

Derivations into various duals of Lau product of Banach algebras

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Abstract. For two Banach algebras A and B , an interesting product $A \times_{\theta} B$, called the θ -Lau product, was recently introduced and studied for some non-zero multiplicative linear functional θ on B . In this paper, by discussing general necessary and sufficient conditions for n -weak amenability of $A \times_{\theta} B$, we extend some results on the n -weak amenability of the unitization $A^{\sharp}$ of A , to the θ -Lau product $A \times_{\theta} B$. In particular, we improve several known results on n -weak amenability of $A \times_{\theta} B$ and answer some questions on this topic.

1. Introduction and some preliminaries

Let A and B be Banach algebras with $\sigma(B) \neq \emptyset$, and let $\theta \in \sigma(B)$, where $\sigma(B)$ is the set of all non-zero multiplicative linear functionals on B . The θ -Lau product $A \times_{\theta} B$ is a Banach algebra which is defined as the vector space $A \times B$ equipped with the algebra multiplication

$$(a_1, b_1)(a_2, b_2) = (a_1a_2 + \theta(b_2)a_1 + \theta(b_1)a_2, b_1b_2) \quad (a_1, a_2 \in A, b_1, b_2 \in B),$$

and the norm $\|(a, b)\| = \|a\| + \|b\|$. This type of product was introduced by LAU [8] for a certain class of Banach algebras known as Lau algebras, and was extended by SANGANI MONFARED [9] for arbitrary Banach algebras. The unitization $A^{\sharp}$ of A can be regarded as the ι -Lau product $A \times_{\iota} \mathbb{C}$, where $\iota \in \sigma(\mathbb{C})$ is the identity map.

This product provides not only new examples of Banach algebras by themselves, but it can also serve as a source of (counter-) examples for various purposes in functional and harmonic analysis. From the homological algebra point

of view, $A \times_{\theta} B$ is a strongly splitting Banach algebra extension of B by A , which means that A is a closed two-sided ideal of $A \times_{\theta} B$, and the quotient $(A \times_{\theta} B)/A$ is isometrically isomorphic to B . The Lau product of Banach algebras enjoys some properties that are not shared in general by arbitrary strongly splitting extensions. For instance, commutativity is not preserved by a generally strongly splitting extension. However, $A \times_{\theta} B$ is commutative if and only if both A and B are commutative.

Many basic properties of $A^{\sharp}$, some notions of amenability and some homological properties are extended to $A \times_{\theta} B$ by many authors; see, for example, [4], [7], [9], [10] and [11]. In particular, GHADERI, NASR-ISFAHANI and NEMATİ [4] extended some results on n -weak amenability of $A^{\sharp}$, obtained by DALES, GHAHRAMANI and GRONBEAK [3], to $A \times_{\theta} B$. They showed that if A and B are $(2n+1)$ -weakly amenable, then $A \times_{\theta} B$ is $(2n+1)$ -weakly amenable, [4, Theorem 4.1]. For a continuous derivation $D : A \times_{\theta} B \rightarrow (A \times_{\theta} B)^{(2n+1)}$ with $D(a, 0) = 0$, they claim that $D(0, b) \in B^{(2n+1)}$. By a careful look at their proof, we could only conclude that $(\iota^{(2n+1)} \circ D)(0, b) = 0$ on the subspace $\langle AA^{(2n)} \cup A^{(2n)}A \rangle$ of $A^{(2n)}$, where $\iota : A \rightarrow A \times_{\theta} B$ is the natural embedding. So, there appear to be some gaps in their proof. This result, under a suitable condition on A , was proved by EBRAHIMI VISHKI and KHODDAMI [12], but the question of whether there is an analogue to this result for the even case was left open. Moreover, in the case when A is unital, it was shown that $A \times_{\theta} B$ is n -weakly amenable if and only if both A and B are n -weakly amenable. It was also left as an open question whether this result holds for the case when A has a bounded approximate identity.

The n -weak amenability of the unitization $A^{\sharp}$ of a Banach algebra A was also studied by ZHANG [13]. As a main result of [13, Section 3.2], he showed that if A is weakly amenable or has a bounded approximate identity, then for each $n \geq 0$, $A^{\sharp}$ is n -weakly amenable if and only if A is n -weakly amenable [13, Theorem 3.16].

In this paper, we discuss general necessary and sufficient conditions for $A \times_{\theta} B$ to be n -weakly amenable, for an integer $n \geq 0$. We extend some results about the n -weak amenability of $A^{\sharp}$, obtained by Zhang, to the θ -Lau product $A \times_{\theta} B$. In particular, we improve several results on n -weak amenability of $A \times_{\theta} B$, fix the gap in Theorem 4.1 of [4], and partially answer some questions on this topic.

