

On inequalities for alternating trigonometric sums

By HORST ALZER (Waldbröl) and MAN KAM KWONG (Hong Kong)

Abstract. We present various inequalities for alternating trigonometric sums. Among others, we prove that the double-inequality

$$\frac{1 - \sqrt{2}}{3} \leq \sum_{k=1}^n (-1)^{k-1} \frac{\sin^2((2k-1)x)}{2k-1} \leq 1$$

is valid for all natural numbers n and real numbers x . Both bounds are sharp.

1. Introduction and statement of results

Two classical results on trigonometric polynomials state that for all $n \in \mathbb{N}$ and $x \in (0, \pi)$,

$$0 < \sum_{k=1}^n \frac{\sin(kx)}{k} \tag{1.1}$$

and

$$-1 < \sum_{k=1}^n \frac{\cos(kx)}{k}. \tag{1.2}$$

Both bounds are sharp. The validity of (1.1) was conjectured by Fejér in 1910, and the first proofs were published by JACKSON [9] and GRONWALL [8] in 1911 and 1912, respectively. The counterpart (1.2) was established by YOUNG [14] in 1913.

Mathematics Subject Classification: 26D05, 42A05.

Key words and phrases: trigonometric sums, inequalities, sharp bounds, absolutely monotonic. The research of the second author is supported by the Hong Kong Government GRF Grant PolyU 5003/12P, and the Hong Kong Polytechnic University Grants G-YBCQ and G-YBKT.

The inequalities of Fejér–Jackson–Gronwall and Young have stimulated the research of many mathematicians, who discovered remarkable new proofs, generalizations, refinements and numerous variants of (1.1) and (1.2). Moreover, it was shown that inequalities for trigonometric polynomials have interesting applications in various fields, like, for example, the theory of univalent functions and Fourier analysis.

Here, we demonstrate that the inequalities (1.1) and (1.2) have an application in the theory of absolutely monotonic functions. An infinitely differentiable function defined on an interval I is said to be absolutely monotonic if the function and all its derivatives are nonnegative on I . A detailed study of absolutely monotonic functions can be found in WIDDER [13, Chapter IV].

For $x \in (-1, 1)$ and $\theta \in (0, \pi)$, we define the two functions

$$f_\theta(x) = \frac{1}{1-x} \arctan \frac{x \sin(\theta)}{1-x \cos(\theta)}$$

and

$$g_\theta(x) = \frac{1}{1-x} \left(x - \frac{1}{2} \log(1 - 2x \cos(\theta) + x^2) \right).$$

Proposition. *For all $\theta \in (0, \pi)$, the functions $f_\theta(x)$ and $g_\theta(x)$ are absolutely monotonic on $(0, 1)$.*

PROOF. Using the identities (see GOULD [7, (2.23), (2.24)])

$$\sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n} x^n = \arctan \frac{x \sin(\theta)}{1-x \cos(\theta)}$$

and

$$\sum_{n=1}^{\infty} \frac{\cos(n\theta)}{n} x^n = -\frac{1}{2} \log(1 - 2x \cos(\theta) + x^2),$$

which are valid for $|x| < 1$, we arrive at the series representations

$$f_\theta(x) = \sum_{n=1}^{\infty} \sum_{k=1}^n \frac{\sin(k\theta)}{k} x^n$$

and

$$g_\theta(x) = \sum_{n=1}^{\infty} \left(1 + \sum_{k=1}^n \frac{\cos(k\theta)}{k} \right) x^n.$$

It follows that if $\theta \in (0, \pi)$, then

$$\frac{d^n}{dx^n} f_\theta(x) > 0 \quad \text{and} \quad \frac{d^n}{dx^n} g_\theta(x) > 0 \quad (0 \leq n \in \mathbf{Z}; 0 < x < 1).$$

This means that f_θ and g_θ are absolutely monotonic on $(0, 1)$. □

We remark that ASKEY and GASPER [4] proved that the function $x \mapsto (1/x)f_\theta(x)$ is absolutely monotonic on $(0, 1)$ if $\theta \in (0, \pi)$.

