_1}(n) + w^+_{S_2}(n) \geq 3$. We describe our method in detail only in the case when $S_1 = \{3\}$, $S_2 = \{2, 5\}$. The other cases can be handled similarly. In this case we have five equations to solve, namely:

$$\begin{aligned}
3^{a_1} &= 2^{b_1}\cdot 5^{c_1} + 2^{b_2}\cdot 5^{c_2},\\
3^{a_1} &= 2^{b_1}\cdot 5^{c_1} + 2^{b_2}\cdot 5^{c_2} + 2^{b_3}\cdot 5^{c_3},\\
3^{a_1} + 3^{a_2} &= 2^{b_1}\cdot 5^{c_1},\\
3^{a_1} + 3^{a_2} &= 2^{b_1}\cdot 5^{c_1} + 2^{b_2}\cdot 5^{c_2},\\
3^{a_1} + 3^{a_2} + 3^{a_3} &= 2^{b_1}\cdot 5^{c_1}.
\end{aligned}$$

To find all solutions of the above equations, we apply the algorithm introduced in [3]. Here we only sketch the method and concentrate on how to use it for our present equations. For the detailed description of the general method, see [3]. First, by an exhaustive search, we find all 'small' solutions of the equations in non-negative integers a_i, b_i, c_i, $(i = 1, 2, 3)$. Then, after modifying the equations appropriately, we try to find a modulus m such that the modified equation has no solutions modulo m. We illustrate the method by solving the equation

$$3^{a_1} = 2^{b_1} \cdot 5^{c_1} + 2^{b_2} \cdot 5^{c_2}.$$

By an exhaustive search, we get that this equation has only five solutions with a_1, b_1, b_2, c_1, $c_2 \leq 100$, namely

$$\begin{aligned}(a_1, b_1, b_2, c_1, c_2) = &(1, 0, 0, 1, 0), (2, 0, 0, 3, 0), (4, 4, 1, 0, 0),\\ &(2, 0, 1, 2, 0), (3, 1, 0, 0, 2),\end{aligned}$$

yielding $n = 3, 9, 27, 81$. Note that $9 = 3^2$ can be represented in two ways, since $9 = 1 + 8 = 5 + 4$. We suspect that the equation has no other solutions. First, it can be seen that if both b_1 and b_2 are greater than zero, then this equation has no solutions modulo 2. The same argument applies for c_1, c_2 modulo 5, thus we conclude that we have to solve the following two equations:

$$3^{a_1} = 1 + 2^{b_2} \cdot 5^{c_2}, \tag{20}$$

$$3^{a_1} = 5^{c_1} + 2^{b_2}. \tag{21}$$

Since in every 'small' solution the exponent of 3 is at most 4, then instead of the equations above, we consider

$$3^5 \cdot 3^{a_1'} = 1 + 2^{b_2} \cdot 5^{c_2}, \tag{22}$$

$$3^5 \cdot 3^{a_1'} = 5^{c_1} + 2^{b_2}, \tag{23}$$

respectively, where every exponent is a non-negative integer. If our expectation is true, then these equations have no solutions. To prove this, we show that these equations are already not solvable locally, modulo an appropriately chosen modulus. Concerning the question how to find such a modulus, we refer once again to [3]. Now we only state that if we choose m to be $3^5 \cdot 7 \cdot 13 \cdot 17 \cdot 19 \cdot 37 \cdot 73 \cdot 97 \cdot 109 \cdot 163 \cdot 193 \cdot 257 \cdot 433 \cdot 487 \cdot 577 \cdot 769 \cdot 1153 \cdot 1297 \cdot 2593 \cdot 3457 \cdot 10369$, then as one can check, equation (22) has no solutions modulo m. Thus, in (20),

a_1 has to be less than or equal to 4. By checking every possibility, we get that this equation has three solutions, namely

$$(a_1, b_1, b_2, c_1, c_2) = (1, 0, 0, 1, 0), (2, 0, 0, 3, 0), (4, 4, 1, 0, 0).$$

Similarly, if $m = 3^5 \cdot 7 \cdot 13 \cdot 17 \cdot 19 \cdot 37 \cdot 73 \cdot 97 \cdot 109 \cdot 163 \cdot 193 \cdot 433 \cdot 577 \cdot 769$, then equation (23) has no solutions modulo m, thus we only have to check (21) with $a_1 \leq 4$. In this case, we get the remaining two 'small' solutions. The other equations can be handled similarly. Finally, we mention that an appropriately chosen divisor of $M = 2^{16} \cdot 3^{10} \cdot 5^8 \cdot 7 \cdot 13 \cdot 17 \cdot 19 \cdot 163 \cdot 37 \cdot 433 \cdot 193 \cdot 97 \cdot 73 \cdot 257 \cdot 109 \cdot 577 \cdot 769 \cdot 487 \cdot 1153 \cdot 1297 \cdot 1459 \cdot 2593 \cdot 2917 \cdot 3457 \cdot 3889 \cdot 10369 \cdot 1373 \cdot 3137 \cdot 12289 \cdot 17497 \cdot 18433 \cdot 39367 \cdot 52489 \cdot 65537 \cdot 50177 \cdot 139969 \cdot 147457 \cdot 209953 \cdot 331777 \cdot 472393 \cdot 114689 \cdot 268913 \cdot 470597 \cdot 629857 \cdot 746497 \cdot 786433 \cdot 839809 \cdot 995329 \cdot 614657$ is sufficient for every equation under investigation. (Certainly, one can take $m = M$ in each case, however, then the computation time would be enormous.) □

Acknowledgements. The authors are grateful to the referees for their helpful comments.