2. $(2n+1)$ -weak amenability

We start this section with some preliminaries about n -weak amenability. Let A be a Banach algebra, and X a Banach A -bimodule. Then the dual space X^*

of X becomes a dual Banach A -bimodule with the module actions defined by

$$(fa)(x) = f(ax), \quad (af)(x) = f(xa),$$

for all $a \in A, x \in X$ and $f \in X^*$. Similarly, the n -th dual $X^{(n)}$ of X is a Banach A -bimodule. In particular, $A^{(n)}$ is a Banach A -bimodule. A derivation from A into X is a linear mapping $D : A \rightarrow X$, satisfying

$$D(ab) = D(a)b + aD(b) \quad (a, b \in A).$$

If $x \in X$, then $d_x : A \rightarrow X$ defined by $d_x(a) = ax - xa$ is a derivation. A derivation D is inner if there is an $x \in X$ such that $D = d_x$.

A Banach algebra A is called n -weakly amenable, for an integer $n \geq 0$, if every continuous derivation from A into $A^{(n)}$ is inner, where $A^{(0)} = A$. The algebra A is said to be weakly amenable if it is 1-weakly amenable. The concept of weak amenability was first introduced by BADE, CURTIS and DALES in [1] for commutative Banach algebras, and was extended to the noncommutative case by JOHNSON [5]. DALES, GHAHRAMANI and GRØNBÆK [3] initiated and intensively developed the study of n -weak amenability of Banach algebras.

Throughout the paper, n is assumed to be a non-negative integer, A and B are assumed to be Banach algebras, and θ an element of $\sigma(B)$. For brevity of notation, we usually identify an element of A with its canonical image in $A^{(2n)}$, as well as an element of A^* with its image in $A^{(2n+1)}$.

The Banach space $(A \times_{\theta} B)^{(2n+1)}$ can be identified with the Banach space $A^{(2n+1)} \times B^{(2n+1)}$ equipped with the maximum norm $\|(f, g)\| = \max\{\|f\|, \|g\|\}$ in the natural way. By induction, we find that the $(A \times_{\theta} B)$ -bimodule actions on $(A \times_{\theta} B)^{(2n+1)}$ are formulated as follows:

$$\begin{aligned} (f, g)(a, b) &= (fa + \theta(b)f, gb + f(a)\theta), \\ (a, b)(f, g) &= (af + \theta(b)f, bg + f(a)\theta), \end{aligned}$$

for $a \in A, b \in B, f \in A^{(2n+1)}$ and $g \in B^{(2n+1)}$.

To clarify the relation between $(2n+1)$ -weak amenability of $A \times_{\theta} B$ and that of A and B , we begin with the following lemma which plays a key role in the sequel. This lemma was proved in [12, Proposition 2.1].

Lemma 2.1. *A mapping $D : A \times_{\theta} B \rightarrow (A \times_{\theta} B)^{(2n+1)}$ is a continuous derivation if and only if*

$$D(a, b) = (D_A(b) + T_A(a), D_B(b) + T_B(a)),$$

for all $a \in A$ and $b \in B$, where

- (a) $D_B : B \rightarrow B^{(2n+1)}$ is a continuous derivation.
- (b) $D_A : B \rightarrow A^{(2n+1)}$ is a bounded linear operator such that $D_A(b_1b_2) = \theta(b_1)D_A(b_2) + \theta(b_2)D_A(b_1)$ for all $b_1, b_2 \in B$, and $D_A(b)a = aD_A(b) = 0$ for all $a \in A$ and $b \in B$.
- (c) $T_B : A \rightarrow B^{(2n+1)}$ is a bounded linear operator such that $bT_B(a) = T_B(a)b$ and $\theta(b)T_B(a) = bT_B(a) + D_A(b)(a)\theta$ for all $a \in A$ and $b \in B$.
- (d) $T_A : A \rightarrow A^{(2n+1)}$ is a continuous derivation such that $(T_A(a_1)(a_2) + T_A(a_2)(a_1))\theta = T_B(a_1a_2)$ for $a_1, a_2 \in A$.

Moreover, $D = d_{(f,g)}$, for some $f \in A^{(2n+1)}$ and $g \in B^{(2n+1)}$, if and only if $D_B = d_g$, $T_A = d_f$, $D_A = 0$ and $T_B = 0$.