The inequalities (1.1) and (1.2) can be refined if we assume that $n \geq 2$. We have

$$x^2 \left(\cot \frac{x}{2} - \frac{\pi - x}{2} \right) < \sum_{k=1}^n \frac{\sin(kx)}{k} \quad (2 \leq n \in \mathbf{N}; 0 < x < \pi) \quad (1.3)$$

and

$$-\frac{5}{6} \leq \sum_{k=1}^n \frac{\cos(kx)}{k} \quad (2 \leq n \in \mathbf{N}; 0 < x < \pi). \quad (1.4)$$

The constant $-5/6$ is best possible. Proofs for (1.3) and (1.4) can be found in ALZER and KOUMANDOS [1], and in BROWN and KOUMANDOS [6], respectively. We note that inequality (1.4) plays an important role in the proof of Theorem 1 below.

Among the many variants and analogues of (1.1) and (1.2) given in the literature, we mention three inequalities which have motivated our work. They are all concerned with sums having only terms involving odd multiples of x . The first is an elegant counterpart of (1.1) due to KOSCHMIEDER [10]:

$$0 < \sum_{k=1}^n \frac{\sin((2k-1)x)}{2k-1} \quad (n \in \mathbf{N}; 0 < x < \pi).$$

This is a special case of a more general result. If $(a_k)_{k \geq 1}$ is a decreasing sequence of positive numbers, then

$$0 < \sum_{k=1}^n a_k \sin((2k-1)x) \quad (n \in \mathbf{N}; 0 < x < \pi).$$

Proofs for the next inequalities are given in ALZER and KOUMANDOS [2]:

$$-\frac{1}{18}\sqrt{3} \leq \sum_{k=1}^n (-1)^{k-1} \frac{\sin((2k-1)x)}{k} \quad (n \in \mathbf{N}; 0 < x < \pi) \quad (1.5)$$

and

$$-\text{Si}(\pi) < \sum_{k=1}^n (-1)^{k-1} \frac{\cos((2k-1)x)}{k} < \text{Si}(\pi) \quad (n \in \mathbf{N}; x \in \mathbf{R}). \quad (1.6)$$

Here, $\text{Si}(x) = \int_0^x (\sin(t)/t) dt$ denotes the sine integral. The constant bounds in (1.5) and (1.6) are sharp. For more information on inequalities connected

with trigonometric sums, we refer to ASKEY [3], ASKEY and GASPER [5], MILOVANOVIĆ, MITRINOVIĆ and RASSIAS [12, Chapter 4], and the references given therein.

In this paper, we present new inequalities for certain alternating trigonometric sums. Our first theorem offers a relative of (1.1).

Theorem 1. *For all natural numbers n and real numbers x , we have*

$$-\frac{1}{8} \leq \sum_{k=1}^n (-1)^{k-1} \frac{\sin^2(kx)}{k}. \quad (1.7)$$

The lower bound is sharp.

Using the summation by parts method, we obtain the following extension of (1.7).

Corollary 1. *Let $(a_k)_{k \geq 1}$ be a sequence of nonnegative real numbers such that $(ka_k)_{k \geq 1}$ is decreasing. Then, for $n \in \mathbf{N}$ and $x \in \mathbf{R}$,*

$$-\frac{a_1}{8} \leq \sum_{k=1}^n (-1)^{k-1} a_k \sin^2(kx).$$

In particular, for any $\alpha \in [0, 1)$,

$$-\frac{1}{8(1-\alpha)} \leq \sum_{k=1}^n (-1)^{k-1} \frac{\sin^2(kx)}{k-\alpha}.$$

The next theorem provides a generalization of (1.7) involving two parameters.

Theorem 2. *Let λ and μ be real numbers with $\lambda \geq \mu \geq 0$. For all natural numbers n and real numbers x , we have*

$$\frac{5\mu - \lambda}{8} \leq \sum_{k=1}^n (-1)^{k-1} \frac{\lambda \sin^2(kx) + \mu \cos^2(kx)}{k}. \quad (1.8)$$

The lower bound is sharp.