References

[1] Zs. Ádám, L. Hajdu and F. Luca, Representing primes as linear combinations of pure powers, *Acta Arith.* **138** (2009), 101–107.

[2] Cs. Bertók, Representing integers as sums or differences of general power products, *Acta Math. Hung.* **141** (2013), 291–300.

[3] Cs. Bertók and L. Hajdu, A Hasse-type principle for exponential Diophantine equations and its applications, *Math. Comp.* **85** (2016), 849–860.

[4] Y. Bugeaud, M. Cipu and M. Mignotte, On the representation of Fibonacci and Lucas numbers in an integer base, *Ann. Math. Québec* **37** (2013), 31–43.

[5] V. S. Dimitrov and E. W. Howe, Lower bounds on the lengths of double-base representations, *Proc. Amer. Math. Soc.* **139** (2011), 3423–3430.

[6] P. Erdős, C. Mauduit and A. Sárközy, On arithmetic properties of integers with missing digits, I, Distribution in residue classes, *J. Number Theory* **70** (1998), 99–120.

[7] P. Erdős, C. Mauduit and A. Sárközy, On arithmetic properties of integers with missing digits, II, Prime factors, *Discrete Math.* **200** (1999), 149–164, Paul Erdős memorial collection.

[8] J.-H. Evertse, The number of solutions of decomposable form equations, *Invent. Math.* **122** (1995), 559–601.

[9] J.-H. Evertse and K. Győry, On the number of solutions of weighted unit equations, *Compositio Math.* **66** (1988), 329–354.

[10] L. Hajdu and F. Luca, On the length of arithmetic progressions in linear combinations of S-units, *Arch. Math.* **94** (2010), 357–363; correction, *ibid.* **103** (2014), 399–400.

[11] L. Hajdu and R. Tijdeman, Representing integers as linear combinations of powers, *Publ. Math. Debrecen* **79** (2011), 461–468.

[12] L. Hajdu and R. Tijdeman, Representing integers as linear combinations of power products, *Arch. Math.* **98** (2012), 527–533.

[13] F. Luca, Distinct digits in base b expansions of linear recurrence sequences, *Quaest. Math.* **23** (2000), 389–404.

[14] E. M. Matveev, An explicit lower bound for a homogeneous rational linear form in the logarithms of algebraic numbers, II, *Izv. Ross. Akad. Nauk Ser. Mat.* **64** (2000), 125–180 (in *Russian*); translation in *Izv. Math.* **64** (2000), 1217–1269.

[15] C. Mauduit, C. Pomerance and A. Sárközy, On the distribution in residue classes of integers with a fixed sum of digits, *Ramanujan J.* **9** (2005), 45–62.

[16] C. Mauduit and A. Sárközy, On the arithmetic structure of sets characterized by sum of digits properties, *J. Number Theory* **61** (1996), 25–38.

[17] C. Mauduit and A. Sárközy, On the arithmetic structure of the integers whose sum of digits is fixed, *Acta Arith.* **81** (1997), 145–173.

[18] M. B. Nathanson, Geometric group theory and arithmetic diameter, *Publ. Math. Debrecen* **79** (2011), 563–572.

[19] A. Pethő and R. F. Tichy, On digit expansions with respect to linear recurrences, *J. Number Theory* **33** (1989), 243–256.

[20] H. G. Senge and E. G. Straus, PV-numbers and sets of multiplicity, *Period. Math. Hungar.* **3** (1973), 93–100.

[21] C. L. Stewart, On the representation of an integer in two different bases, *J. Reine Angew. Math.* **319** (1980), 63–72.

[22] S. G. Wagner, Numbers with fixed sum of digits in linear recurrent number systems, *Ramanujan J.* **14** (2007), 43–68.

CSANÁD BERTÓK
INSTITUTE OF MATHEMATICS
UNIVERSITY OF DEBRECEN
H-4002 DEBRECEN
P. O. BOX 400
HUNGARY

E-mail: bertok.csanad@science.unideb.hu

FLORIAN LUCA
SCHOOL OF MATHEMATICS
UNIVERSITY OF THE WITWATERSRAND
PRIVATE BAG X3
WITS 2050
JOHANNESBURG
SOUTH AFRICA

E-mail: florian.luca@wits.ac.za

LAJOS HAJDU
INSTITUTE OF MATHEMATICS
UNIVERSITY OF DEBRECEN
H-4002 DEBRECEN
P. O. BOX 400
HUNGARY

E-mail: hajdul@science.unideb.hu

DIVYUM SHARMA
SCHOOL OF MATHEMATICS
TATA INSTITUTE OF
FUNDAMENTAL RESEARCH
HOMI BHABHA ROAD
MUMBAI – 400 005
INDIA

E-mail: divyum@math.tifr.res.in

(Received December 8, 2015; revised June 14, 2016)