As a first result, we give general necessary and sufficient conditions for $A \times_\theta B$ to be $(2n+1)$ -weakly amenable.

Theorem 2.2. *The θ -Lau product $A \times_\theta B$ is $(2n+1)$ -weakly amenable if and only if*

- (1) B is $(2n+1)$ -weakly amenable.
- (2) The only bounded linear operator $S : A \rightarrow B^{(2n+1)}$, such that $S(a_1a_2) = 0$ for all $a_1, a_2 \in A$ and $bS(a) = S(a)b = \theta(b)S(a)$ for all $b \in B$ and $a \in A$, is zero.
- (3) If $T : A \rightarrow A^{(2n+1)}$ is a continuous derivation such that there exists a bounded linear operator $S : A \rightarrow B^{(2n+1)}$ satisfying $(T(a_1)(a_2) + T(a_2)(a_1))\theta = S(a_1a_2)$ for all $a_1, a_2 \in A$, then T is inner.

Before we prove this theorem, we need the following lemmas.

Lemma 2.3. *Condition (2) in Theorem 2.2 is equivalent to the density of A^2 in A .*

PROOF. Let condition (2) in Theorem 2.2 hold, and let $f \in A^*$ be such that $f|_{A^2} = 0$. Define $S : A \rightarrow B^{(2n+1)}$ by $S(a) = f(a)\theta$ for all $a \in A$. Then S is a bounded linear operator such that $S(a_1a_2) = 0$ for all $a_1, a_2 \in A$, and $bS(a) = S(a)b = \theta(b)S(a)$ for all $b \in B$ and $a \in A$. So, $S = 0$. This shows that $f = 0$. Therefore, A^2 is dense in A . The converse is clear. $\square$

Lemma 2.4. *Let B be a weakly amenable Banach algebra, and X be a Banach space. If $D : B \rightarrow X$ is a bounded linear operator such that $D(b_1b_2) = \theta(b_1)D(b_2) + \theta(b_2)D(b_1)$ for all $b_1, b_2 \in B$, then $D = 0$.*

PROOF. Let $f \in X^*$. Then $f \circ D : B \rightarrow \mathbb{C}$ is a continuous point derivation at θ , so it is zero [3, Proposition 1.3]. This shows that $D = 0$. $\square$

Now, we are ready to prove Theorem 2.2.

PROOF. To prove the necessity, suppose that $A \times_{\theta} B$ is $(2n+1)$ -weakly amenable. Let $D : B \rightarrow B^{(2n+1)}$ be a continuous derivation. Then $\overline{D} : A \times_{\theta} B \rightarrow (A \times_{\theta} B)^{(2n+1)}$ defined by $\overline{D}(a, b) = (0, D(b))$ is a continuous derivation, and hence it is inner. Lemma 2.1 implies that D is also inner. So B is $(2n+1)$ -weakly amenable.

To prove (2), let $S : A \rightarrow B^{(2n+1)}$ be a bounded linear operator such that $S(a_1 a_2) = 0$ for all $a_1, a_2 \in A$ and $bS(a) = S(a)b = \theta(b)S(a)$ for all $b \in B$ and $a \in A$. Define $\overline{D} : A \times_{\theta} B \rightarrow (A \times_{\theta} B)^{(2n+1)}$ by $\overline{D}(a, b) = (0, S(a))$. Then $\overline{D}$ is a continuous derivation, by Lemma 2.1. Thus $S = 0$ by the innerness of $\overline{D}$.

By a similar argument, we can prove (3). Indeed, suppose that $T : A \rightarrow A^{(2n+1)}$ is a continuous derivation, and $S : A \rightarrow B^{(2n+1)}$ is a bounded linear operator satisfying

$$(T(a)(c) + T(c)(a))\theta = S(ac),$$

for all $a, c \in A$. This, together with Lemma 2.3, implies that $bS(a) = S(a)b = \theta(b)S(a)$ for all $a \in A$ and $b \in B$. Define $\overline{D} : A \times_{\theta} B \rightarrow (A \times_{\theta} B)^{(2n+1)}$ by $\overline{D}(a, b) = (T(a), S(a))$. Then Lemma 2.1 implies that $\overline{D}$ is a continuous derivation, so it is inner. Therefore T is inner, as required. This completes the proof of necessity.