Remark 1. We denote the sum given in (1.8) by $D_n(\lambda, \mu; x)$. If $\lambda > \mu$, then

$$\lim_{n \rightarrow \infty} D_{2n}(\lambda, \mu; \pi/2) = \mu \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} + (\lambda - \mu) \sum_{k=1}^{\infty} \frac{1}{2k-1} = \infty.$$

This implies that there are no constant upper bounds for the sums in (1.8) and (1.7).

An application of Theorem 2 leads to a class of nonnegative trigonometric sums.

Theorem 3. *Let c be a real number. The inequality*

$$0 \leq \sum_{k=1}^n (-1)^{k-1} \frac{\sin^2(kx) + c \cos^2(kx)}{k} \quad (1.9)$$

holds for all natural numbers n and real numbers x if and only if $c \in [1/5, 1]$.

The following result is a striking companion of (1.7).

Theorem 4. *For all natural numbers n and real numbers x , we have*

$$\frac{1 - \sqrt{2}}{3} \leq \sum_{k=1}^n (-1)^{k-1} \frac{\sin^2((2k-1)x)}{2k-1} \leq 1. \quad (1.10)$$

Both bounds are sharp.

Again, the summation by parts method yields the next corollary.

Corollary 2. *Let $(b_k)_{k \geq 1}$ be a sequence of nonnegative real numbers such that $((2k-1)b_k)_{k \geq 1}$ is decreasing. Then, for $n \in \mathbf{N}$ and $x \in \mathbf{R}$,*

$$\frac{1 - \sqrt{2}}{3} b_1 \leq \sum_{k=1}^n (-1)^{k-1} b_k \sin^2((2k-1)x) \leq b_1.$$

In particular, for any $\beta \in [1, 2]$,

$$\frac{1 - \sqrt{2}}{3(2 - \beta)} \leq \sum_{k=1}^n (-1)^{k-1} \frac{\sin^2((2k-1)x)}{2k - \beta} \leq \frac{1}{2 - \beta}.$$

Finally, we offer an analogue of (1.8), an extension of (1.10).

Theorem 5. *Let λ and μ be real numbers with $\lambda \geq \mu \geq 0$. For all natural numbers n and real numbers x , we have*

$$\begin{aligned} & \frac{(1 + \sqrt{2})\mu + (1 - \sqrt{2})\lambda}{3} \\ & \leq \sum_{k=1}^n (-1)^{k-1} \frac{\lambda \sin^2((2k-1)x) + \mu \cos^2((2k-1)x)}{2k-1} \leq \lambda. \end{aligned} \quad (1.11)$$

Both bounds are sharp.

In the next section, we collect some lemmas which we need to establish Theorem 4. The proofs of our theorems are given in Section 3.

2. Lemmas

Throughout this paper, we maintain the following notations:

$$\begin{aligned} C_n(x) &= \sum_{k=1}^n (-1)^{k-1} \frac{1 - \cos((2k-1)x)}{2k-1} \\ &= \int_0^x \sum_{k=1}^n (-1)^{k-1} \sin((2k-1)t) dt = (-1)^{n-1} \int_0^x \frac{\sin(2nt)}{2 \cos(t)} dt, \\ F_m(x) &= \int_0^x \frac{\sin(mt)}{2 \sin(t)} dt = C_n(x + \pi/2) - C_n(\pi/2) \quad (m = 2n), \end{aligned}$$

and

$$x_k = \frac{k\pi}{m}.$$

Lemma 1. *If $m \geq 6$, then*

$$F_m(x_1) - F_m(x_2) < 0.263.$$

PROOF. We obtain

$$F_m(x_1) - F_m(x_2) = \frac{1}{2m} \int_{\pi}^{2\pi} \frac{-\sin(t)}{t} \frac{t}{\sin(t/m)} dt.$$