To prove the sufficiency, suppose that conditions (1)–(3) hold. Let $\overline{D} : A \times_{\theta} B \rightarrow (A \times_{\theta} B)^{(2n+1)}$ be a continuous derivation. Then

$$\overline{D}(a, b) = (D_A(b) + T_A(a), D_B(b) + T_B(a)), \quad (a \in A, b \in B),$$

in which the component mappings D_A, D_B, T_A and T_B are satisfying the conditions (a)–(d) of Lemma 2.1. By condition (1) and [3, Proposition 1.2], B is weakly amenable, and so Lemma 2.4 implies that $D_A = 0$. By conditions (1) and (3), D_B and T_A are inner derivations. Since $D_A = 0$, it follows that $bT_B(a) = T_B(a)b = \theta(b)T_B(a)$ for all $b \in B$ and $a \in A$. Moreover, since T_A is inner, $T_A(a_1)(a_2) + T_A(a_2)(a_1) = 0$, for all $a_1, a_2 \in A$. It follows that $T_B(a_1 a_2) = 0$ for all $a_1, a_2 \in A$. From condition (2), $T_B = 0$. Therefore, $\overline{D}$ is inner, by Lemma 2.1. This proves that $A \times_{\theta} B$ is $(2n+1)$ -weakly amenable, as claimed. $\square$

As an immediate consequence of Theorem 2.2, we have the next result, which was proved in [12, Proposition 2.3]. Before, we recall that A is called $(2n+1)$ -cyclicly weakly amenable if every continuous derivation $D : A \rightarrow A^{(2n+1)}$ for which $D(a_1)(a_2) + D(a_2)(a_1) = 0$, for all $a_1, a_2 \in A$, is inner. This result, for the case $n = 0$, was also proved in [9, Theorem 2.11].

Corollary 2.5. *If $A \times_{\theta} B$ is $(2n + 1)$ -weakly amenable, then B is $(2n + 1)$ -weakly amenable and A is $(2n + 1)$ -cyclicly weakly amenable.*

PROOF. By condition (1) of Theorem 2.2, B is $(2n + 1)$ -weakly amenable. The $(2n + 1)$ -cyclic weak amenability of A follows from condition (3) of Theorem 2.2, if we take $S = 0$. $\square$

Using Theorem 2.2, for $B = \mathbb{C}$ and $\theta = \iota$, we get the next result, which was already proved in [13, Proposition 3.9].

Corollary 2.6. *$A^{\#}$ is $(2n + 1)$ -weakly amenable if and only if*

- (1) *$\langle A^2 \rangle$, the linear span of A^2 , is dense in A .*
- (2) *Every continuous derivation $D : A \rightarrow A^{(2n+1)}$, with the condition that there is a $T \in A^*$ such that $D(a_1)(a_2) + D(a_2)(a_1) = T(a_1 a_2)$ for all $a_1, a_2 \in A$, is inner.*

We know from [3, Proposition 1.3] that if A is $(2n + 1)$ -weakly amenable, then A^2 is dense in A . Thus, as a consequence of Lemma 2.3 and Theorem 2.2, we have the next result which extends the related results on $(2n + 1)$ -weak amenability of $A^{\#}$ [3, Proposition 1.4], and improves [12, Proposition 2.4]. This result has been proved in [4, Theorem 4.1], but the proof contains a gap that we fix here.

Proposition 2.7. *Let A and B be $(2n + 1)$ -weakly amenable. Then $A \times_{\theta} B$ is $(2n + 1)$ -weakly amenable.*

Using Theorem 2.2 and Proposition 2.7, with $B = A$, we have the following.

Corollary 2.8. *A is $(2n + 1)$ -weakly amenable if and only if $A \times_{\theta} A$ is $(2n + 1)$ -weakly amenable.*

It was shown in [13, Corollary 3.10 and 3.12] and [3, Proposition 1.4] that if A is commutative or weakly amenable, or has a bounded approximate identity, then $A^{\#}$ is $(2n + 1)$ -weakly amenable if and only if A is $(2n + 1)$ -weakly amenable. In the next result, which is a consequence of Theorem 2.2 and Proposition 2.7, we extend it to $A \times_{\theta} B$.

Theorem 2.9. *Suppose that one of the following statements holds:*

- (i) *A has a bounded approximate identity.*
- (ii) *A is weakly amenable.*
- (iii) *A and B are commutative.*

Then $A \times_{\theta} B$ is $(2n + 1)$ -weakly amenable if and only if both A and B are $(2n + 1)$ -weakly amenable.