Since $t \mapsto t/\sin(t)$ is increasing on $(0, \pi]$, we obtain for $t \in [\pi, 2\pi]$:

$$\frac{t}{\sin(t/m)} \leq \frac{\pi}{3 \sin(\pi/3)} m \leq 1.21m.$$

Thus,

$$F_m(x_1) - F_m(x_2) \leq -\frac{1.21}{2} \int_{\pi}^{2\pi} \frac{\sin(t)}{t} dt = \frac{1.21}{2} (\text{Si}(\pi) - \text{Si}(2\pi)) = 0.2624 \dots \quad \square$$

Lemma 2. *If $m = 2n \geq 2$, then*

$$F_m(\pi/2) - F_m(x_1) = C_n(\pi/2 - x_1).$$

PROOF. We have

$$\begin{aligned} &2(-1)^{n-1} (F_m(\pi/2) - F_m(x_1) - C_n(\pi/2 - x_1)) \\ &= 2(-1)^{n-1} (C_n(\pi) - C_n(\pi/2 + x_1) - C_n(\pi/2 - x_1)) \end{aligned}$$

$$\begin{aligned}
&= \left(\int_0^\pi - \int_0^{\pi/2+x_1} - \int_0^{\pi/2-x_1} \right) \frac{\sin(2nt)}{\cos(t)} dt \\
&= \left(\int_{\pi/2-x_1}^\pi - \int_0^{\pi/2+x_1} \right) \frac{\sin(2nt)}{\cos(t)} dt = 0. \quad \square
\end{aligned}$$

Lemma 3. *Let $m \geq 6$ be an even integer. If $r = [(m-2)/4]$ and $s = [m/4]$, then*

$$F_m(x_1) > F_m(x_3) > \cdots > F_m(x_{2r+1}) \quad (2.1)$$

and

$$0 = F_m(x_0) < F_m(x_2) < F_m(x_4) < \cdots < F_m(x_{2s}). \quad (2.2)$$

PROOF. As shown in KWONG [11], integration by parts yields, for $1 \leq k \leq r$,

$$\begin{aligned}
F_m(x_{2k+1}) - F_m(x_{2k-1}) &= \int_{x_{2k-1}}^{x_{2k+1}} \frac{\sin(mt)}{2 \sin(t)} dt \\
&= - \int_{(2k-1)\pi/m}^{(2k+1)\pi/m} \frac{(1 + \cos(mt)) \cos(t)}{2m \sin^2(t)} dt < 0
\end{aligned}$$

and, for $0 \leq k \leq s-1$,

$$F_m(x_{2k+2}) - F_m(x_{2k}) = \int_{2k\pi/m}^{(2k+2)\pi/m} \frac{(1 - \cos(mt)) \cos(t)}{2m \sin^2(t)} dt > 0.$$

This leads to (2.1) and (2.2). $\square$

Lemma 4. *Let $m \geq 6$ be an even integer. Then, for $x \in [0, \pi/2]$,*

$$0 \leq F_m(x) \leq F_m(x_1). \quad (2.3)$$

PROOF. Since F_m is increasing on $[x_{2k}, x_{2k+1}]$ ($k = 0, \dots, [(m-2)/4]$), and decreasing on $[x_{2k+1}, x_{2k+2}]$ ($k = 0, \dots, [(m-4)/4]$), we conclude that F_m attains a local maximum at x_{2k+1} , and a local minimum at x_{2k} . Let $s = [m/4]$. If $m/2$ is odd, then $x_{2s+1} = \pi/2$, and if $m/2$ is even, then $x_{2s} = \pi/2$. This means that $\pi/2$ is either a local maximum or a local minimum of F_m . Using (2.1) and (2.2) gives

$$F_m(0) = 0 \leq F_m(\pi/2) \leq F_m(x_1).$$

This implies that (2.3) is valid for all $x \in [0, \pi/2]$. $\square$

Remark 2. A proof of the right-hand side of (2.3) is also given in ALZER and KOUMANDOS [1].