PROOF. By Proposition 2.7 and Theorem 2.2, in all three cases we have only to show that the $(2n+1)$ -weak amenability of $A \times_{\theta} B$ implies that A is weakly amenable. So, assume that $A \times_{\theta} B$ is $(2n+1)$ -weakly amenable. Assume that (i) holds and $D : A \rightarrow A^{(2n+1)}$ is a continuous derivation. Let $\{e_{\alpha}\}$ be a bounded approximate identity of A , and $E \in A^{**}$ be a weak* cluster point of $\{e_{\alpha}\}$. Define $S : A \rightarrow B^{(2n+1)}$ by $S(a) = D(a)(E)\theta$. Then

$$\begin{aligned} S(a_1 a_2) &= (D(a_1)a_2 + a_1 D(a_2))(E)\theta \\ &= (D(a_1)(a_2 E) + D(a_2)(E a_1))\theta = (D(a_1)(a_2) + D(a_2)(a_1))\theta, \end{aligned}$$

for all $a_1, a_2 \in A$. So, condition (3) of Theorem 2.2 implies that D is inner, as required.

Assume that (ii) holds and $D : A \rightarrow A^{(2n+1)}$ is a continuous derivation. Let $P : A^{(2n+1)} \rightarrow A^*$ be the projection with kernel $A^{\perp}$. Then $P \circ D : A \rightarrow A^*$ is an inner derivation. On the other hand, the continuous derivation $(I - P) \circ D : A \rightarrow A^{\perp} \subseteq A^{(2n+1)}$ satisfies condition (3) of Theorem 2.2, with $S = 0$. So, $(I - P) \circ D$ is inner. This shows that D is inner. So A is $(2n+1)$ -weakly amenable.

Finally, assume that (iii) holds. Since $A \times_{\theta} B$ is commutative and A is a closed ideal of $A \times_{\theta} B$, it is enough to show that A^2 is dense in A , see [2, Theorem 2.8.69]. For this, we are assuming that $A \times_{\theta} B$ is $(2n+1)$ -weakly amenable. Therefore, condition (2) of Theorem 2.2 is satisfied. By Lemma 2.3, we have that A^2 is dense in A , and this completes the proof. $\square$

3. $(2n)$ -weak amenability

In this section, we examine the conditions in which $A \times_{\theta} B$ is $(2n)$ -weakly amenable. First, we recall that the Banach space $(A \times_{\theta} B)^{(2n)}$ can be also identified with the Banach space $A^{(2n)} \times B^{(2n)}$ equipped with the norm $\|(f, g)\| = \|f\| + \|g\|$ in the natural way. By induction, we find that the $(A \times_{\theta} B)$ -bimodule actions on $(A \times_{\theta} B)^{(2n)}$ are formulated as follows:

$$\begin{aligned} (f, g)(a, b) &= (fa + g(\theta)a + \theta(b)f, gb), \\ (a, b)(f, g) &= (af + g(\theta)a + \theta(b)f, bg), \end{aligned}$$

for $a \in A, b \in B, f \in A^{(2n)}$ and $g \in B^{(2n)}$.

To clarify the relation between $(2n)$ -weak amenability of $A \times_{\theta} B$ and that of A and B , we need the following lemma that was proved in [12, Proposition 2.2].

Lemma 3.1. *A mapping $D : A \times_{\theta} B \rightarrow (A \times_{\theta} B)^{(2n)}$ is a continuous derivation if and only if*

$$D(a, b) = (D_A(b) + T_A(a), D_B(b) + T_B(a)),$$

for all $a \in A$ and $b \in B$, where

- (a) $D_B : B \rightarrow B^{(2n)}$ is a continuous derivation.
- (b) $D_A : B \rightarrow A^{(2n)}$ is a bounded linear operator such that $D_A(b_1 b_2) = \theta(b_1)D_A(b_2) + \theta(b_2)D_A(b_1)$ for all $b_1, b_2 \in B$ and $D_A(b)a = aD_A(b) = -D_B(b)(\theta)a$ for all $a \in A$ and $b \in B$.
- (c) $T_B : A \rightarrow B^{(2n)}$ is a bounded linear operator such that $T_B(a_1 a_2) = 0$ for all $a_1, a_2 \in A$ and $bT_B(a) = T_B(a)b = \theta(b)T_B(a)$ for all $a \in A$ and $b \in B$.
- (d) $T_A : A \rightarrow A^{(2n)}$ is a bounded linear operator such that $T_A(a_1 a_2) = a_1 T_A(a_2) + T_A(a_1)a_2 + T_B(a_2)(\theta)a_1 + T_B(a_1)(\theta)a_2$ for $a_1, a_2 \in A$.

Moreover, $D = d_{(f,g)}$ for some $f \in A^{(2n)}$ and $g \in B^{(2n)}$ if and only if $D_B = d_g$, $T_A = d_f$, $D_A = 0$ and $T_B = 0$.

As a first result for $(2n)$ -weak amenability of $A \times_{\theta} B$, we give the following characterization, which extends the related results in [12].

Theorem 3.2. *The θ -Lau product $A \times_{\theta} B$ is $(2n)$ -weakly amenable if and only if*

- (1) A is $(2n)$ -weakly amenable.
- (2) If $T : B \rightarrow B^{(2n)}$ is a continuous derivation such that there is a bounded linear operator $D : B \rightarrow A^{(2n)}$ satisfying $D(b_1 b_2) = \theta(b_1)D(b_2) + \theta(b_2)D(b_1)$ for all $b_1, b_2 \in B$, and $D(b)a = aD(b) = -T(b)(\theta)a$ for all $a \in A$ and $b \in B$, then T is inner.
- (3) The only bounded linear operator $D : B \rightarrow A^{(2n)}$, such that $D(b_1 b_2) = \theta(b_1)D(b_2) + \theta(b_2)D(b_1)$ for all $b_1, b_2 \in B$ and $aD(b) = D(b)a = 0$ for all $a \in A$ and $b \in B$, is zero.
- (4) If $S : A \rightarrow B^{(2n)}$ is a bounded linear operator such that $S(a_1 a_2) = 0$ for all $a_1, a_2 \in A$ and $bS(a) = S(a)b = \theta(b)S(a)$ for all $a \in A$ and $b \in B$, and there is a bounded linear operator $T : A \rightarrow A^{(2n)}$ satisfying $T(a_1 a_2) = a_1 T(a_2) + T(a_1)a_2 + S(a_1)(\theta)a_2 + S(a_2)(\theta)a_1$ for all $a_1, a_2 \in A$, then $S = 0$.

PROOF. To prove the necessity, suppose that $A \times_{\theta} B$ is $(2n)$ -weakly amenable. Let $D : A \rightarrow A^{(2n)}$ be a continuous derivation. Then $\overline{D} : A \times_{\theta} B \rightarrow (A \times_{\theta} B)^{(2n)}$ defined by $\overline{D}(a, b) = (D(a), 0)$ is a continuous derivation, and hence it is inner. It follows from Lemma 3.1 that D is inner. Therefore, A is $(2n)$ -weakly amenable.

To prove (2), let $T : B \rightarrow B^{(2n)}$ be a continuous derivation, and $D : B \rightarrow A^{(2n)}$ be a bounded linear operator satisfying (2). We define $\overline{D} : A \times_\theta B \rightarrow (A \times_\theta B)^{(2n)}$ by $\overline{D}(a, b) = (D(b), T(b))$. Then Lemma 3.1 implies that $\overline{D}$ is a continuous derivation, so it is inner. Hence, T is inner, as required.

Let $D : B \rightarrow A^{(2n)}$ be a bounded linear operator such that $D(b_1 b_2) = \theta(b_1)D(b_2) + \theta(b_2)D(b_1)$ for all $b_1, b_2 \in B$ and $aD(b) = D(b)a = 0$ for all $a \in A$ and $b \in B$. By Lemma 3.1, we conclude that $\overline{D} : A \times_\theta B \rightarrow (A \times_\theta B)^{(2n)}$, defined by $\overline{D}(a, b) = (D(b), 0)$, is a continuous derivation, and so it is inner. Hence, $D = 0$. This proves (3).

To prove (4), we use a similar argument. Indeed, if $S : A \rightarrow B^{(2n)}$ and $T : A \rightarrow A^{(2n)}$ are bounded linear operators satisfying (4), then Lemma 3.1 implies that $\overline{D} : A \times_\theta B \rightarrow (A \times_\theta B)^{(2n)}$, given by $\overline{D}(a, b) = (T(a), S(a))$, is a continuous derivation, and so it is inner. Hence, $S = 0$. This completes the proof of necessity.