Lemma 5. *If $n \geq 3$ and $x \in [0, \pi]$, then*

$$C_n(\pi/2 - x_1) \leq C_n(x).$$

PROOF. Let $m = 2n$. We show that

$$F_m(x - \pi/2) \geq F_m(-x_1). \quad (2.4)$$

Case 1. $0 \leq x \leq \pi/2$. Then, $0 \leq \pi/2 - x \leq \pi/2$. Applying $F_m(-x) = -F_m(x)$ and Lemma 4 gives

$$F_m(x - \pi/2) = -F_m(\pi/2 - x) \geq -F_m(x_1) = F_m(-x_1).$$

Case 2. $\pi/2 \leq x \leq \pi$. We have $0 \leq x - \pi/2 \leq \pi/2$, so that Lemma 4 yields

$$F_m(x - \pi/2) \geq 0 \geq -F_m(x_1) = F_m(-x_1).$$

From (2.4) we conclude that

$$C_n(x) = F_m(x - \pi/2) + C_n(\pi/2) \geq F_m(-x_1) + C_n(\pi/2) = C_n(\pi/2 - x_1). \quad \square$$

Lemma 6. *Let $n \geq 1$ be an odd integer. Then, for $x \in [0, \pi]$,*

$$C_n(x) \leq 2.$$

PROOF. We have

$$C_1(x) = 1 - \cos(x) \leq 2.$$

Let $n \geq 3$ be odd and $m = 2n$. We consider two cases.

Case 1. $0 \leq x \leq \pi/2$. Then, $0 \leq \pi/2 - x \leq \pi/2$. Applying $F_m(-x) = -F_m(x)$, Lemma 4 and

$$C_n(\pi/2) = \sum_{k=1}^n \frac{(-1)^{k-1}}{2k-1} \leq C_3(\pi/2) = \frac{13}{15} \quad (2.5)$$

gives

$$C_n(x) = C_n(\pi/2) - F_m(\pi/2 - x) \leq C_n(\pi/2) \leq \frac{13}{15}.$$

Case 2. $\pi/2 \leq x \leq \pi$. From Lemma 4 we obtain

$$C_n(x) = C_n(\pi/2) + F_m(x - \pi/2) \leq C_n(\pi/2) + F_m(x_1). \quad (2.6)$$

If $s \in (0, \pi]$, then $x \mapsto x \sin(s/x)$ is increasing on $[1, \infty)$, so that for $m \geq 6$,

$$F_m(x_1) = \frac{1}{2} \int_0^\pi \frac{\sin(s)}{m \sin(s/m)} ds \leq \frac{1}{2} \int_0^\pi \frac{\sin(s)}{6 \sin(s/6)} ds = \frac{14}{15}. \quad (2.7)$$

Combining (2.5), (2.6) and (2.7) gives

$$C_n(x) \leq \frac{27}{15}.$$

This completes the proof of Lemma 6. $\square$

3. Proofs of theorems

PROOF OF THEOREM 1. Let $A_n(x)$ be the sum in (1.7). Since

$$A_n(x + \pi) = A_n(x) \quad \text{and} \quad A_n(\pi/2 - x) = A_n(\pi/2 + x),$$

it suffices to prove (1.7) for $x \in [0, \pi/2]$. We have

$$A_1(x) = \sin^2(x), \quad A_2(x) + \frac{1}{8} = \frac{1}{8}(4 \cos^2(x) - 3)^2, \quad A_3(x) = A_2(x) + \frac{1}{3} \sin^2(3x).$$

It follows that (1.7) holds for $n = 1, 2, 3$. Let $n \geq 4$. Applying

$$\sin^2(y) = \frac{1}{2} - \frac{1}{2} \cos(2y) \quad (3.1)$$

gives

$$-2A_n(x) = G_n + H_n(2x),$$

where

$$G_n = \sum_{k=1}^n \frac{(-1)^k}{k} \quad \text{and} \quad H_n(x) = \sum_{k=1}^n (-1)^{k-1} \frac{\cos(kx)}{k}.$$

We set $x = (\pi - z)/2$. Then, $0 \leq z \leq \pi$. Using

$$G_n \leq G_4 = -\frac{7}{12}$$

and (1.4) yields

$$-2A_n(x) \leq -\frac{7}{12} + H_n(2x) = -\frac{7}{12} - \sum_{k=1}^n \frac{\cos(kz)}{k} \leq -\frac{7}{12} + \frac{5}{6} = \frac{1}{4}.$$

This leads to (1.7). Moreover, since $A_2(\pi/6) = -1/8$, we conclude that $-1/8$ is the best possible constant lower bound. $\square$

PROOF OF THEOREM 2. As in Section 1, we denote the sum in (1.8) by $D_n(\lambda, \mu; x)$. Then,

$$D_n(\lambda, \mu; x) = -\mu G_n + (\lambda - \mu)A_n(x).$$

We obtain

$$D_1(\lambda, \mu; x) = \mu + (\lambda - \mu) \sin^2(x) \geq \mu \geq \frac{5\mu - \lambda}{8}.$$

Let $n \geq 2$. We apply (1.7) and

$$\frac{1}{2} = -G_2 \leq -G_n.$$

Thus,

$$D_n(\lambda, \mu; x) \geq \frac{1}{2}\mu - \frac{1}{8}(\lambda - \mu) = \frac{5\mu - \lambda}{8} = D_2(\lambda, \mu; \pi/6).$$

This completes the proof of Theorem 2. $\square$

PROOF OF THEOREM 3. Let $c \in [1/5, 1]$. Applying Theorem 2, we obtain

$$D_n(1, c; x) \geq \frac{5c - 1}{8} \geq 0.$$

This settles (1.9). We assume that (1.9) is valid for all n and x . Then,

$$D_2(1, c; \pi/6) = \frac{5}{8} \left(c - \frac{1}{5} \right) \geq 0.$$

Thus, $c \geq 1/5$. We have

$$D_n(1, c; \pi/2) = \frac{1+c}{2} \sum_{k=1}^n \frac{(-1)^{k-1}}{k} + \frac{1-c}{2} \sum_{k=1}^n \frac{1}{k} \geq 0.$$

If $c > 1$, then

$$\lim_{n \rightarrow \infty} D_n(1, c; \pi/2) = -\infty.$$

A contradiction. Hence, $c \leq 1$. $\square$

PROOF OF THEOREM 4. Let $S_n(x)$ be the sine sum in (1.10). Since

$$S_n(x + \pi) = S_n(x) \quad \text{and} \quad S_n(\pi/2 - x) = S_n(\pi/2 + x),$$

it remains to prove (1.10) for $x \in [0, \pi/2]$. Using (3.1), we obtain the representation

$$S_n(x) = \frac{1}{2} C_n(y) \quad \text{with} \quad y = 2x \in [0, \pi].$$

Thus, we have to show that

$$-0.276\dots = \frac{2}{3}(1 - \sqrt{2}) \leq C_n(y) \leq 2. \quad (3.2)$$

Applying Lemma 6 and

$$C_{2N}(y) \leq C_{2N+1}(y) \quad (N \geq 0)$$

reveals that the right-hand side of (3.2) is valid for all $n \geq 1$.

We find that

$$C_1(y) = 1 - \cos(y) \geq 0 > \frac{2}{3}(1 - \sqrt{2})$$

and

$$C_2(y) = \frac{2}{3}(1 - \sqrt{2}) + \frac{1}{3}(\sqrt{2} + \cos(y))(\sqrt{2} - 2\cos(y))^2 \geq \frac{2}{3}(1 - \sqrt{2}).$$

Let $n \geq 3$. Applying Lemmas 5, 2, 4 and 1 leads to

$$-C_n(y) \leq -C_n(\pi/2 - x_1) = F_m(x_1) - F_m(\pi/2) < F_m(x_1) - F_m(x_2) < 0.263.$$

Hence,

$$C_n(y) \geq -0.263 > \frac{2}{3}(1 - \sqrt{2}).$$

This completes the proof of the left-hand side of (3.2). Since

$$S_1(\pi/2) = 1 \quad \text{and} \quad S_2(\pi/8) = \frac{1 - \sqrt{2}}{3},$$

we conclude that the bounds given in (1.10) are sharp. $\square$

PROOF OF THEOREM 5. Let $R_n(\lambda, \mu; x)$ denote the sum given in (1.11). Then we have

$$R_n(\lambda, \mu; x) = (\lambda - \mu)S_n(x) + \mu \sum_{k=1}^n \frac{(-1)^{k-1}}{2k-1}.$$