For sufficiency, suppose that $\overline{D} : A \times_\theta B \rightarrow (A \times_\theta B)^{(2n)}$ is a continuous derivation. By Lemma 3.1, $\overline{D}$ is in the form

$$\overline{D}(a, b) = (D_A(b) + T_A(a), D_B(b) + T_B(a)), \quad (a \in A, b \in B),$$

in which the component mappings D_A, D_B, T_A and T_B satisfy the conditions (a)–(d) of Lemma 3.1. By condition (2), D_B is an inner derivation. Thus, there is $g \in B^{(2n)}$ such that $D_B = d_g$. Since $D_B(b)(\theta) = (bg - gb)(\theta) = g(\theta b - b\theta) = \theta(b)g(\theta - \theta) = 0$, from condition (3), we get $D_A = 0$. By condition (4), $T_B = 0$. This, together with condition (d) of Lemma 3.1, implies that T_A is a continuous derivation. From condition (1), it follows that there is $f \in A^{(2n+1)}$ such that $T_A = d_f$. Therefore, $\overline{D}$ is inner. This proves that $A \times_\theta B$ is $(2n)$ -weakly amenable, as claimed. $\square$

As an immediate consequence of Theorem 3.2, we have the next result, which has already been proved in [12, Proposition 2.5]; see also [3, Proposition 1.4]. Recall that B is called $\theta_{(2n)}$ -null weakly amenable if every continuous derivation $D : B \rightarrow B^{(2n)}$ for which $D(b)(\theta) = 0$, for all $b \in B$, is inner.

Corollary 3.3. *If $A \times_\theta B$ is $(2n)$ -weakly amenable, then A is $(2n)$ -weakly amenable and B is $\theta_{(2n)}$ -null weakly amenable.*

PROOF. By condition (1) of Theorem 3.2, A is $(2n)$ -weakly amenable. The $\theta_{(2n)}$ -null weak amenability of B follows from condition (2) of Theorem 3.2, by taking $D = 0$. $\square$

Proposition 3.4. *Suppose that A has a bounded approximate identity and $n \geq 1$. Then condition (2) in Theorem 3.2 is equivalent to $(2n)$ -weak amenability of B . If A is unital, then the equivalence is also true for $n = 0$.*

PROOF. If B is $(2n)$ -weakly amenable, then it is trivial that condition (2) of Theorem 3.2 holds. For the converse, first let $n \geq 1$, and let $\{e_\alpha\}$ be a bounded approximate identity of A . Suppose that $T : B \rightarrow B^{(2n)}$ is a continuous derivation. Define $D : B \rightarrow A^{(2n)}$ by $D(b) = -T(b)(\theta)E$, where $E \in A^{**}$ is a weak* cluster point of $\{e_\alpha\}$. Then $aD(b) = D(b)a = -T(b)(\theta)a$ for all $a \in A$ and $b \in B$. Moreover,

$$\begin{aligned} D(b_1b_2) &= -T(b_1b_2)(\theta)E = (-T(b_1)b_2 - b_1T(b_2))(\theta)E \\ &= -\theta(b_2)T(b_1)(\theta)E - \theta(b_1)T(b_2)(\theta)E = \theta(b_2)D(b_1) + \theta(b_1)D(b_2), \end{aligned}$$

for all $b_1, b_2 \in B$. So, condition (2) of Theorem 3.2 implies that T is inner, as required.

Now, let $n = 0$ and $\mathbf{1}$ be the unit of A . If $T : B \rightarrow B$ is a continuous derivation, then $D : B \rightarrow A$, defined by $D(b) = -T(b)(\theta)\mathbf{1}$, satisfies condition (2) of Theorem 3.2, and so T is inner. Therefore, B is (0) -weakly amenable. $\square$

Proposition 3.5. *Condition (3) of Theorem 3.2 holds if and only if $\langle AA^{(2n-1)} \cup A^{(2n-1)}A \rangle$ is dense in $A^{(2n-1)}$, or every continuous point derivation at θ is zero.*

PROOF. It is clear that condition (3) of Theorem 3.2 holds if $\langle AA^{(2n-1)} \cup A^{(2n-1)}A \rangle$ is dense in $A^{(2n-1)}$. So, assume that every continuous point derivation at θ is zero, and $D : B \rightarrow A^{(2n)}$ is a bounded linear map satisfies condition (3) of Theorem 3.2. If $f \in A^{(2n+1)}$, then $f \circ D$ is a continuous point derivation at θ , so it is zero. This implies that $D = 0$.

For the converse, take a non-zero $f \in A^{(2n)}$ with $af = fa = 0$ for all $a \in A$, and let $d : B \rightarrow \mathbb{C}$ be a continuous point derivation at θ . Then $D : B \rightarrow A^{(2n)}$ defined by $D(b) = d(b)f$ satisfies condition (3) of Theorem 3.2, so it is zero. Thus $d = 0$, as required. $\square$

Using Theorem 3.2, with $B = \mathbb{C}$ and $\theta = \iota$, we get the next result which extends [13, Proposition 3.13].