Applying (1.10) and

$$\frac{2}{3} \leq \sum_{k=1}^n \frac{(-1)^{k-1}}{2k-1} \leq 1$$

leads to

$$\begin{aligned} R_2(\lambda, \mu; \pi/8) &= \frac{(1 + \sqrt{2})\mu + (1 - \sqrt{2})\lambda}{3} = \frac{1 - \sqrt{2}}{3}(\lambda - \mu) + \frac{2}{3}\mu \leq R_n(\lambda, \mu; x) \\ &\leq (\lambda - \mu) + \mu = \lambda = R_1(\lambda, \mu; \pi/2). \end{aligned}$$

This settles (1.11) and reveals that the given bounds are the best possible. $\square$

References

- [1] H. ALZER and S. KOUMANDOS, Inequalities of Fejér–Jackson type, *Monatsh. Math.* **139** (2003), 89–103.
- [2] H. ALZER and S. KOUMANDOS, Sharp estimates for various trigonometric sums, *Analysis* **33** (2012), 9–26.
- [3] R. ASKEY, Orthogonal Polynomials and Special Functions, Reg. Conf. Ser. Appl. Math., Vol. **21**, SIAM (Society for Industrial and Applied Mathematics), Philadelphia, PA, 1975.
- [4] R. ASKEY and G. GASPER, Positive Jacobi polynomial sums, II, *Amer. J. Math.* **98** (1976), 709–737.
- [5] R. ASKEY and G. GASPER, Inequalities for polynomials, In: The Bieberbach Conjecture (A. Baernstein II et al., eds.), Mathematical Surveys and Monographs, Vol. **2**, Amer. Math. Soc., Providence, RI, 1986, 7–32.
- [6] G. BROWN and S. KOUMANDOS, On a monotonic trigonometric sum, *Monatsh. Math.* **123** (1997), 109–119.
- [7] H. W. GOULD, Combinatorial identities: Table III: Binomial identities derived from trigonometric and exponential series, 26 pp, www.math.wvu.edu/~gould.
- [8] T. H. GRONWALL, Über die Gibbssche Erscheinung und die trigonometrischen Summen $\sin x + \frac{1}{2} \sin 2x + \dots + \frac{1}{n} \sin nx$, *Math. Ann.* **72** (1912), 228–243.
- [9] D. JACKSON, Über eine trigonometrische Summe, *Rend. Circ. Mat. Palermo* **32** (1911), 257–262.
- [10] L. KOSCHMIEDER, Vorzeicheneigenschaften der Abschnitte einiger physikalisch bedeutsamer Reihen, *Monatsh. Math. Phys.* **39** (1932), 321–344.
- [11] M. K. KWONG, Improved Vietoris sine inequalities for non-monotone, non-decaying coefficients, 2015, arXiv:1504.06705 [math.CA].
- [12] G. V. MILOVANOVIĆ, D. S. MITRINOVIĆ and TH. M. RASSIAS, Topics in Polynomials: Extremal Problems, Inequalities, Zeros, *World Scientific Publishing Co., Singapore*, 1994.
- [13] D. V. WIDDER, The Laplace Transform, Princeton Mathematical Series, Vol. **6**, Princeton University Press, Princeton, 1946.
- [14] W. H. YOUNG, On certain series of Fourier, *Proc. London Math. Soc. (2)* **11** (1913), 357–366.

HORST ALZER
MORSBACHER STR. 10
51545 WALDBRÖL
GERMANY

E-mail: h.alzer@gmx.de

MAN KAM KWONG
DEPARTMENT OF APPLIED
MATHEMATICS
THE HONG KONG
POLYTECHNIC UNIVERSITY
HUNG HONG
HONG KONG

E-mail: mankwong@polyu.edu.hk

(Received December 19, 2015)