Corollary 3.6. *$A^\#$ is $(2n)$ -weakly amenable if and only if*

- (1) *A is $(2n)$ -weakly amenable.*
- (2) *Every $f \in A^*$, with the conditions that $f|_{A^2} = 0$, and for which there is a bounded linear operator $T : A \rightarrow A^{(2n)}$ such that $T(a_1a_2) = a_1T(a_2) + T(a_1)a_2 + f(a_1)a_2 + f(a_2)a_1$ for all $a_1, a_2 \in A$, is zero.*

We recall from [6] that B is called left (resp. right) θ -amenable if every continuous derivation from B into X^* is inner, for every Banach B -bimodule X with $b \cdot x = \theta(b)x$ (resp. $x \cdot b = \theta(b)x$); ($b \in B, x \in X$). This notion of amenability is a generalization of the left amenability of a class of Banach algebras studied by LAU in [8], known as Lau algebras. Example of left (resp. right) θ -amenable Banach algebras include amenable Banach algebras and the Fourier algebra $A(G)$ for a locally compact group G .

In the next proposition, which extends the related results on $(2n)$ -weak amenability of $A^{\sharp}$ [13, Proposition 3.13 and Corollary 3.14], we give an analogue to [12, Proposition 2.4] for the even case. This answers a question raised by EBRAHIMI VISHKI and KHODDAMI in [12].

Proposition 3.7. *Let A and B be $(2n)$ -weakly amenable, and let $\langle A^2 \rangle$ be dense in A . Then $A \times_{\theta} B$ is $(2n)$ -weakly amenable if one of the following statements holds:*

- (i) *There is no non-zero continuous point derivation at θ .*
- (ii) *$\langle AA^{(2n-1)} \cup A^{(2n-1)}A \rangle$ is dense in $A^{(2n-1)}$.*
- (iii) *B is weakly amenable.*
- (iv) *B is left (resp. right) θ -amenable.*

PROOF. This follows from Theorem 3.2, Proposition 3.5 and the fact that if B is either weakly amenable or left (resp. right) θ -amenable, then there is no non-zero continuous point derivation at θ [3, Proposition 1.3] and [6, Remark 2.4]. $\square$

For the converse of Proposition 3.7, we have the following.

Proposition 3.8. *Suppose that $A \times_{\theta} B$ is $(2n)$ -weakly amenable and $n \geq 1$. Then A and B are $(2n)$ -weakly amenable if one of the following statements holds:*

- (i) *A has a bounded approximate identity.*
- (ii) *B is (2) -weakly amenable.*

PROOF. (i) It follows from Theorem 3.2 and Proposition 3.4. (ii) In view of Theorem 3.2, we have to show that B is $(2n)$ -weakly amenable. To do this, let $T : B \rightarrow B^{(2n)}$ be a continuous derivation, and let $P : B^{(2n)} \rightarrow B^{**}$ be the projection with the kernel $B^{*\perp}$. Then $P \circ T : B \rightarrow B^{**}$ is an inner derivation. On the other hand, the continuous derivation $(I - P) \circ T : B \rightarrow B^{*\perp} \subseteq B^{(2n)}$ satisfies condition (2) of Theorem 3.2, with $D = 0$. So, $(I - P) \circ T$ is also inner. This shows that T is inner. So, B is $(2n)$ -weakly amenable. $\square$

From Propositions 3.7 and 3.8, we obtain also the following result which extends [12, Theorem 3.1].

Theorem 3.9. *Suppose that A has a bounded approximate identity, B is either weakly amenable or left (right) θ -amenable and $n \geq 1$. Then $A \times_{\theta} B$ is $(2n)$ -weakly amenable if and only if both A and B are $(2n)$ -weakly amenable.*

It was shown in [12, Theorem 3.1] that if A is unital, then the n -weak amenability of $A \times_{\theta} B$ is equivalent to the n -weak amenability of both A and B . It was left as an open question for the case when A has a bounded approximate identity; see [12, Remark 3.1]. If we combine Theorems 2.9 and 3.9, we have the following theorem which partially answers this question.

Theorem 3.10. *Suppose that A has a bounded approximate identity and B is either weakly amenable or left (right) θ -amenable. Then $A \times_{\theta} B$ is n -weakly amenable, for $n \geq 1$, if and only if both A and B are n -weakly amenable. If A is unital, then the equivalence is also true for $n = 0$.*

As a consequence of the above theorem, with $A = \mathbb{C}$ and $\theta \in \sigma(B)$, we have the next result.

Corollary 3.11. *The θ -Lau product $\mathbb{C} \times_{\theta} B$ is n -weakly amenable if and only if B is n -weakly amenable.*

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